Waves: from mathematical analysis to natural phenomena*

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March 14, 2017

1 Wave equation

Wave equation describes a numerous phenomena of very different nature and origin. This equation governs the evolution of disturbances near an equilibrium of a continuum. In this section, we start with a simple derivation of wave equation for a one-dimensional "crystal", and then proceed to a more general derivation.

1.1 A system of masses and sprigs

Let us consider a simple model for infinite number of identical point masses m places along the axis x at equal distances ℓ and connected with identical springs, Fig. 1. Let us denote by $x_n = n\ell$ a position of the nth point mass at the equilibrium, where $n \in \mathbb{Z}$ numbers the masses. Oscillations of this system can be described using the infinite-dimensional vector of horizontal displacements:

$$(\ldots, u_{n-1}, u_n, u_{n+1}, u_{n+2}, \ldots). \tag{1.1}$$

Second Newton's law for the nth mass yields

$$m\ddot{u}_n = F_n^- + F_n^+, \tag{1.2}$$

where the two forces F_n^- and F_n^+ are applied from both sides. Since the deformation of a spring between the masses n and $n-1$ equals $u_{n+1} - u_n$, the forces are determined by Hooke's law

$$F_n^- = -k(u_n - u_{n-1}), \quad F_n^+ = k(u_{n+1} - u_n), \tag{1.3}$$

where k is the elastic constant of each spring. Substituting (1.3) into (1.2) yields

$$m\ddot{u}_n = k(u_{n+1} + u_{n-1} - 2u_n). \tag{1.4}$$

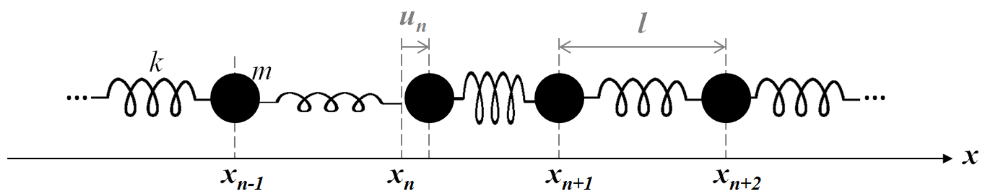

Figure 1: A system of masses and springs modeling oscillations of a one-dimensional crystal.

Let us assume now that the distances ℓ are very small. Then, one can consider a "macroscopic picture", where only large-scale oscillations are present. Mathematically, this means that we assume a solution to be close to a smooth deformation function $u(x,t)$, such that $u_n(t) = u(x_n,t)$, see Fig. 2. For the right-hand side of (1.4) this means

$$u_{n+1} + u_{n-1} - 2u_n = u(x_{n+1}) + u(x_{n-1}) - 2u(x_n) \tag{1.5}$$

(all terms are taken at the same time t). Now we recall that $x_n = n\ell$ and expand the terms with the arguments $x_{n+1} = x_n + \ell$ and $x_{n-1} = x_n - \ell$ in Taylor series for small ℓ. The results of this calculation is

$$u_{n+1}+u_{n-1}-2u_n = \left[u(x_n) + \frac{\partial u}{\partial x}\,\ell + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,\ell^2 + o(\ell^2)\right] + \left[u(x_n) - \frac{\partial u}{\partial x}\,\ell + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,\ell^2 + o(\ell^2)\right] - 2u(x_n). \tag{1.6}$$

After the terms cancellation, we have

$$u_{n+1} + u_{n-1} - 2u_n = \left(\frac{\partial^2 u}{\partial x^2}\right)_{x_n} \ell^2 + o(\ell^2), \tag{1.7}$$

where the derivative is taken at $x = x_n$. Substituting this term back into (1.4) with $u_n(t) = u(x_n,t)$ and dividing both sides by m, we obtain

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{x_n} = \frac{k\ell^2}{m}\left[\left(\frac{d^2 u}{dx^2}\right)_{x_n} + o(1)\right]. \tag{1.8}$$

In the limit of small distances $\ell \to 0$, we can define the mass density per unit length $\rho = m/\ell$ and the elastic coefficient per unit length $K = k\ell$. Then the coefficient in (1.8) can be written as

$$\frac{k\ell^2}{m} = \frac{k\ell}{m/\ell} \to \frac{K}{\rho}. \tag{1.9}$$

*The author is grateful to Marlon M. López F. and Vítor de Oliveira Sudbrack for their help in preparation of these lecture notes.

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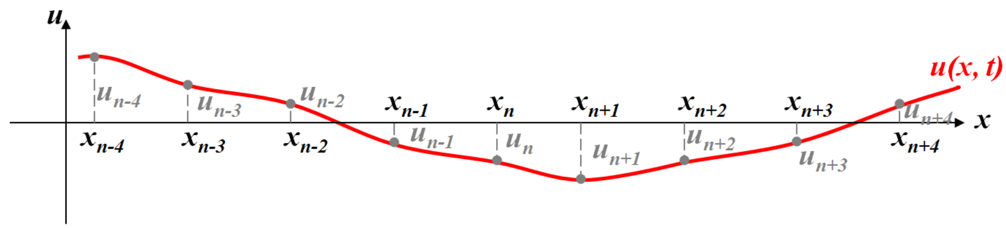

Figure 2: Microscopic, $u_n(t)$, and macroscopic, $u(x,t)$, description of the system.

In this limit, it is natural to assume that both K and ρ are fixed. Thus, denoting $a^2 = K/\rho$ and dropping the vanishing term $o(1)$ in (1.8), we get the limiting equation as

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.10}$$

This is the *wave equation*, and the parameter a is called the *sound speed*.

1.2 General derivation

Now let us show that the wave equation (1.10) may be derived heuristically, based on a set of simple and very general hypotheses. We will focus on the case when the state of the system at every time t is determined by a scalar (real) smooth function $u(x)$. We will also consider a one-dimensional space $x \in \mathbb{R}$, and comment later on the extension to higher (two or three) space dimensions. The hypotheses we need are:

(H1) For any $u_0 \in \mathbb{R}$, the constant state $u(x) \equiv u_0$ is a stable equilibrium.

(H2) We consider small oscillations near the constant-state equilibrium $u(x,t) \equiv 0$.

(H3) The system is homogeneous in space and time.

(H4) The system has parity symmetry, $x \mapsto -x$.

(H5) The system is time-reversible, $t \mapsto -t$.

(H6) Oscillations are large-scale in space and time (long waves).

The exact meaning and role of each hypothesis will become clear during the derivation below.

In order to construct the most general equation of motion for the function $u(x,t)$, we assume that this function can be represented by its Taylor series. In other words, equation of motion can be written in terms of all derivatives $\partial^{n+m}u/\partial x^n \partial t^m$, $n,m \geq 0$, taken at a specific point (x,t):

$$\mathcal{F}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \dots\right) = 0. \tag{1.11}$$

Here $\mathcal{F}$ is a (still unknown) function of an infinite number of arguments. Note that this function cannot depend explicitly on x or t due to the homogeneity hypothesis (H3), because homogeneity implies that properties of the system are exactly the same at every point in space and time.

