Obs For different Γ_j, the system is Hamiltonian for

$$\frac{dx_j'}{dt} = \frac{\partial H}{\partial y_j'}, \quad \frac{dy_j'}{dt} = -\frac{\partial H}{\partial x_j'}$$

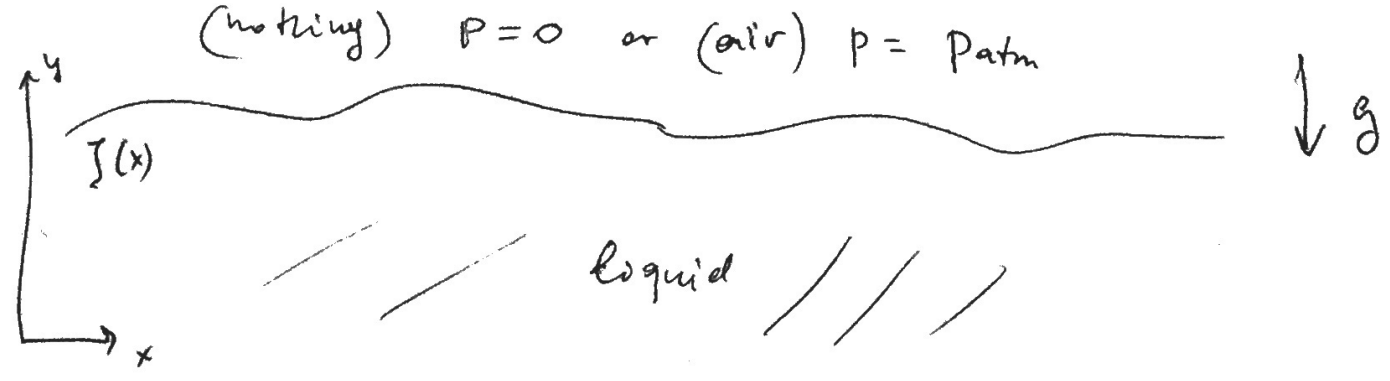
with $x_j' = \sqrt{|\Gamma_j|}\, x_j$, $y_j' = \sqrt{|\Gamma_j|}\, y_j \operatorname{sgn} \Gamma_j$

and $H = -\frac{1}{4\pi} \sum_{i \neq j} \Gamma_i \Gamma_j \log |x_i - x_j|$ written for x_j' and y_j'

Obs Methods of Hamiltonian mechanics and statistical mechanics can be used.

Gravity (surface) waves

(1) for potential incompressible ideal flow.
(2) semi-infinite fluid, (3) gravity g.

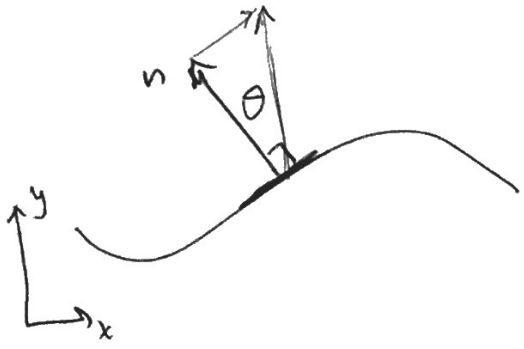


Equations for the potential φ:

$$\nabla\varphi = 0 \quad \text{for} \quad y < \zeta(x,t)$$

$$(\ u = \nabla\varphi\ ,\ \nabla\cdot u = \Delta\varphi = 0\ )$$

Conditions on <u>free</u> surface, $y = \zeta(x,t)$:



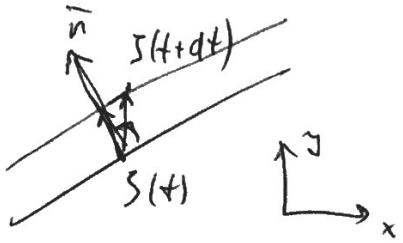
Normal speed of flow =

= normal speed of boundary

$$\vec{n} = \frac{1}{\sqrt{1-(\zeta_x)^2}}\,(-\zeta_x, 1)\ , \qquad \zeta_x = \frac{\partial\zeta}{\partial x}$$

$$u_n = \frac{\partial\varphi}{\partial n} = \nabla\varphi\cdot n = \frac{1}{\sqrt{1+\zeta_x^2}}\left(-\frac{\partial\varphi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\varphi}{\partial y}\right) \quad \text{(flow)}$$

$$\frac{\partial\zeta}{\partial t}\cos\theta = \frac{\partial\zeta}{\partial t}\frac{1}{\sqrt{1+\zeta_x^2}} \quad \text{(boundary)}$$



$$\Rightarrow \boxed{\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\zeta}{\partial t}}$$

kinematic b.c. at $y = \zeta(x,t)$.

For irrotational flow (see prev. lectures)

$$\frac{\partial \varphi}{\partial t} + \underbrace{\frac{|u|^2}{2}}_{\left(\frac{u^2+v^2}{2}\right)} + \frac{p}{\rho} + g y = f(t)$$

$p = \text{const}$ at free boundary $\Rightarrow$ $\frac{p}{\rho}$, $f(t)$ can be "included" into φ.

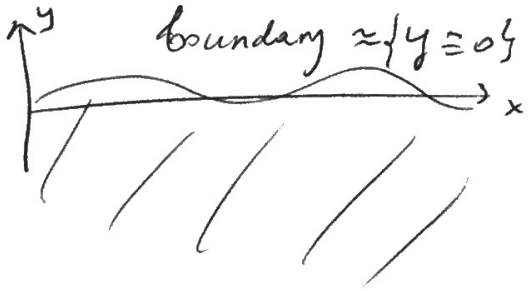
$$\boxed{\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi}{\partial y}\right)^2 + g\,\zeta(x,t) = 0} \quad \text{at } y = \zeta(x,t)$$

Dynamic bound. cond.

+ conditions of no flow as $y \to -\infty$: $\nabla\varphi \to 0$.

Stationary state : $\zeta \equiv 0$, $\varphi \equiv 0$.

