Linear Algebra and Applications
First List
Deadline: April 1, 2020

The list below includes exercises from the textbook and some additional problems. Solving all problems is strongly recommended. The problems that must be written and sent to the monitor (scanned or photographed) by email **jose14manuel14@gmail.com by April 1** are: **Exercises: 16,19,21 (Chapter 1, Textbook) and Problems: 2,4,7.**

List of Problems

Textbook Exercises: 4,9,11-12,16,18-21 (Chapter 1) and 3,6-7 (Chapter 2).

Problem 1. *Let* $V = \{(a, b) : a, b \in \mathbb{R}^+\}$. *Define addition on* V *by*

$$(a_1, b_1) + (a_2, b_2) = (a_1 a_2, b_1 b_2).$$

Further, define scalar multiplication for $c \in \mathbb{R}$ *by*

$$c \cdot (a, b) = (a^c, b^c).$$

Prove that V *with these operations is a vector space over* $\mathbb{R}$.

Problem 2. *Let* V *be a vector space,* $\mathcal{F}$ *a collection of subspaces of* V *with the following property: If* $X, Y \in \mathcal{F}$, *then there exists a* $Z \in \mathcal{F}$ *such that* $X \cup Y \subset Z$. *Prove that* $\cup_{U \in \mathcal{F}} U$ *is a subspace of* V.

Problem 3. *Let* X *be a set and* K *a field. The* ***support*** *of a function* $f \in \mathbb{M}(X, K) = \{f : X \to K \mid f \text{ is a function}\}$ *is denoted by* $spt(f)$ *and is defined to be* $\{x \in X \mid f(x) \neq 0\}$. *We will say that* $f \in \mathbb{M}(X, K)$ *has finite support if* $spt(f)$ *is a finite set. We will denote by* $\mathbb{M}_{fin}(X, K)$ *the collection of all functions* $f : X \to K$ *which have finite support.*
a). *Prove that* $\mathbb{M}_{fin}(X, K)$ *is subspace of* $\mathbb{M}(X, K)$.
b). *For* $Y \subset X$, *let* $\chi_Y : X \to K$ *be the characteristic function of* Y, *that is:*

$$\chi_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y. \end{cases}$$

Prove that $\{\chi_{\{x\}} : x \in X\}$ *is a basis of* $\mathbb{M}_{fin}(X, K)$.

Problem 4. *Let* $V = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in F\}$ *and* $f, g, h \in V'$ *be defined by*

$$f(p) = p(-1), \;\; g(p) = p(0), \;\; h(p) = p(1),$$

with $p(x) \in V$. *Show that* $\{f, g, h\}$ *is basis for* V'.

Problem 5. *Let* V *be a finite-dimensional vector space and* $f, g \in V'$. *For any* $v \in V$, $f(v) = 0$ *if and only if* $g(v) = 0$. *Show that* f *and* g *are linearly dependent.*

Problem 6. *Let* $X = C[a,b]$ *(*$a, b \in \mathbb{R}$ *and* $a < b$*) be the vector space of all real-valued continuous functions over the interval* $[a,b]$ *and*

$$Y = \{f \in X \mid \int_a^b f(t)dt = 0\}$$

a). *Prove that if* $\mathbb{R}$ *is identified with the set of all constant functions over* $[a,b]$ *then* $X = \mathbb{R} \oplus Y$*.*
b). *For* $a = 0$*,* $b = 1$*, and* $f(t) = t^2 + t - 1$*, find the unique* $c \in \mathbb{R}$ *and* $g \in Y$ *such that* $f(t) = c + g(t)$ *for all* $t \in [0,1]$*.*

Problem 7. *Let* $V = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{R}\}$ *(the space of polynomials of degrees up to 2 and coefficients in* $\mathbb{R}$*) and* U *the subset of* V *satisfying*

$$\int_{-1}^{1} p(t)dt = 0$$

for all $p \in V$*.*
a). *Show that* U *is a subspace of* V *and find a basis to describe* U*.*
b). *Find a basis for the quotient space* V/U *and verify the Theorem 6 in Chapter 1 in the textbook.*