By the hypothesis (H2), we are only interested in small oscillations, which means that all arguments of function $\mathcal{F}$ in equation (1.11) are small. This permits us to linearize this equation (keeping linear terms and neglecting higher-order nonlinear terms):

$$c_{00}u + c_{10}\frac{\partial u}{\partial x} + c_{01}\frac{\partial u}{\partial t} + c_{20}\frac{\partial^2 u}{\partial x^2} + +c_{11}\frac{\partial^2 u}{\partial x \partial t} + \cdots = \sum_{n,m=0}^{\infty} c_{nm}\frac{\partial^{n+m} u}{\partial x^n \partial t^m} = 0, \tag{1.12}$$

where c_{nm} are real coefficients. The hypothesis (H1) for the equilibrium at arbitrary constant state implies that

$$c_{00} = 0, \tag{1.13}$$

i.e., there is no term proportional to u in (1.12). Similarly, the symmetry hypotheses (H4) and (H5) imply that the coefficient vanishes for an every odd derivative with respect to x or t (otherwise, this term is not invariant with respect to parity of time-reversal):

$$c_{nm} = 0 \quad \text{for odd } n \text{ or odd } m. \tag{1.14}$$

In long-wave approximation, which is the last hypothesis (H6), we assume that the dependence of $u(x,t)$ on both variables is slow. This means that the wave has large size in space and, thus, changes slowly in time. Formally, this condition can be written as the expression

$$u(x,t) = U\left(\frac{x}{L}, \frac{t}{T}\right) \tag{1.15}$$

for large parameters L (wave length) and T (wave time-period) and a function $U(\xi,\tau)$ with typical scales $\delta\xi \sim \delta\tau \sim 1$, see Fig. 3. In this case, each derivative becomes

$$\frac{\partial^{n+m} u}{\partial x^n \partial t^m} = \frac{1}{L^n T^m}\frac{\partial^{n+m} U}{\partial \xi^n \partial \tau^m}. \tag{1.16}$$

The long-wave approximation in this representation is understood as the limit of large L and T, which means that we only need to keep the largest terms (1.16). These are the terms with smallest k and m, and according to (1.13) and (1.14) the larges terms are given by $(n,m) = (2,0)$ and $(n,m) = (0,2)$. With only these two terms kept, we write (1.12) as

$$c_{20}\frac{\partial^2 u}{\partial x^2} + c_{02}\frac{\partial^2 u}{\partial t^2} = 0. \tag{1.17}$$

Depending on the sign of the ratio c_{20}/c_{02}, this equation can be written as

$$\frac{\partial^2 u}{\partial t^2} \pm a^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.18}$$

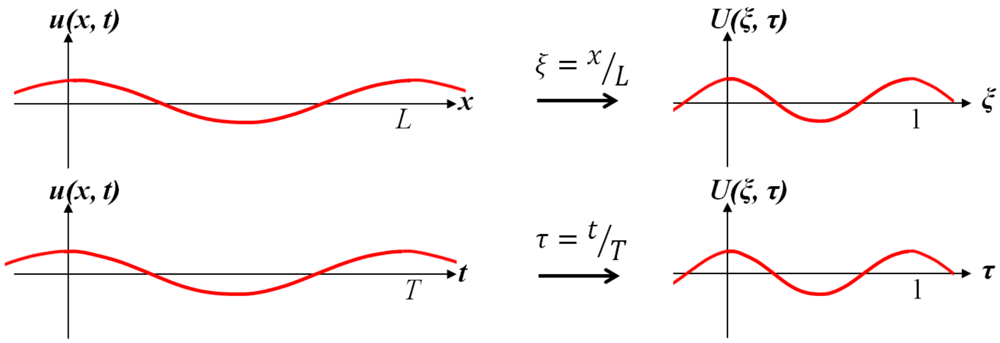

Figure 3: Long-wave approximation with typical spatial scale L (wave length) and temporal scale T.

where $a^2 = |c_{20}/c_{02}|$.

Finally, let us show that the positive sign in (1.18) is ruled out by the stability hypothesis (H1). Indeed, this equation with the positive sign has a solution

$$u(x,t) = e^{akt} \cos kx, \tag{1.19}$$

which is limited in space at any given time, but its solution grows exponentially in time. Such behavior indicates that the constant-state equilibrium $u(x,t) \equiv 0$ is unstable. As a result, only the negative sign is allowed in (1.18) and we arrive the *wave equation*:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.20}$$

In the case of three-dimensional space, a similar argument yields an equation that contains second derivatives for all spatial coordinates, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$ and $\partial^2 u/\partial z^2$, with different coefficients. One can impose an extra hypothesis by assuming isotropy of the space (invariance of equations in all space directions). In this case, coefficients of these second-derivative terms must be equal and we get the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \Delta u = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1.21}$$

A similar argument yields the wave equation in any other space dimension too.

1.3 Examples of wave equations in nature

Due to a very general nature of the hypotheses (H1-H6), examples of wave equation in natural sciences and engineering are very numerous. We list some examples below:

- Vibrations of a string with tension T and linear density ρ: u is a displacement, $a = \sqrt{T/\rho}$.

- Sound waves in gas of liquid: u is the longitudinal displacement and a is the sound speed.
- Electromagnetic waves: u is the electromagnetic field variable and a is the light speed.
- Transverse or longitudinal waves in solids: u is the displacement, a is the wave speed.
- Shallow water waves (waves much longer than the water depth H): u is the surface elevation, $a = \sqrt{gH}$.

2 D'Alembert's solution of the wave equation

We start by solving the one-dimensional wave equation (1.20) on infinite line $x \in \mathbb{R}$. For thus purpose, let is perform a change of variables

$$u(x,t) = v(\xi,\eta), \quad \xi = x - at, \quad \eta = x + at. \tag{2.1}$$

The derivatives are now have to be computed using the chain rule as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) v, \tag{2.2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = -a\,\frac{\partial v}{\partial \xi} + a\,\frac{\partial v}{\partial \eta} = a\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) v. \tag{2.3}$$

Similarly, for the second-order derivatives, we have

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 v = \frac{\partial^2 v}{\partial \xi^2} + 2\,\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}, \tag{2.4}$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)^2 v = a^2\left(\frac{\partial^2 v}{\partial \xi^2} - 2\,\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right). \tag{2.5}$$

Substituting (2.4) and (2.5) into the wave equation (1.20), after cancelations, we find

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0. \tag{2.6}$$

A general solution of this equation can be obtained, by interpreting (2.6) as

$$\frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \xi}\right) = 0. \tag{2.7}$$