Small-amplitude waves :



Linearizing all conditions:

$$\left.\frac{\partial \zeta}{\partial t}\right|_{y=\zeta} \approx \left.\frac{\partial \zeta}{\partial t}\right|_{y=0}, \quad \left.\frac{\partial \varphi}{\partial y}\right|_{y=\zeta} \approx \left(\frac{\partial \varphi}{\partial y}\right)_{y=0}$$

$\Rightarrow$ $\Delta\varphi = 0$ for $y < 0$

$$\left.\begin{aligned} &\frac{\partial \varphi}{\partial y} = \frac{\partial \zeta}{\partial t} \quad \text{(kinematic b.c. at } y=0) \\ &\frac{\partial \varphi}{\partial t} + g\zeta = 0 \quad \text{(dynamic b.c. at } y=0) \end{aligned}\right\} \quad \frac{\partial \varphi}{\partial y} + \frac{1}{g}\frac{\partial^2 \varphi}{\partial t^2} = 0 \quad \text{at } y = 0.$$

Solutions $\varphi = f(y)\cos(kx - \omega t)$

(components of Fourier transform in x)

$$\Delta\varphi = 0 \Rightarrow \frac{d^2 f}{dy^2} - k^2 f = 0 \Rightarrow f = Ae^{ky} + Be^{-ky}.$$

Since $e^{-ky} \to \infty$ as $ky \to -\infty \Rightarrow B = 0$ (otherwise $\nabla\varphi \to \infty$)

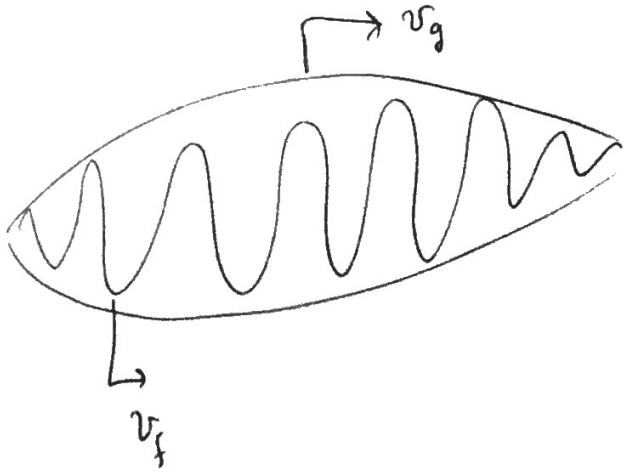
Hence, $\varphi = Ae^{ky}\cos(kx - \omega t)$.

At $y = 0$: $\frac{\partial\varphi}{\partial y} + \frac{1}{g}\frac{\partial^2\varphi}{\partial t^2} = 0 \Rightarrow k - \frac{\omega^2}{g} = 0$

$$\Rightarrow \boxed{\omega = \pm\sqrt{gk}}$$ dispersion relation for deep water.

Phase speed $v_p = \frac{\omega}{k} = \pm\sqrt{g/k}$

Group speed $v_g = \frac{d\omega}{dk} = \pm\frac{1}{2}\sqrt{g/k}$



Longer waves $L = \frac{2\pi}{k}$

(smaller k)

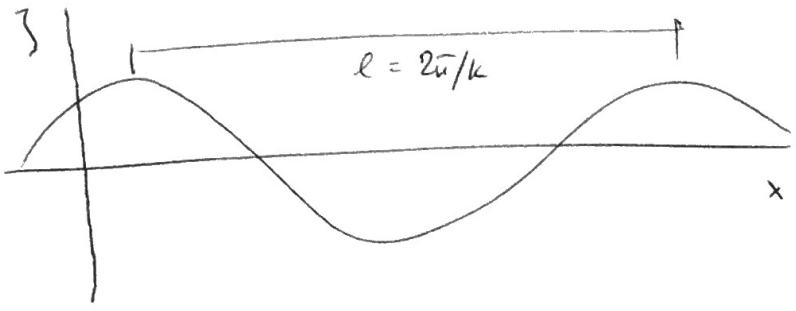
travel faster!

Phase speed $v_p = \omega/k$

solution : $\varphi = A e^{ky} \cos(kx - \omega t + \theta_0)$

$$\zeta = -\frac{1}{g}\left(\frac{\partial \varphi}{\partial t}\right)_{y=0} = -\frac{\omega A}{g} \sin(kx - \omega t + \theta_0)$$

$$= -\frac{\omega A}{g} \sin\left[k(x - v_p t) + \theta_0\right]$$



wave length = $2\pi/k$

wave speed = $v_p = \omega/k$

k is a wave number

ω is a frequency.

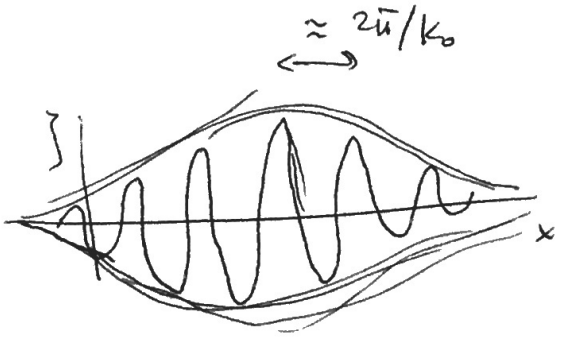
In complex form:

$$\zeta = \mathrm{Re}\left[b \exp\left[i(kx - \omega t)\right]\right], \quad b = \frac{\omega A i}{g} e^{i\theta_0}$$

Group speed:

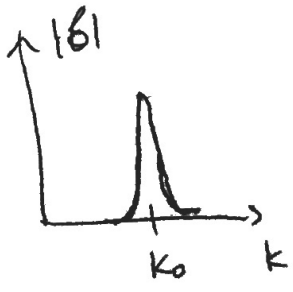
wave packet

$k \approx k_0$



$$\zeta = \mathrm{Re} \int b(k)\, e^{i(kx - \omega t)}\, dk$$

where $b(k)$ is "supported" near k_0

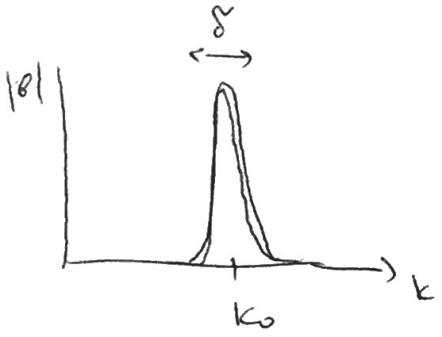


$t=0$: $\zeta(x) = \mathrm{Re} \int b(k)\, e^{ikx}\, dk =$

$= \mathrm{Re}\; e^{ik_0 x} \int b(k)\, e^{i(k-k_0)x}\, dk =$

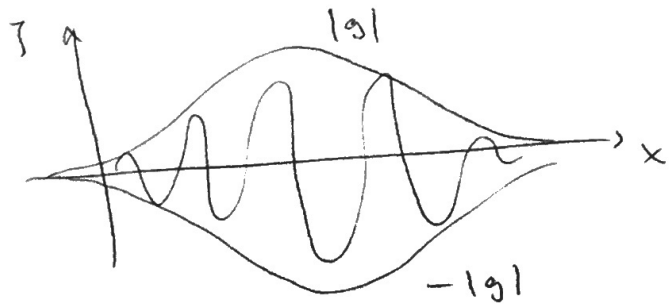
$= \mathrm{Re}\; e^{ik_0 x} \int b(k_0+\delta)\, e^{i\delta x}\, d\delta$

$= \mathrm{Re}\; e^{ik_0 x}\, g(x).$



since only small δ enter the integral

$\Rightarrow$ $g(x)$ changes at length $\frac{2\pi}{\delta} \gg \frac{2\pi}{k_0}$ (slow change)



