Linear Algebra and Applications
List 4
Deadline: May 20, 2020

The list below includes exercises from the textbook and some additional problems. **All problems** must be written and sent to the monitor (scanned or photographed) by email **jose14manuel14@gmail.com by May 20**.

List of Problems

Textbook Exercises: 1,2,3,4,5,6 (Chapter 7)

Problem 1. *Let A be a square matrix.*

a). If $A^n = 0$ for some positive integer n, show that $I - A$ is invertible.

b). Suppose that $A^2 + 2A + I = 0$. Show that $I - A$ is invertible.

Problem 2. *Let $A = (a_{ij})$ be a $n \times n$ invertible matrix such that*

$$\sum_{j=1}^{n} a_{ij} = a, \quad i = 1, 2, ..., n$$

a). Show that a must be an eigenvalue of A and that $a \neq 0$..

b). Show that if $A^{-1} = (b_{ij})$ then $\sum_{j=1}^{n} b_{ij} = 1/a$, for $i = 1, 2, ..., n$.

Problem 3. *Let A be an 2×2 matrix and suppose that $A^2 = 0$. Show that for $c \in \mathbb{R}$ we have $det(cI - A) = c^2$.*

Problem 4. *Let*

$$A = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}$$

Prove that the characteristic polynomial for A is $p(s) = s^3 - as^2 - bs - c$ and that this is also the minimal polynomial for A.

Problem 5. *Let $A = (a_{ij})$ be a $n \times n$ matrix satisfying the following positive diagonally dominant condition:*

$$a_{ii} > \sum_{j \neq i} |a_{ij}|, \quad i = 1, ..., n$$

Show that $det(A) > 0$. ***Hint:*** *Use the Problem 10 given in the List 3.*

Problem 6. *Let A be a $n \times n$ matrix such that $AA^T = I$ and $det(A) < 0$. Show that $det(A + I) = 0$.*

Problem 7. *Consider the real vector space* $\mathbb{R}^2$ *and define the function*

$$< x, y >= x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2$$

for $x = (x_1, y_1)$ *and* $y = (y_1, y_2)$.

a). *Show that* $<, >$ *is an inner product on* $\mathbb{R}^2$.

b). *Show that* $|x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2| \leq \sqrt{(x_1 - x_2)^2 + 3x_2^2}\sqrt{(y_1 - y_2)^2 + 3y_2^2}$

Problem 8. *Let* $A = (a_{ij})$ *be a* 2×2 *matrix. For* X, Y *in* $\mathbb{R}^{2\times1}$ *let*

$$f_A(X, Y) = Y^T AX.$$

Show that f_A *is an inner product on* $\mathbb{R}^{2\times1}$ *if and only if* $A = A^T$, $a_{11} > 0, a_{22} > 0$, *and* $det(A) > 0$.

Problem 9. *Consider the vectors*

$$\beta_1 = (3, 0, 4)$$

$$\beta_2 = (-1, 0, 7)$$

$$\beta_3 = (2, 9, 11)$$

in $\mathbb{R}^3$ *equipped with the standard inner product. Using the* ***Gram-Schmidt process*** *find a orthonormal basis for* $\mathbb{R}^3$ *and express an arbitrary vector* (x_1, x_2, x_3) *as a linear combination of elements in this new basis.*

Problem 10. *Let* V *be a vector space equipped with the inner product* $<, >$.

a). *Let* S *be a non-empty subset of* V. *Show that* $S^\perp$ *is a subspace of* V *and* $S \subset (S^\perp)^\perp$.

b). *Let* S_1 *and* S_2 *be two non-empty subsets of* V. *If* $S_1 \subset S_2$ *then* $S_2^\perp \subset S_1^\perp$.

c). *Let* U, W *be two subspaces of* V. *Show that* $(U + W)^\perp = U^\perp \cap W^\perp$.

Problem 11. *Let* V *be the vector space of all real* $n \times n$ *symmetric matrices.*

a). *Prove that* $< A, B >= tr(AB)$ *is an inner product on* V.

b). *In the case* $n = 2$, *find the distance between the matrices* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 4 \\ 4 & 10 \end{pmatrix}$