This equation implies that the expression in the parentheses does not depend on η and, thus,

$$\frac{\partial v}{\partial \xi} = F(\xi) \tag{2.8}$$

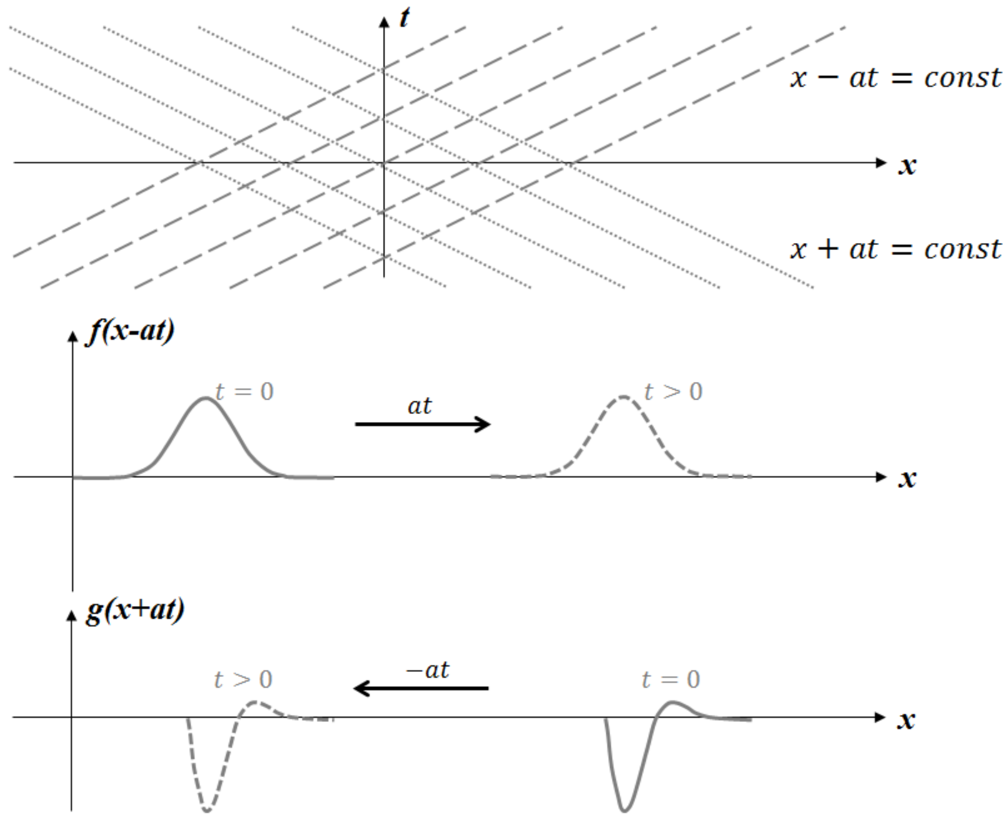

Figure 4: Traveling-wave solutions of the wave equation.

for an arbitrary function $F(\xi)$. Integrating this expression with respect to ξ at fixed η yields

$$v(\xi, \eta) = f(\xi) + g(\eta), \tag{2.9}$$

where $f(\xi) = \int F(\xi)d\xi$ and $g(\eta)$ is an arbitrary integration constant. It is easy to see that (2.9) is indeed a solution of (2.6). In original variables (2.1) we have

$$u(x, t) = f(x - at) + g(x + at). \tag{2.10}$$

We showed that the general solution can be represented as a sum of two arbitrary functions $f(\xi)$ and $g(\eta)$, which are constant along the lines $x - at = const$ and $x + at = const$, respectively. These two families of straight lines are called characteristic lines or, simply, *characteristics*, see Fig. 4. The function $f(x - at)$ is a traveling-wave solution that moves with constant speed a keeping the same shape. Similarly, the function $g(x + at)$ is a traveling-wave solution that moves with constant speed $-a$ in opposite direction.

Now let us consider a Cauchy problem: finding the solution that satisfies arbitrary initial conditions

$$t = 0: \quad u = \varphi(x), \quad \frac{\partial u}{\partial t} = \psi(x). \tag{2.11}$$

Here $\varphi(x)$ describes the initial shape and $\psi(x)$ is the initial speed of the solution. Using (2.10) at $t = 0$ and the chain rule, we get

$$\varphi(x) = f(x) + g(x), \quad \psi(x) = -af'(x) + ag'(x), \tag{2.12}$$

where prime denotes a derivative of the function. From these two relations, one can express the functions

$$f'(x) = \frac{\varphi'(x)}{2} - \frac{\psi(x)}{2a}, \quad g'(x) = \frac{\varphi'(x)}{2} + \frac{\psi(x)}{2a}. \tag{2.13}$$

Integrating these equalities with respect to x, yields

$$f(x) = \frac{\varphi(x)}{2} - \int_0^x \frac{\psi(x)}{2a}dx + c_1, \quad g(x) = \frac{\varphi(x)}{2} + \int_0^x \frac{\psi(x)}{2a}dx + c_2, \tag{2.14}$$

with some integration constants c_1 and c_2. Substituting (2.14) into the first relation of (2.12), we obtain the relation

$$c_1 + c_2 = 0. \tag{2.15}$$

The final d'Alembert's solution is obtained after the substitution of (2.14) and (2.15) into (2.10), which yields

$$u(x,t) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(x)dx. \tag{2.16}$$

One can see that, at a given point x and time t, the solution $u(x,t)$ depends only on the initial shape at two points $x \pm at$ and the initial speed in the interval between these two points.

As a specific example, we consider the case when the initial displacement $\varphi(x)$ is nonzero, but there is no initial speed, $\psi(x) \equiv 0$. Solution in this case is a sum of two waves $\frac{1}{2}\varphi(x - at)$ and $\frac{1}{2}\varphi(x + at)$. Thus, it gets wider first and eventually separates into two waves moving in opposite directions, see Fig. 5(a). Each wave has the same shape as the initial condition but a twice smaller amplitude.

3 Fourier series

For solving the wave equation in a finite interval, we need the concept of Fourier series, which we describe in this section. The Fourier series (in complex form) is given by

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \tag{3.1}$$

where c_n are complex coefficients and x is a real parameter. We will assume that the series converges absolutely: $\sum |c_n| < \infty$. The function $f(x)$ is real for all x if the coefficients corresponding to n and $-n$ are complex conjugate, i.e.,

$$c_{-n} = \overline{c_n}, \tag{3.2}$$

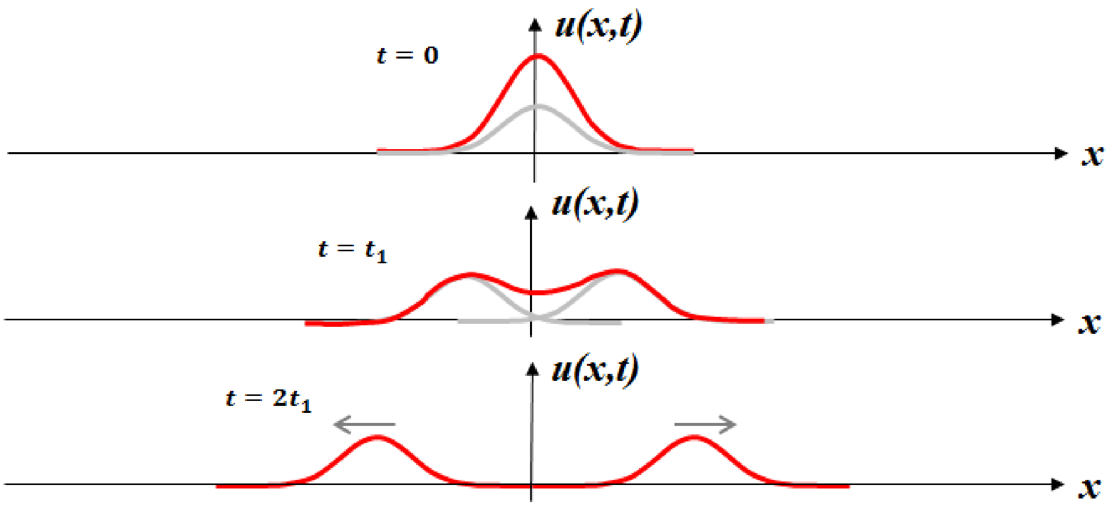

Figure 5: Solution of the wave equation for the Cauchy problem in the case $\psi(x) = 0$.