$t>0$: $\zeta(x) = \mathrm{Re} \int b(k)\, e^{i(kx-\omega t)}\, dk$

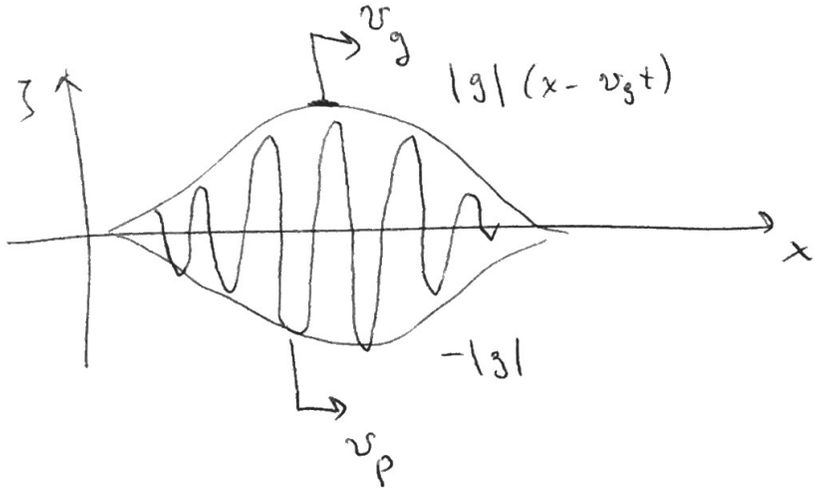
$\omega(k) \approx \omega_0 + \frac{d\omega}{dk}(k-k_0)$, $\omega_0 = \omega(k_0)$

$\zeta(x) \approx \mathrm{Re} \int b(k)\, e^{i(kx - \omega_0 t - v_g(k-k_0)t)}\, dk =$

$$= \mathrm{Re}\, e^{i(k_0 x - \omega_0 t)} \int B(k)\, e^{i[(k-k_0)x - v_g(k-k_0)t]}\, dk$$

$$= \mathrm{Re}\, e^{ik_0(x - v_p t)} \int B(k_0+\delta)\, e^{i\delta(x - v_g t)}\, d\delta$$

$$= \mathrm{Re}\, e^{ik_0(x - v_p t)}\, g(x - v_g t), \quad \left(\begin{array}{l} v_p = \omega_0/k_0 \\ v_g = d\omega/dk \end{array} \right)$$



group speed v_g is a speed of the wave envelope.

For deep water

$$\omega = \sqrt{gk} \quad \Rightarrow \quad v_p = \frac{\omega}{k} = \sqrt{g/k} \quad \Rightarrow v_p = 2v_g$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{g/k}$$

Longer waves (smaller k) travel faster.
Wave crests travel twice faster than a whole wave

DISPERSION: $v_p \neq v_g$.

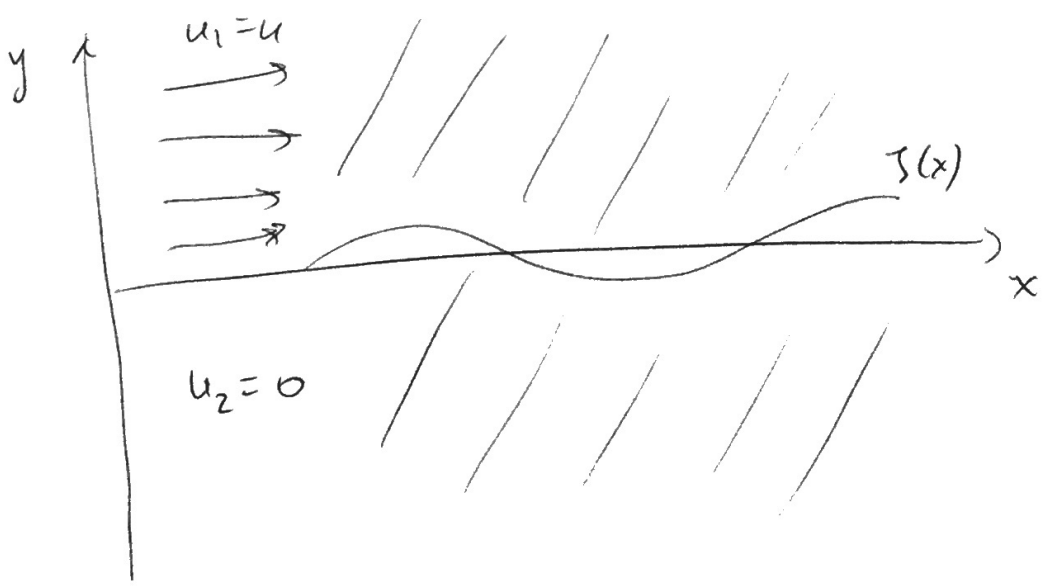
Obs: Wave breaking

nonlinear phenomenon

(linear theory is valid for finite times only!)

Kelvin-Helmholtz instability

(contact discontinuity)



fluid of speed u over fluid of zero speed

(in general, $u_1 \neq u_2$)

Question: dynamics of small perturbations $\zeta(x)$ and stability

Unperturbed solution:

$$\varphi = \begin{cases} ux, & y>0 \\ 0, & y<0 \end{cases}, \qquad \zeta(x) \equiv 0.$$

Small perturbation:

$$\varphi(x,y) = \begin{cases} ux + \tilde{\varphi}(x,y), & y > \zeta(x) \\ \tilde{\varphi}(x,y), & y < \zeta(x) \end{cases}$$

with small $\tilde{\varphi}$ and ζ.

For small ζ we can compute all quantities at the interface $y = \zeta(x) \approx 0$ in first approximation.

Kinematic boundary condition (normal speed = = normal speed of interface)

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial x}\frac{d\zeta}{dx} + \frac{\partial \zeta}{\partial t}$$

, see same eq. for gravitational waves.

$$\frac{\partial \varphi}{\partial x} \approx \begin{cases} u + \frac{\partial \tilde{\varphi}}{\partial x}, & y>0 \\ \frac{\partial \tilde{\varphi}}{\partial x}, & y<0 \end{cases}, \qquad \frac{\partial \varphi}{\partial y} = \frac{\partial \tilde{\varphi}}{\partial y}.$$

After linearization (keeping only first-order terms) we have two conditions: above ($y=0^+$) and below ($y=0^-$) the interface.

$y=0^+$: $\frac{\partial \tilde{\varphi}}{\partial y} = u\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial t}$; $y=0^-$: $\frac{\partial \tilde{\varphi}}{\partial y} = \frac{\partial \zeta}{\partial t}$.

Dynamic boundary condition:

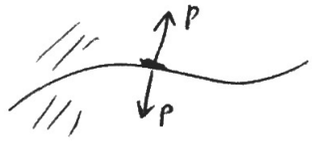
Bernoulli for potential flow (for $y>0$, and $y<0$)

$$\frac{\partial \varphi}{\partial t} + \frac{u^2+v^2}{2} + \frac{p}{\rho} = \text{const}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi}{\partial y}\right)^2 + \frac{p}{\rho} = \text{const}$$

Pressure is continuous across the interface

(3rd Newton's law)



$$\Rightarrow \left[\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \varphi}{\partial y}\right)^2\right]_{y=0^+} = \left[\text{—— } n \text{ ——}\right]_{y=0^-} + \text{const.}$$

After linearization: $\frac{\partial \varphi}{\partial t} = \frac{\partial \tilde{\varphi}}{\partial t}$, $\frac{\partial \varphi}{\partial x} = \begin{cases} u + \frac{\partial \tilde{\varphi}}{\partial x}, & y>0 \\ \frac{\partial \tilde{\varphi}}{\partial x}, & y<0 \end{cases}$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \tilde{\varphi}}{\partial y}$$

$$\left(\frac{\partial \tilde{\varphi}}{\partial t} + u\frac{\partial \tilde{\varphi}}{\partial x}\right)_{y=0^+} = \left(\frac{\partial \tilde{\varphi}}{\partial t}\right)_{y=0^-}$$

(here we took const $= \frac{1}{2}u^2$)

Summary

$\Delta\tilde{\varphi} = 0$ for $y > 0$ and $y < 0$

At $y = 0$:

$y = 0^+$: $\dfrac{\partial\tilde{\varphi}}{\partial y} = u\dfrac{\partial\zeta}{\partial x} + \dfrac{\partial\zeta}{\partial t}$

$y = 0^-$: $\dfrac{\partial\tilde{\varphi}}{\partial y} = \dfrac{\partial\zeta}{\partial t}$

$$\left(\frac{\partial\tilde{\varphi}}{\partial t} + u\frac{\partial\tilde{\varphi}}{\partial x}\right)_{y=0^+} = \left(\frac{\partial\tilde{\varphi}}{\partial t}\right)_{y=0^-}$$

Solution:

$$\tilde{\varphi} = f(y)\, e^{i(kx-\omega t)}, \qquad \zeta = C e^{i(kx-\omega t)}$$

(solution for a Fourier component in x-direction)

$\mathrm{Re}\,\tilde{\varphi}, \mathrm{Im}\,\tilde{\varphi}, \ldots \rightarrow$ real solutions (linear equations).

$$\Delta\tilde{\varphi} = \left(f''(y) - k^2 f\right) e^{i(kx-\omega t)} = 0$$

$$\Rightarrow f = Ae^{ky} + Be^{-ky}$$

For a bounded solution we must have $f = \begin{cases} Be^{-ky}, & y > 0 \\ Ae^{ky}, & y < 0 \end{cases}$

Hence, $\tilde{\varphi} = \begin{cases} B e^{-ky} e^{i(kx-\omega t)}, & y>0 \\ A e^{ky} e^{i(kx-\omega t)}, & y<0. \end{cases}$

Substituting into conditions at $y=0$:

$y=0^+$: $\frac{\partial \tilde{\varphi}}{\partial y} = y\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} \Rightarrow$

$$\Rightarrow -kB e^{i(kx-\omega t)} = cuki e^{i(kx-\omega t)} - cwie^{i(kx-\omega t)}$$

$$\Rightarrow kB - (\omega - uk)iC = 0$$

Similarly: $y=0^-$: $\frac{\partial \tilde{\varphi}}{\partial y} = \frac{\partial \eta}{\partial t} \Rightarrow kA + \omega iC = 0$

$$(\ldots)_{y=0^+} = (\ldots)_{y=0^-} \Rightarrow -\omega A + (\omega - uk)B = 0$$

All together we have 3 linear equations

$$\begin{pmatrix} 0 & k & -(\omega-uk)i \\ k & 0 & \omega i \\ -\omega & \omega-uk & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

$\exists (A,B,C) \neq (0,0,0) \Leftrightarrow \det[\text{matrix}] = 0 \Rightarrow$

$$\Rightarrow -\omega^2 ki - (\omega-uk)^2 ki = 0 \Rightarrow 2\omega^2 - 2uk\omega + u^2k^2 = 0 \Rightarrow$$

$$\boxed{\omega = \frac{uk}{2} \pm i\frac{uk}{2}}$$

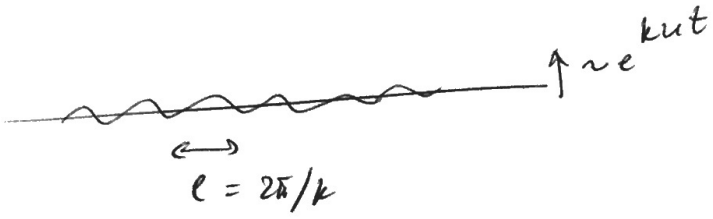
Solutions $\sim e^{-i\omega t} = e^{iukt/2} e^{\pm ukt}$

$|\beta| \sim e^{ukt} \nearrow \infty$ with increasing t.

This growth is faster for larger k

(smaller wavelength: $\ell = 2\pi/k$) $\Rightarrow$

$\Rightarrow$ small-scale perturbations "explode"



This means that 1) contact discontinuity is unstable

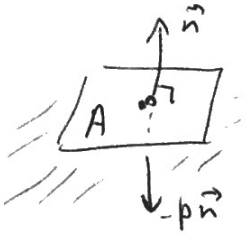
2) the problem is ill-posed (requires regularization, e.g., adding viscous term of the Navier-Stokes eq.)

3) Turbulent solutions when regularized!

VISCOUS FLOW

Ideal fluid is the fluid with properties (forces) corresponding to thermodynamic equilibrium:

$$\text{Force} = -p\vec{n}A$$



Thermod. equilibrium is possible, when fluid moves with $u \equiv \text{const}$ (as a "rigid body") or, rotates with constant angular velocity (as a "rigid body"):

total momentum / angular momentum = const.

Otherwize (i.e. in any nontrivial flow)

$$\text{Force} = -p\vec{n}A + \boxed{\text{nonequilibrium term.}}$$

Local velocity function:

$$u(r) = u_0 + \frac{\partial u}{\partial r} r + \frac{1}{2} \frac{\partial^2 u}{\partial r^2} r^2 + \dots$$

Using $r = (x_1, x_2, x_3)$,

$$u(r) = u_0 + \sum_i \frac{\partial u}{\partial x_i} x_i + \frac{1}{2} \sum_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} x_i x_j + \dots$$

For microscopic motion, macroscopic derivatives are small $\Rightarrow$ in first approximation

$$u(r) \simeq u_0 + \frac{\partial u}{\partial r} r \Rightarrow \text{noneq. term is a linear function of } \frac{\partial u}{\partial r}$$

This will give the Navier-Stokes equations

(higher-order approximations $\longrightarrow$ different models)

Obs $\frac{\partial u}{\partial r} = D + S$ with $S = \frac{1}{2}\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$

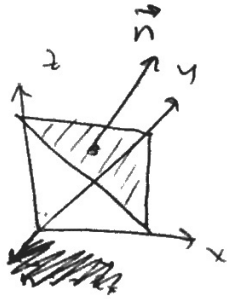
symm. part — antisymm. part

describing rotation (not violating therm. equil.)