in particular, c_0 must be real. A real version of the Fourier series is obtained by taking real part of (3.1), which yields

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{3.3}$$

with real coefficients a_n and b_n. Comparison of (3.1) and (3.2) with (3.3) provides the relation of the coefficients in these two representations as

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2} \quad \text{for} \quad n > 0, \quad c_{-n} = \overline{c_n}. \tag{3.4}$$

Teorem 3.1. *The n^{th} Fourier coefficient is given by*

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx. \tag{3.5}$$

Proof. Using expression (3.1) with n substituted by by m in the integral (3.5) yields

$$\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}} c_m e^{i(m-n)x} dx = \sum_{m \in \mathbb{Z}} c_m \int_{-\pi}^{\pi} e^{i(m-n)x} dx. \tag{3.6}$$

The integral for integer m and n is computed as

$$\int_{-\pi}^{\pi} e^{i(m-n)x} dx = \begin{cases} 2\pi, & m = n; \\ 0, & m \neq n; \end{cases} \tag{3.7}$$

which leads to the formula (3.5). □

For coefficients of the real Fourier series (3.4) we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx. \tag{3.8}$$

Teorem 3.2. *For a smooth periodic function $f(x) \in C^\infty(\mathbb{S}^1)$ and a given power $a \geq 0$, there exists a constant C (depending on a function f and power a) such that*

$$|c_n| < C|n|^{-a} \quad \text{for} \quad |n| \neq 0. \tag{3.9}$$

Proof. Using Theorem 3.1, we have

$$|c_n| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x)e^{-inx}\right| dx = C, \quad C = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|\, dx. \tag{3.10}$$

This proves the theorem for $a = 0$. Now, let us perform the integration by parts first, which yields

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}f(x)\frac{e^{-inx}}{-in}\bigg|_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \frac{1}{2\pi in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx, \end{aligned} \tag{3.11}$$

where the terms at integration limits vanish because of periodicity of the function $f(x)e^{-inx}$. Similarly to (3.10), we obtain

$$|c_n| \leq \frac{1}{2\pi|n|}\int_{-\pi}^{\pi}|f'(x)|\, dx = C|n|^{-1}, \quad C = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f'(x)|\, dx. \tag{3.12}$$

This proves the theorem for $a = 1$. Repeating such integrations by parts in (3.11), the statement of the theorem can be proved by induction for an arbitrary integer power $a > 0$. □

Corollary 3.1. *For any smooth periodic function, $f \in C^\infty(\mathbb{S}^1)$, the Fourier series (3.1) with coefficients (3.5) converges.*

Proof. By Theorem 3.2, we have

$$\left|\sum_{n\in\mathbb{Z}} c_n e^{ikn}\right| < \sum_{n\in\mathbb{Z}}|c_n| < |c_0| + C_a\sum_{|n|>0}|n|^{-a}.$$

The last expression converges for a sufficiently large a, for example, $a = 2$. This implies that the series converges absolutely for any x. □

Teorem 3.3. *Fourier series of a twice-differentiable function, $f \in C^2(\mathbb{S}^1)$, converges to the function:*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} \quad \text{for} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx. \tag{3.13}$$

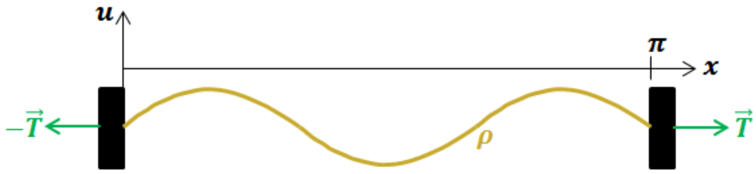

Figure 6: Vibrations of a string.

For the proof we refer to the book: V.I. Arnold. Lectures on partial differential equations (Springer, 2013). In the theory of functional analysis, this theorem is extended to differentiable periodic functions $f \in C^1(\mathbb{S}^1)$ or further to even larger functional spaces.

One can interpret the statement of this theorem in the geometrical way, by introducing the scalar product of two periodic functions $f(x)$ and $g(x)$ as

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx. \tag{3.14}$$

According to (3.7), the set of functions $\{e^{inx}, n \in \mathbb{Z}\}$ forms an orthogonal basis in the space of periodic functions. Then Fourier coefficients c_n in (3.13) are projections of $f(x)$ on the corresponding basis element, and the Fourier series represents the expansion of $f(x)$ in this basis.

4 String vibrations: spectral method

Let us consider vibrations of a string described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{4.1}$$

in which case the wave speed is $a = \sqrt{T/\rho}$ with the string tension T and linear density ρ. The function $u(x,t)$ describes the shape of the string, Fig. 6. For our analysis, it is convenient to rescale the longitudinal axis such that the string length is $\ell = \pi$. The end points of the sting are fixed providing the boundary conditions

$$u|_{x=0} = u|_{x=\pi} = 0. \tag{4.2}$$

4.1 Frequencies and vibration modes

Let us look for a solution in the complex form

$$u = \varphi(x)e^{i\omega t}, \tag{4.3}$$

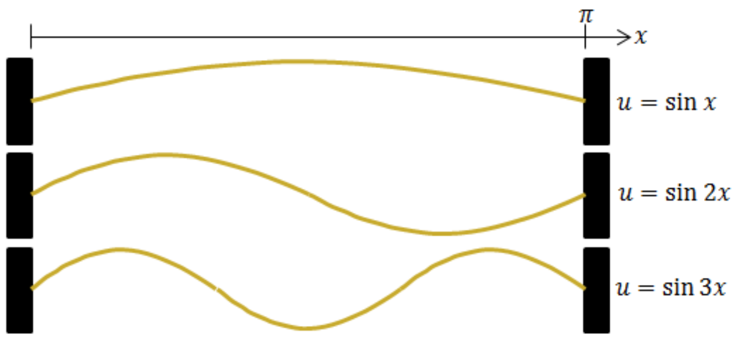

Figure 7: Fundamental mode $\sin x$, and overtones $\sin nx$ with $n = 2, 3, \ldots$.