$\Rightarrow$ nonequil. force term depends only on D

$$D = \frac{1}{2}\left[\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial t}\right)^T\right] \quad \left(\text{linear function of } D\right)$$

Obs Consider a small volume



$F^{(n)}$ force on face $\perp \vec{n}$

$-F^{(x)}$ force on face $\perp \vec{e}_x$

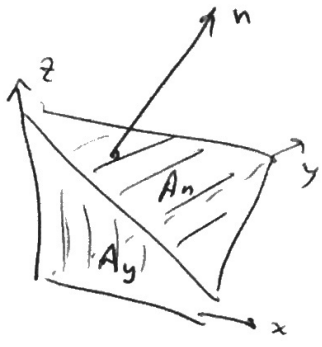
$-F^{(y)}$ " " " $\perp \vec{e}_y$

$-F^{(z)}$ " " " $\perp \vec{e}_z$

$$m\ddot{a} = F^{(n)} - F^{(x)} - F^{(y)} - F^{(z)}.$$

Force is proportional to the (small) area

$$F^{(n)} = T^{(n)} A_n, \quad F^{(x,y,z)} = T^{(x,y,z)} A_{x,y,z}$$



$$\frac{m}{A_n}\ddot{a} = [illegible]$$

$$= T^{(n)} - \underbrace{\frac{A_x}{A_n}}_{n_x} T^{(x)} - \underbrace{\frac{A_y}{A_n}}_{n_y} T^{(y)} - \underbrace{\frac{A_z}{A_n}}_{n_z} T^{(z)}$$

$$= T^n - n_x T^{(x)} - n_y T^{(y)} - n_z T^{(z)}.$$

In the limit of volume $\to 0$, we have $\frac{m}{A_n} \to 0$.

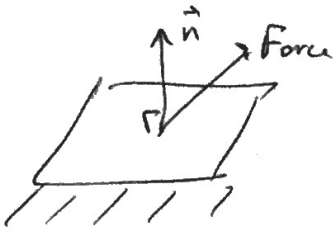
Hence, $T^{(n)} = n_x T^{(x)} + n_y T^{(y)} + n_z T^{(z)} = \sigma \vec{n}$

with (3×3) matrix σ (balance of forces at a point).

Similarly, using balance of angular momenta, one can show that $\sigma^T = \sigma$ (symmetric matrix).

(Cauchy's theorem).

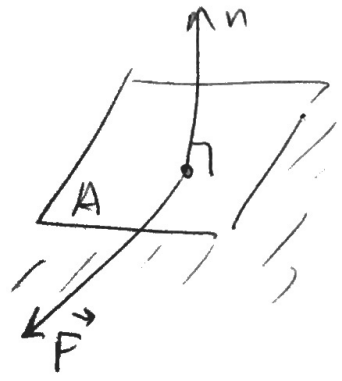
We found: Force $= (-p\vec{n} + \sigma\vec{n})A$



Cauchy's theorem: the force applied to a small surface element ΔA in a continuum is equal to

$$F = \sigma \vec{n} A$$

where σ is a symmetric $n \times n$ matrix (tensor) and $\vec{n} \in \mathbb{R}^n$ is the normal vector (n is the space dimension)

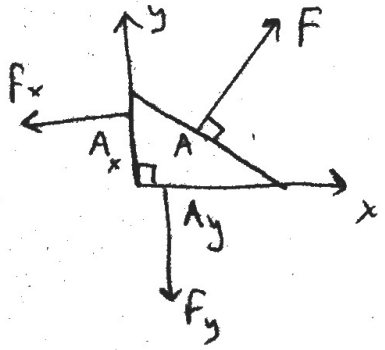


Obs 1: F depends linearly on $\vec{n}$!

Obs 2: $\sigma^T = \sigma$!

Proof (for $n=2$)

(a) Consider a small triangle with sides A, A_x, A_y



By 2d Newton's law

$$ma = F + F_x + F_y$$

Dividing by A and taking $\vec{n} = \left(\frac{A_x}{A}, \frac{A_y}{A}\right)$

$$\frac{m}{A} a = \frac{F}{A} + \frac{F_x}{A_x} n_x + \frac{F_y}{A_y} n_y$$

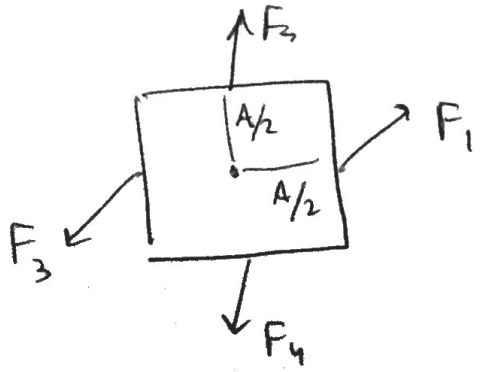
In the limit of $\triangle \to \bullet$ (size of triangle $\to 0$), $\frac{m}{A} \to 0$ ($m \sim A^2$)

$\Rightarrow$ [illegible]

$$\frac{F}{A} \to T \;,\quad \frac{F_x}{A_x} \to T_x \;,\quad \frac{F_y}{A_y} \to T_y \;,\quad T = -T_x n_x - T_y n_y = \sigma \vec{n}$$

$$\Rightarrow F = \sigma \vec{n} A.$$

(6) Consider a square region of size $A \times A$ with small A.



$$F_1 = \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} A \,, \quad F_2 = \sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} A$$

$$F_3 = \sigma \begin{pmatrix} -1 \\ 0 \end{pmatrix} A \,, \quad F_4 = \sigma \begin{pmatrix} 0 \\ -1 \end{pmatrix} A$$

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$$

For the angular momentum we have

$$I \ddot{\varphi} = \begin{pmatrix} A/2 \\ 0 \end{pmatrix} \times F_1 + \underbrace{\begin{pmatrix} 0 \\ A/2 \end{pmatrix} \times F_2}_{\text{torque}} + \begin{pmatrix} -A/2 \\ 0 \end{pmatrix} \times F_3 + \begin{pmatrix} 0 \\ -A/2 \end{pmatrix} \times F_4$$

I: moment of inertia; φ: rotation angle

Substituting forces and dividing by A^2, we have

$$\frac{I}{A^2} \ddot{\varphi} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \sigma_{xx} \\ \sigma_{yx} \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \times \begin{pmatrix} \sigma_{xy} \\ \sigma_{yy} \end{pmatrix} + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sigma_{xx} \\ -\sigma_{yx} \end{pmatrix} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \times \begin{pmatrix} -\sigma_{xy} \\ -\sigma_{yy} \end{pmatrix} =$$

$$\frac{I}{A^2} \ddot{\varphi} = \sigma_{yx} - \sigma_{xy} \,, \quad \text{in the limit } A \to 0$$

$$I \sim A^4 \Rightarrow \frac{I}{A^2} \to 0.$$

$$\Rightarrow \sigma_{xy} = \sigma_{yx} \Rightarrow \sigma^T = \sigma.$$

We need a symmetric matrix σ, which is linear function of $D = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ * & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ * & * & \frac{\partial w}{\partial z} \end{bmatrix}$

(* symmetric, * sym, * sym)

But this expression must be rotation-invariant due to isotropy of the fluid.

Symmetry theory provides two such functions:

$$D \quad \text{and} \quad (\operatorname{tr} D)\,\bar{I} = (\nabla \cdot u)\,\bar{I}.$$

Def

$$\sigma = \lambda(\nabla \cdot u)\,\bar{I} + 2\mu D$$

$$= \zeta(\nabla \cdot u)\,\bar{I} + \mu\left(2D - \frac{2}{3}(\nabla \cdot u)\,I\right)$$

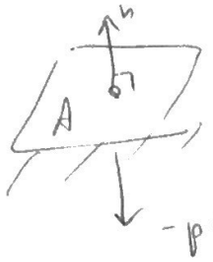
where $\zeta = \lambda + \frac{2}{3}\mu$.

Here λ is the bulk viscosity
μ is the dynamic viscosity
ζ is the second viscosity

} 2 independent viscous parameters.

In Euler eqs: $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\frac{\nabla p}{\rho} + b$

~~In Viscous flow~~ when force $= -p\vec{n}A$.



In viscous flow Force $= -p\vec{n}A + \sigma\vec{n}A =$

$= -(+pI - \sigma)\vec{n}A.$

In derivations, we substitute $pI \to pI - \tau$,

(derivations with Ostrogradsky-Gauss theorem or divergence th.)

which yields

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\frac{\nabla p}{\rho} + \frac{\nabla\tau}{\rho} + b$$

$$\nabla\sigma = (\partial_x, \partial_y, \partial_z)\boxed{\tau}$$

3×3

Substituting σ from Def $\Rightarrow$

$$\rho\frac{Du}{Dt} = \boxed{\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = -\nabla p + \tfrac{2}{3}\mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u)}$$

Navier-Stokes eq. for compressible flow

Obs $(\nabla\tau)_i = \partial_j\sigma_{ij} = \partial_j\left[\lambda(\nabla\cdot u)\delta_{ij} + \mu\partial_j u_i + \mu\partial_i u_j\right] =$

$= \lambda\partial_i(\nabla\cdot u) + \mu\partial_j^2 a_i + \mu\underbrace{\partial_j\partial_i u_j}_{\partial_i(\nabla\cdot u)} \Rightarrow \mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u)$

Compressible flow :

N.S. eq. + continuity eq. + constitutive eq. ($p = p(\rho)$)

Incompressible flow :

$$\boxed{\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho} \Delta u, \quad \nabla \cdot u = 0}$$

For $\rho = \text{const}$, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity

New type of boundary condition is necessary :

$u = 0$ on fixed boundary (no-slip)

(or $u = u_b$, flow speed = boundary speed for moving boundaries)

Good news : N.S. eq. is well-conditioned,
$\exists!$ solution locally in time,
continuously depending on initial condit.

Bad news : we have no proof of $\exists$ of smooth solution globally in time, $t > 0$.

(millenium problem).

Energy dissipation

$$E_{km} = \frac{\rho}{2} \int_D |u|^2 dV$$

$$\frac{d}{dt} E_{km} = \frac{\rho}{2} \frac{d}{dt} \int_D |u|^2 dV = \rho \int_D u \cdot \frac{\partial u}{\partial t} dV.$$

$(|u| = u \cdot u)$

For incompressible N.S. (no body force)

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla) u - \frac{\nabla p}{\rho} + \nu \Delta u \quad , \quad \nabla \cdot u = 0$$

$$u \cdot \frac{\partial u}{\partial t} = -u \cdot \left[(u \cdot \nabla) u \right] - \frac{1}{\rho} (u \cdot \nabla) p + \nu u \cdot \Delta u.$$

(a) Exercise (from the past): $u \cdot (u \cdot \nabla) u = \frac{1}{2} (u \cdot \nabla) |u|^2$

if $\nabla \cdot u = 0 \Rightarrow u \cdot (u \cdot \nabla) u = \nabla \cdot \left(\frac{|u|^2}{2} u \right)$

(b) $(u \cdot \nabla) p = \nabla \cdot (pu)$ if $\nabla \cdot u = 0$

(c) $u \cdot \Delta u = \nabla \cdot (u \cdot \nabla u) - |\nabla u|^2$ (check!)

where $|\nabla \vec{u}|^2 \overset{\text{def}}{=} |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2$.

Hence,

$$\frac{d}{dt} E_{km} = \rho \int_D u \cdot \frac{\partial u}{\partial t} dV = \rho \int_D \nabla \cdot \left[-\frac{|u|^2}{2} u - \frac{p}{\rho} u + \nu \, u \cdot \nabla u \right] dV$$

$$- \rho \nu \int_D |\nabla u|^2 dV$$

$$\int_D \nabla \cdot (\dots)\, dV = \int_{\partial D} (\dots)\cdot n\, ds = 0 \qquad (u = 0 \text{ at } \partial D)$$

$$\Rightarrow \frac{d}{dt} E_{kin} = -\rho\nu \int_D |\nabla u|^2 dV = -\mu \int_D |\nabla u|^2 dV.$$

Thermodynamic condition : $\mu \geq 0$.

In compressible case, similar analysis yields : $\mu \geq 0$,

$$\lambda + \frac{2}{3}\mu \geq 0.$$

Exercise

Show that in incomp. flow in finite domain D (or infinite domain with $u \to 0$ at ∞ sufficiently fast),

$$\frac{d}{dt} E_{kin} = \frac{\mu}{2} \sum_{i,j=1}^{3} \int_D \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 dV - \mu \int_D \|\omega\|^2 dV =$$

$$= -2\mu\Omega,$$

where $\Omega = \frac{1}{2}\int \|\omega\|^2 dV$ is called the enstrophy.