where ω is a vibration frequency and $\varphi(x)$ is the eigenfunction. From (4.1) we have

$$-\omega^2 \varphi e^{i\omega t} - a^2 \varphi''(x) e^{i\omega t} = 0. \tag{4.4}$$

Together with boundary conditions (4.2), this yields

$$\varphi'' + \left(\frac{\omega}{a}\right)^2 \varphi = 0, \quad \varphi(0) = \varphi(\pi) = 0. \tag{4.5}$$

A general solution of the first equation is

$$\varphi = A \cos\left(\frac{\omega}{a} x\right) + B \sin\left(\frac{\omega}{a} x\right). \tag{4.6}$$

Then the first boundary condition $\varphi(x) = 0$ requires $A = 0$ and the seconds boundary condition $\varphi(\pi) = 0$ yields

$$B \sin\left(\frac{\omega}{a} \pi\right) = 0. \tag{4.7}$$

Since we are looking for nontrivial solutions $u(x, t)$, the frequencies and corresponding eigenvectors are found as

$$\omega = \pm an, \quad \varphi(x) = \sin nx, \quad n = 1, 2, \ldots \tag{4.8}$$

Both real imaginary parts of (4.3), proportional to $\cos \omega t$ and $\sin \omega t$, are solutions of the same problem. Taking their linear combination with arbitrary coefficients A_n and B_n yields the general solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos ant + B_n \sin ant) \sin nx. \tag{4.9}$$

This solution represents a combination of vibrational modes with frequencies $\omega = ak$, which are also known as the fundamental tone ($n = 1$) and overtones ($n > 1$), Fig. 7.

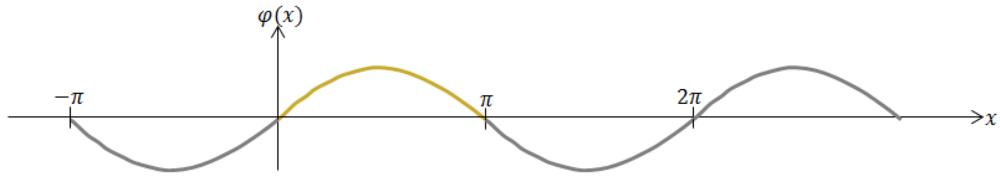

Figure 8: Extension of $\varphi(x)$ to a 2π-periodic odd function.

4.2 Cauchy problem

Let us consider the initial value problem

$$t = 0: \quad u = \varphi(x), \quad \frac{\partial u}{\partial t} = \psi(x), \tag{4.10}$$

for given initial form of the string $\varphi(x)$ and initial velocity denoted by $\psi(x)$. Both $\varphi(x)$ and $\psi(x)$ are assumed to be smooth functions satisfying the boundary conditions $\varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0$. These properties allow extending the function $\varphi(x)$ first to an odd function $\varphi(-x) = -\varphi(x)$ in the interval $-\pi \leq x \leq \pi$, and then by periodicity to the whole line $x \in \mathbb{R}$, see Fig. 8. This means that the function $\varphi(x)$ can be represented in the form of Fourier series (3.3). The same, of course, can be done for $\psi(x)$. Because odd functions $\varphi(x)$ and $\psi(x)$ may contain only odd terms in the Fourier series, which are $b_n \sin nx$, we have

$$\varphi(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \psi(x) = \sum_{n=1}^{\infty} \tilde{b}_n \sin nx. \tag{4.11}$$

Using expressions (3.8) for the Fourier coefficients, we find

$$b_n = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin nx \, dx, \quad \tilde{b}_n = \frac{2}{\pi} \int_0^{\pi} \psi(x) \sin nx \, dx, \tag{4.12}$$

where we reduced the integration interval from $[-\pi, \pi]$ to $[0, \pi]$ because the products $\varphi(x) \sin nx$ and $\psi(x) \sin nx$ are even functions; this yields the extra factor 2 in the coefficient.

Comparing $\varphi(t)$ from (4.11) and $u(x,t)$ from (4.9) at $t = 0$, we obtain

$$\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} A_n \sin nx. \tag{4.13}$$

Similarly, comparing $\psi(t)$ from (4.11) and $\partial u/\partial t$ from (4.9) evaluated at $t = 0$, yields

$$\sum_{n=1}^{\infty} \tilde{b}_n \sin nx = \sum_{n=1}^{\infty} anB_n \sin nx. \tag{4.14}$$

Equalities (4.13) and (4.14) are satisfies by choosing

$$A_n = b_n, \quad B_n = \frac{\tilde{b}_n}{an}. \tag{4.15}$$

As a result, we obtain the solution of the Cauchy problem in the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(b_n \cos ant + \frac{\tilde{b}_n}{an} \sin ant \right) \sin nx. \tag{4.16}$$

Exercise 4.1. *Find a Fourier-series solution for the string vibrations with the initial conditions: $\varphi(x) = x$ for $0 \le x \le \pi/2$, $\varphi(x) = \pi - x$ for $\pi/2 \le x \le \pi$ and $\psi(x) = 0$ (pinched string).*

This vibration theory can be extended to higher space dimensions, for example, vibrations of an elastic membrane ($\mathbf{r} \in \mathbb{R}^2$) or acoustic vibrations in a resonator ($\mathbf{r} \in \mathbb{R}^3$). In these examples, equations of motion are given by the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \Delta u = 0 \tag{4.17}$$

with the Dirichlet boundary condition

$$u|_{\partial\Omega} = 0. \tag{4.18}$$

The latter means that the function $u(x,t)$ is defined inside the domain $\Omega \in \mathbb{R}^n$ ($n = 2$ or 3), and it vanishes at the boundary of this domain denoted by $\partial\Omega$. A solution of the form $u(\mathbf{r},t) = \varphi(\mathbf{r}) e^{i\omega t}$ leads to the eigenvalue problem for ω and $\varphi(\mathbf{r})$ as

$$\Delta\varphi + \frac{\omega^2}{a^2}\varphi = 0, \quad \varphi|_{\partial\Omega} = 0. \tag{4.19}$$

Solving this problem yields natural frequencies and vibrational modes of the system. A general solution is commonly found as a linear combination of such vibrational modes.

5 Wave equation with dissipation

Dissipation is the irreversible process, violating the time-reversibility hypothesis (H5) in our derivation of the wave equation in Section 1.2. This hypothesis was used to justify the absence of derivatives of odd order with respect to time. Therefore, for taking into account a small dissipation, we have to put these odd time-derivative terms back into the equation.

In the long-wave approximation, the most important terms are those with lowest-order derivatives. Therefore, the largest term with odd time-derivative is $\partial u/\partial t$. With this term taken into account, the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2} + \varepsilon \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{5.1}$$

where $\varepsilon > 0$ is a small dissipation coefficient. One can check that the positive sign before the dissipation term is due to the stability requirement; see the exercise below. The dissipation term in (5.1) depends only on the displacement of the system, but not its deformation (a change of u with respect to x). This type of dissipation is called external. Such external dissipation originates, for example, from the resistance of air during the string vibrations.

In many cases external dissipation is very small or does not exist at all. For example, dissipation in the continuum requires its deformation. This means that the dissipative mechanism is described by the higher-order derivatives (with odd order of time-derivative), which are

$$\frac{\partial^3 u}{\partial t \partial x^2} \quad \text{and} \quad \frac{\partial^3 u}{\partial t^3}. \tag{5.2}$$

Here we used the parity symmetry ($x \mapsto -x$) that allows only even-order derivatives in x. When dissipation is small, one can use the wave equation to show that

$$\frac{\partial^3 u}{\partial t^3} = \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) \approx \frac{\partial}{\partial t}\left(a^2 \frac{\partial^2 u}{\partial x^2}\right) = a^2 \frac{\partial^3 u}{\partial t \partial x^2}. \tag{5.3}$$

This means that both terms in (5.2) are equivalent in the leading-order approximation. Hence, the corresponding dissipative wave equation can written as

$$\frac{\partial^2 u}{\partial t^2} - 2\gamma \frac{\partial^3 u}{\partial t \partial x^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{5.4}$$

where $\gamma > 0$ is a small dissipation coefficient. The dissipation of this type is called internal, because it is triggered by the deformation (second derivative of u with respect to x). The negative sign follows from the stability condition, as we will see below.

Let us now describe the general solution of the wave equation with internal dissipation. We consider the complex solutions in the form

$$u = e^{\lambda t} \sin kx, \quad k = 1, 2, \ldots, \tag{5.5}$$

with the complex eigenvalue λ. These functions already satisfy the boundary conditions $u|_{x=0} = u|_{x=\pi} = 0$. Substituting this expression into (5.4) and cancelling the common factors, we obtain the characteristic equation

$$\lambda^2 + 2\gamma k^2 \lambda + a^2 k^2 = 0. \tag{5.6}$$

Its solutions are

$$\lambda = -\gamma k^2 \pm \sqrt{D}, \quad D = \gamma^2 k^4 - a^2 k^2. \tag{5.7}$$

Note that $\operatorname{Re}\lambda < 0$ for all modes if $\gamma > 0$, implying asymptotic stability of the equilibrium state (for $\gamma < 0$ the equilibrium would be unstable). As in the previous section, the general solution is obtained as a linear combination of all vibratiom modes with $k = 1, 2, \ldots$ (one has to take a real part for getting a real solution)

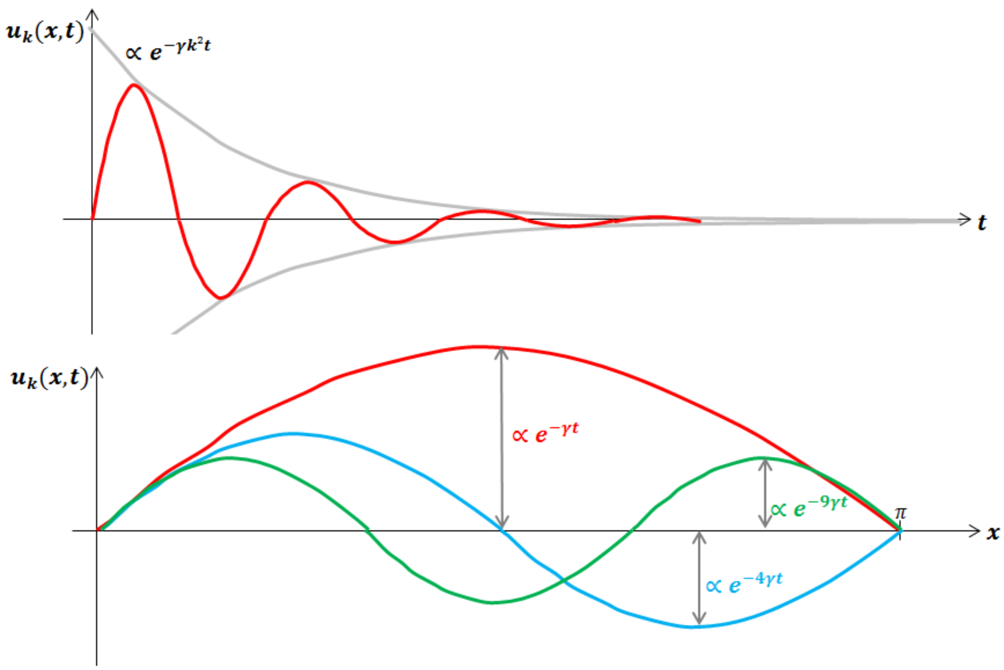

Figure 9: Decay of amplitudes for a string with internal dissipation for modes with $k < a/\gamma$.

The sign of the discriminant in (5.7) depends on k as

$$D < 0 \quad \text{if} \quad k < a/\gamma; \qquad D > 0 \quad \text{if} \quad k > a/\gamma. \tag{5.8}$$

In these two typical cases, we have

(a) If $k < a/\gamma$, then $\lambda = -\sigma \pm i\omega \in \mathbb{C}$ with $\sigma = \gamma k^2 > 0$ and $\omega = \sqrt{a^2k^2 - \gamma^2 k^4}$.

(b) If $k > a/\gamma$, then both λ_+ and λ_- are real and negative.

Therefore, only first several modes with $k < a/\gamma$ oscillate: their time-dependence is given by real and imaginary parts of $e^{\lambda t}$, which are $e^{\sigma t}\cos\omega t$ and $e^{\sigma t}\sin\omega t$, see Fig. 9. A physical string emits sound with the corresponding frequencies ω. Higher overtones, with $k > a/\gamma$, do not oscillate at all but decay exponentially with the rates $e^{\lambda_\pm t}$.

Exercise 5.1. *Perform a similar analysis for a string with only external dissipation (5.1).*

6 Nonlinear effects

Another approximation used in our derivation of the wave equation in Section 1.2 is related to the hypothesis (H2) of small oscillations. We used this assumption to linearize the equation. In this section we describe how a small effect of these neglected terms can be taken into account. For this purpose, let us write the wave equation as

$$\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right) u = 0. \tag{6.1}$$

As we saw in Section 2, this representation yields the general solution as a sum of two waves $u(x,t) = f(x-at) + g(x+at)$ propagating with speed a in opposite directions. Separately, these two wave solutions $u(x,t) = f(x-at)$ and $u(x,t) = g(x+at)$ satisfy, respectively, the two equations following from (6.1) as

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = 0. \tag{6.2}$$

In our analysis of nonlinear effects, we will focus only on the wave $u(x,t) = f(x-at)$, which propagates in the direction of positive x. It satisfies the equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0. \tag{6.3}$$

Nonlinear terms introduce small changes of the wave profile with time. Such changes of the wave profile may accumulate at large time intervals. Also a small nonlinear interaction exists with the wave $g(x+at)$ propagating in the opposite direction. This interaction, however, is limited to a finite interaction time (the waves pass by each other with the relative speed $2a$), preventing such changes to accumulate. This argument suggests that within first-order approximation, we can focus on nonlinear correction in equation (6.3) only, and neglect the interaction between the two equations in (6.2).