Dimensional analysis

Typically, flows can be characterized by

L (characteristic length), and

V (characteristic speed), $T = \frac{L}{V}$ characteristic time

Moving to dimensionless coordinates:

$$\tilde{x} = \frac{x}{L}, \quad \tilde{u} = \frac{u}{V}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{p} = \frac{p}{\rho V^2}$$

$$\tilde{y} = y/L$$

$$\tilde{z} = z/L$$

($\rho \equiv$ const)

we have for incompressible N.S. eq.

$$\frac{V}{T}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{V}{T}(\tilde{u}\cdot\tilde{\nabla})\tilde{u} = -\frac{V}{T}\tilde{\nabla}\tilde{p} + \nu\frac{V}{L^2}\tilde{\Delta}\tilde{u}$$

or

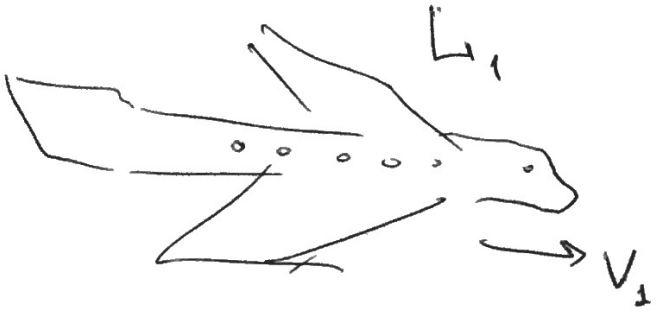
$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + (\tilde{u}\cdot\tilde{\nabla})\tilde{u} = -\tilde{\nabla}\tilde{p} + \frac{1}{R}\tilde{\Delta}\tilde{u}$$

where R (or Re) $= \frac{VL}{\nu}$ is the dimensionless Reinolds number.

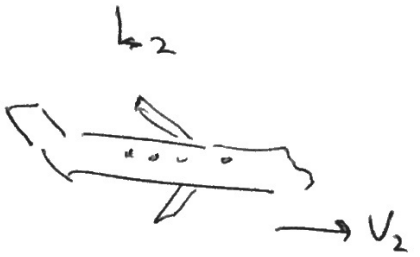
(the only parameter in dimensionless N.S. eq!)

Thus, two flows are <u>similar</u>, if they are described by the same dimensionless solution (same Re).

Example



Real size plane



Small model

Solutions are similar

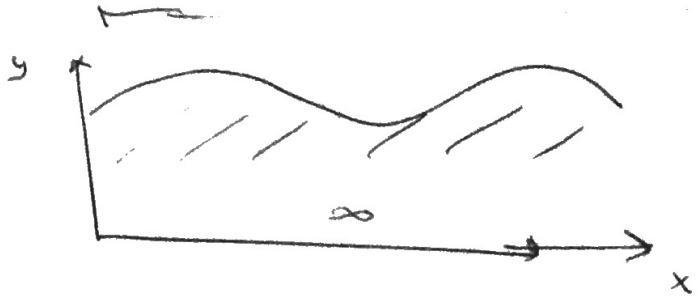
if $Re_1 = \frac{L_1 V_1}{\nu_1} = \frac{L_2 V_2}{\nu_2}$

or if $\nu_2 = \nu_1 \frac{L_2 V_2}{L_1 V_1}$

if $L_2 < L_1$, $V_2 < V_1 \Rightarrow \nu_2 < \nu_1$

(use less viscous fluid)

Decay of gravity waves (deep water)



$$\frac{dE}{dt} = -\mu \int |\nabla u|^2 \, dV$$

Obs: $\frac{dE}{dt} \neq -\mu \int |\omega|^2 dV$ because of free surface.

In 2D (ideal fluid): $|\nabla u|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$

$u = \frac{\partial \varphi}{\partial x}$, $v = \frac{\partial \varphi}{\partial y}$ $\Rightarrow |\nabla u|^2 = \left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 \varphi}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 \varphi}{\partial y^2}\right)^2$

for gravity waves $\varphi = \varphi_0 \cos(kx - \omega t)\, e^{ky}$

$\Rightarrow |\nabla u|^2 = \left(\varphi_0 e^{ky} k^2\right)^2 \left[\cos^2 + 2\sin^2 + \cos^2\right] = 2\varphi_0^2 e^{2ky} k^4$.

Assuming a slow decay in N.S. eqs (solution $\approx$ ideal f. solution)

$$\Rightarrow \frac{dE}{dt} \approx -\mu \int |\nabla u|^2 dV = -\mu 2\varphi_0^2 k^4 \int e^{2ky} dV =$$

$$= -\mu \varphi_0^2 k^3 L \quad \left(\begin{array}{c}\text{channel}\\ \text{length } L\end{array}\right)$$

$$E = \frac{\rho}{2} \int \left(|u|^2 + |v|^2\right) dV = \frac{\rho}{2} \int \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] dV =$$

$$= \frac{\rho}{2} \varphi_0^2 k^2 \int \left[\sin^2 + \cos^2\right] e^{2ky} dV = \frac{\rho}{4} \varphi_0^2 k L.$$

$\Rightarrow$ [illegible]

$$\frac{dE}{dt} \simeq -4\beta k^2 \frac{\mu}{\rho} \langle E \rangle$$

$$\Rightarrow \langle E \rangle \simeq e^{-2\gamma t} \langle E \rangle_0 \quad , \quad \gamma = \frac{2k^2 \mu}{\rho} = 2\nu k^2$$

In deep water $\omega^2 = gk \Rightarrow k = \frac{\omega^2}{g} \Rightarrow \boxed{\gamma = \frac{2\nu\omega^4}{g^2}}$

Short waves $\ell = 2\pi/k$ (large wavenumbers k) decay faster!

Helmholtz decomposition

Irrotational vector field: $\nabla \times u = 0 \Rightarrow u = \nabla p$

for some function p unique up to + const.

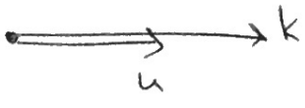
Solenoidal (divergence-free) vector field: $\nabla \cdot u = 0$

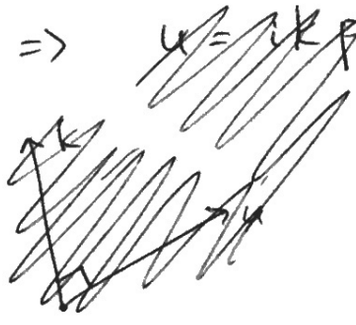
This field can be written (locally) as $u = \nabla \times A$

for some vector field A.

In Fourier space $\nabla \longrightarrow ik$ $\left(u(k) = \int u(r) e^{-ik \cdot r} dr \right)$

Irrotational v.f.: $k \times u = 0 \Rightarrow u = ikp$ $u \parallel k \Rightarrow$

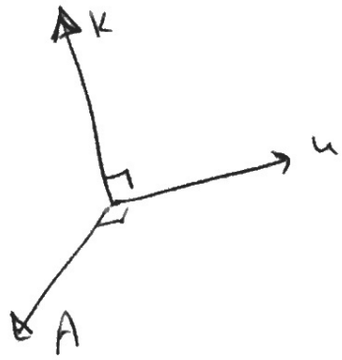




$\Rightarrow u = ikp$

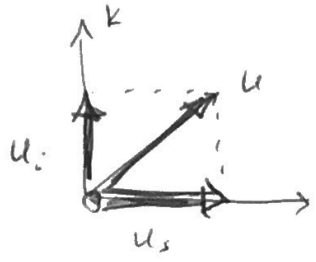
Solenoidal v.s.: $k \cdot u = 0 \Rightarrow u \perp k \Rightarrow$

$u = ik \times A$



(local orthogonal decomposition)

For a general field:



$$u = u_i + u_s$$

↑ irrotational ↑ solenoidal

Theorem (Helmholtz decomposition)

Any (smooth) vector field w in a finite domain D (with smooth boundary ∂D) can be uniquely decomposed as

$$w = u_s + \nabla p \quad \text{in} \quad D$$

where $u_i = \nabla p$ is an irrotational field ($\nabla \times u_i = 0$) and u_s is a solenoidal field ($\nabla \cdot u_s = 0$) with $u_s \cdot n$ on the boundary ∂D.