6.1 Burgers equation

Now we can decide on a specific form of the largest non-linear correction to equation (6.3). Assuming that the nonlinear terms are small, it is natural to consider only quadratic terms in u. Also, in the long-wave approximation, such terms must have a minimum possible number of derivatives. Note that the term u^2 is not allowed, because any constant state must be the equilibrium by hypothesis (H1), see Section 1.2. We conclude that the largest nonlinear terms are given by

$$u\frac{\partial u}{\partial x} \quad \text{and} \quad u\frac{\partial u}{\partial t}. \tag{6.4}$$

Using (6.3), one can use the argument similar to (5.3) to show that these two terms are, in fact, equivalent within the first-order approximation:

$$u\frac{\partial u}{\partial t} \approx -au\frac{\partial u}{\partial x}. \tag{6.5}$$

Adding this term to (6.3), we obtain a universal nonlinear equation for the wave with the leading-order nonlinear correction as

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \varepsilon u\,\frac{\partial u}{\partial x} = 0, \tag{6.6}$$

where ε is a small parameter. Note that we did not use the parity-symmetry ($x \mapsto -x$), which required all derivatives to be even-order in Section 1.2. This is because such a symmetry interchanges the two equations in (6.2) and, hence, it is not a symmetry of equation (6.3) alone.

Let us consider the change of coordinates (for $\varepsilon > 0$)

$$\tilde{x} = x - at, \quad \tilde{t} = \varepsilon t, \tag{6.7}$$

which corresponds to a reference frame $\tilde{x}$ moving with wave speed a, and the evolution observed in fast time $\tilde{t}$. Using the chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tilde{x}}, \quad \frac{\partial u}{\partial t} = \varepsilon\frac{\partial u}{\partial \tilde{t}} - a\frac{\partial u}{\partial \tilde{x}}. \tag{6.8}$$

Substituting (6.8) into (6.6) yields

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\,\frac{\partial \tilde{u}}{\partial \tilde{x}} = 0. \tag{6.9}$$

For $\varepsilon < 0$, the same result is obtained by taking $\tilde{x} = x - at$, $\tilde{t} = |\varepsilon|t$ and $\tilde{u} = -u$. We will now drop the tildes (for simplicity) and obtain the equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0, \tag{6.10}$$

known as the inviscid *Burgers equation.*

6.2 Finite-time blowup

Let us write (6.10) in the form

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u = 0. \tag{6.11}$$

Note that

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) \tag{6.12}$$

represents the full time-derivative of $u(x,t)$ along the trajectory given by the equation $dx/dt = u$ on the (x,t) plane; use the chain rule to verify this fact. Since this derivative in (6.11) is zero, it follows that $u(x,t)$ does not change along such a trajectory. This, in turn, means that the trajectory $dx/dt = u$ itself is a straight line. Let $u = u_0(x)$ be the initial condition at $t = 0$. Then for any given x_0, the constant value $u = u_0(x_0)$ propagates along the corresponding line (see Fig. 10)

$$x = x_0 + u_0(x_0)t. \tag{6.13}$$

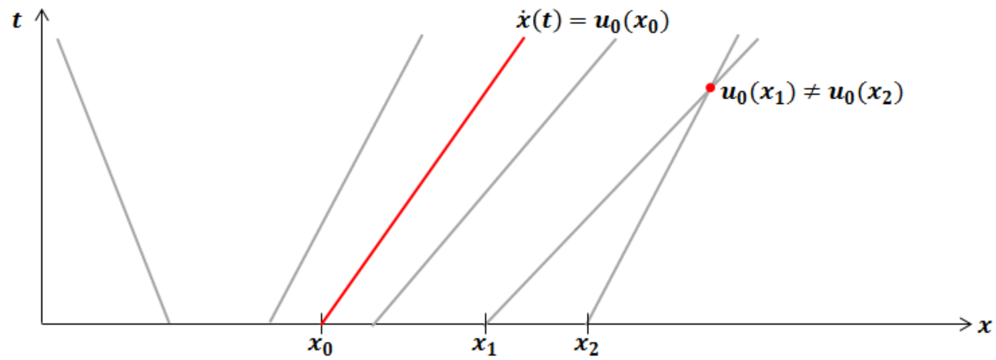

Figure 10: Characteristics of the inviscid Burgers equation. Intersection of characteristics carrying distinct values of $u_0(x_1) \neq u_0(x_2)$ imply the nonexistence of smooth solution at large times.

By similarity with the analysis in Section 2, we call such straight lines *characteristics*.

We can summarize our findings as a system of two equations

$$x = x_0 + u_0(x_0)t, \quad u = u_0(x_0). \tag{6.14}$$

Here $u_0(x)$ is the initial condition at $t = 0$ and x_0 is an auxiliary variable. These two equation provide the implicit form for a solution $u(x,t)$ of the Cauchy problem for the Burgers equation. Fig. 11 shows evolution of a single wave in the Burgers equation. According to (6.13), every value in the initial condition $u_0(x)$ propagates with the constant speed $u_0(x)$. This means that larger values propagate with larger speed, which leads to the gradual inclination of the whole wave profile to the right. Note that such behavior must eventually (in finite time) lead to a singularity. This is already clear from the fact that different characteristics starting from the right side of the wave intersect, while they carry different values of the dependent variable u, see both Fig. 10 and 11. Such a singularity is called a finite-time blowup.

Let us find the exact time of the blowup. As one can infer from Fig. 11, the derivative $\partial u/\partial x$ becomes infinite at the blowup. To compute this derivative, we use the representation $u = u_0(x_0)$ as

$$\frac{\partial u}{\partial x} = u_0'(x_0)\frac{\partial x_0}{\partial x}. \tag{6.15}$$

where we consider the auxiliary variable x_0 as a function of x at fixed time. The derivative $\partial x_0/\partial x$ can be obtained from the first relation in (6.14) as

$$\frac{\partial x_0}{\partial x} = \left(\frac{\partial x}{\partial x_0}\right)^{-1} = \frac{1}{1 + u_0'(x_0)t}. \tag{6.16}$$