Proof

For $u_i = \nabla p$ and solenoidal $\nabla \cdot u_s = 0$, with $u_s \cdot n = 0$ on ∂D,

we have $\nabla \cdot (p u_s) = p \nabla \cdot u_s + u_s \cdot \nabla p = u_s \cdot \nabla p$

$$\int_D u_s \cdot u_i \, dV = \int_D u_s \cdot \nabla p \, dV = \int_D \nabla (p u_s) \, dV =$$

$$= \int_{\partial D} p u_s \cdot n \, dA = 0 \quad \Rightarrow \text{"orthogonality" for}$$

any fields ∇p and u_s

Uniqueness

Let $w = u_s + u_i = \tilde{u}_s + \tilde{u}_i$

$$(u_s - \tilde{u}_s) + (u_i - \tilde{u}_i) = 0 \qquad | \cdot (u_s - \tilde{u}_s) dV$$

$$\int_D \| u_s - \tilde{u}_s \|^2 dV + \underbrace{\int_D (u_s - \tilde{u}_s) \cdot (u_i - \tilde{u}_i) dV}_{\to 0 \text{ (orthogonality)}} = 0$$

$$\Rightarrow \quad u_s = \tilde{u}_s \quad \Rightarrow \quad u_i = \tilde{u}_i .$$

Existence

If $w = u_s + u_i$, $u_i = \nabla p$.

Then $\nabla \cdot w = \cancel{\nabla \cdot u_s}^{\,0} + \nabla \cdot u_i = \Delta p$ in D

$$w \cdot n = u_s \cdot n + u_i \cdot n \underset{\text{on } \partial D}{=} u_i \cdot n$$

$$\Rightarrow \quad u_i \cdot n = \nabla p \cdot n = \frac{\partial p}{\partial n} = w \cdot n \quad \text{on } \partial D$$

$$\begin{cases} \Delta p = \nabla \cdot w & \text{in } D \\ \dfrac{\partial p}{\partial n} = w \cdot n & \text{on } \partial D \end{cases}$$

(Neumann problem)

$\Rightarrow \exists p .$

By construction, $u_i = \nabla p$, $u_s = w - \nabla p$. ■

and $\nabla \cdot u_s = \nabla \cdot w - \Delta p = 0$. ■

Implications for the Navier-Stokes and Euler equations for incompressible flow

Let us define a projector $\mathbb{P}$ (linear operator) such that

$$\mathbb{P} w = u_s \quad \text{for} \quad \forall\, w \in D.$$

Consider the N.S. eqs. (in dimensionless form)

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \frac{1}{R} \Delta u$$

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla) u - \nabla p + \frac{1}{R} \Delta u \quad \Big| \cdot \mathbb{P}$$

$\mathbb{P}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial u}{\partial t}$ because $\frac{\partial u}{\partial t}$ is solenoidal ($\nabla \cdot u = 0$)

and $\frac{\partial u}{\partial t} \cdot n = \frac{\partial}{\partial t}(u \cdot n) = 0$ on ∂D

(because $u = 0$ on ∂D)

$\mathbb{P}(\nabla p) = 0$ (no solenoidal part)

Hence, we rewrite the N.S. equations as

$$\frac{\partial u}{\partial t} = \mathbb{P}\left(-(u \cdot \nabla) u + \frac{1}{R} \Delta u\right).$$

No pressure involved!

Similarly, for the Euler eqs: $\frac{\partial u}{\partial t} = -\mathbb{P}((u \cdot \nabla) u)$.

Obs $\mathbb{P}(\Delta u)$ in general not equal to Δu

($\nabla \cdot (\Delta u) = \Delta (\nabla \cdot u) = 0$ but $(\Delta u) \cdot n \neq 0$ at ∂D in general)

Obs No pressure in eqs. and no need for the second equation $\nabla \cdot u = 0$, This condition ($\nabla \cdot u = 0$) is automatically satisfied by the solution.

Exercises: (1) derive $\frac{dE}{dt} = -2\nu\rho\Omega$, $\Omega = \frac{1}{2}\int |\omega|^2 dV$ enstrophy

(2) Show how the form of stress tensor

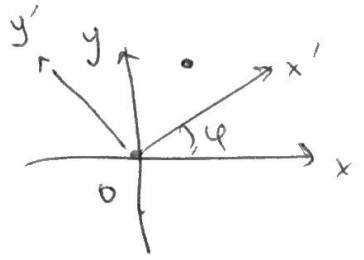
$$\sigma = a\, tr\left(\frac{\partial u}{\partial r}\right) \bar{I} + b\left[\frac{\partial u}{\partial r} + \left(\frac{\partial u}{\partial r}\right)^T\right]$$

follows from isotropy of fluid. (for linear function)

Exercises

(1) Rotation (2D) $\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x' \\ y' \end{pmatrix}$, $R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$(R^{-1} = R^T)$



for speeds: $\vec{u} : \begin{pmatrix} u \\ v \end{pmatrix} = R \begin{pmatrix} u' \\ v' \end{pmatrix}$

$$\frac{\partial \vec{u}}{\partial \vec{r}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = R \frac{\partial \vec{u}'}{\partial \vec{r}'} \frac{\partial \vec{r}'}{\partial \vec{r}} = R \frac{\partial \vec{u}'}{\partial \vec{r}'} R^{-1}$$

or $\left(\frac{\partial \vec{u}'}{\partial \vec{r}'} = R^T \frac{\partial \vec{u}}{\partial \vec{r}} R \right)$

tensor rotation

Show that

Some properties hold for the stress tensor

$$\sigma' = R^T \sigma R$$

Show that if $\sigma = f\left(\frac{\partial u}{\partial r}\right) + \left(\frac{\partial u}{\partial r}\right)^T$ is a linear function, invariant under rotation $\left(\sigma' = f\left(\frac{\partial u'}{\partial r'}\right)\right)$

$$\Rightarrow \sigma = a \frac{1}{3} \mathrm{Tr}\left(\frac{\partial \vec{u}}{\partial \vec{r}}\right) \mathbf{I} + b \left[\frac{\partial \vec{u}}{\partial \vec{r}} + \left(\frac{\partial u}{\partial r}\right)^T\right]$$

In general

(2) Finalize the proof of $\frac{dE}{dt} = -2\nu\Omega$ enstrophy