Combining (6.15) and (6.16), we have

$$\frac{\partial u}{\partial x} = \frac{u_0'(x_0)}{1 + u_0'(x_0)t}. \tag{6.17}$$

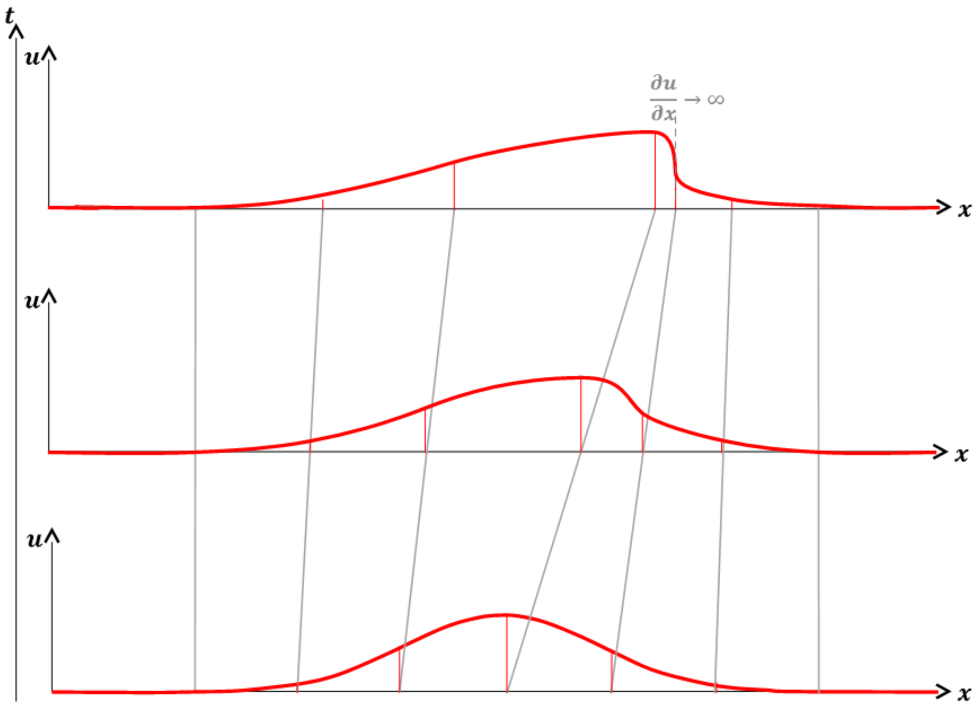

Figure 11: Evolution of a wave $u(x,t)$ for the inviscid Burgers equation at different times: initial time $t = 0$, intermediate time t_1, and the blowup time t_{blowup}. The grey lines in the background show the characteristics on the (x,t) plane. The profile gets steeper on the right side until it forms a blowup: an infinite derivative at one point.

We now explicitly see that, if the initial condition has $u_0' < 0$ at some x_0, then the derivative $\partial u/\partial x \to \infty$ explodes along the corresponding characteristic at time $t = -1/u_0'(x_0) > 0$. Blowup time corresponds to the earliest of such times:

$$t_{blowup} = \min_{u_0'<0} \left[\frac{1}{-u_0'(x_0)} \right], \tag{6.18}$$

where the minimum is taken over all values x_0 that correspond to negative u_0'.

For interpretation of the obtained results, let us recall that the inviscid Burgers equation describes the slow evolution of wave profile in the reference frame moving with speed a, see (6.7). Our results show that, due to nonlinear terms, the wave profile gets steeper with time in the region of negative slopes and becomes less steep in the region of positive slopes. This corresponds to $\varepsilon > 0$ and the opposite tendency can be shown if $\varepsilon < 0$. Though the Burgers equation develops a finite-time blowup with the infinite slope, such behavior is not a universal property of all systems. Indeed, this equation is not valid any more when the slope becomes very steep, because the long-wave hypothesis of slow change of $u(x,t)$ is violated. Despite of this, many system indeed develop a blowup similar to the one we described above: for example, the blowup is typical for gas dynamics, which is a starting point of the shock wave. We can also mention the blowup as a starting point of a traffic jam in transport models.

7 Dispersion. Phase and group speeds

In this section, we will relax the assumption of long-wave approximation, i.e., the hypothesis (H6) in Section 1.2. This assumption was used to identify the leading terms as the terms with the smallest number of derivatives. When the waves are not long, derivatives of all orders play the role, which means that the equation of motion has the general form (1.12) with extra conditions (1.13) and (1.14) due to symmetries. Solutions of this equation can be found in the form

$$u = e^{i(kx-\omega t)}, \tag{7.1}$$

where ω is the frequency and k is called the wavenumber. Since the derivative of even order with respect to x yields the factor $(ik)^n = (-k^2)^{n/2}$ and the derivative of even order with respect to t yields the factor $(-i\omega)^m = (-\omega^2)^{m/2}$, the equation of motion reduces to a relation $F(k^2, \omega^2) = 0$ for some nonlinear function F. Let us assume that this equation can be solved with respect to ω^2 as

$$\omega^2 = f(k^2). \tag{7.2}$$

Since both k and ω can be taken with different signs, we have four solutions for each real $k > 0$ of the form

$$e^{i(kx-\omega t)}, \quad e^{i(kx+\omega t)}, \quad e^{i(-kx-\omega t)}, \quad e^{i(-kx+\omega t)}, \tag{7.3}$$

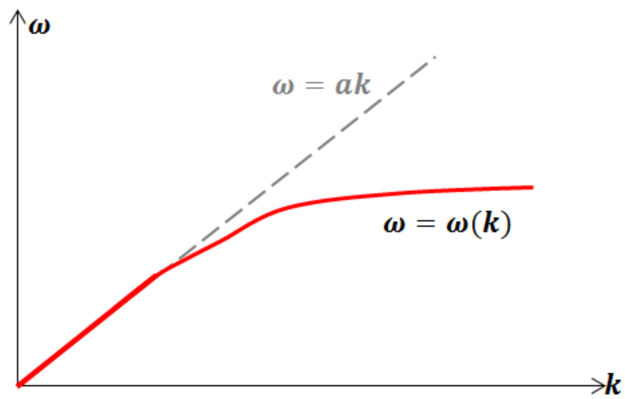

Figure 12: Dispersion relation.

where

$$\omega = \omega(k) = \sqrt{f(k^2)}. \tag{7.4}$$

One can infer from (7.3) that ω must be a real number, otherwise some solutions grow exponentially with time. This means that $f(k^2)$ must be real and nonnegative in order to satisfy the stability requirement. When both k and ω are real, the four real solutions are given by real and imaginary parts of (7.3) as

$$\cos(kx - \omega t), \quad \sin(kx - \omega t), \quad \cos(kx + \omega t), \quad \sin(kx + \omega t). \tag{7.5}$$

These functions are the sinusoidal waves moving with constant speeds $dx/dt = \pm v_f$, where

$$v_f(k) = \frac{\omega(k)}{k} \tag{7.6}$$

is called the *phase speed* at wavenumber k.

Note that $f(0) = 0$ as it follows from (1.13). The long-wave approximation corresponds to $k \to 0$, when e^{ikx} changes slowly in space. In this approximation, the wave equation (1.20) yields $\omega^2 \approx a^2k^2$. These two properties can be summarized as (see Fig. 12)

$$\omega(0) = 0, \quad \omega'(0) = a. \tag{7.7}$$

Deviation of the function $\omega(k)$ from the linear form ak is called the *dispersion*, and $\omega = \omega(k)$ is called the *dispersion relation*.

7.1 Group speed

In order to see the effect of dispersion, let us consider the solution as a combination of two modes with different wavenumbers k_1 and k_2:

$$\begin{aligned} u &= \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) \\ &= 2\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_2-k_1}{2}x - \frac{\omega_2-\omega_1}{2}t\right). \end{aligned} \tag{7.8}$$

Let us now assume that k_1 and k_2 are very close, such that (7.8) is a superposition of two almost identical modes. Then we can write $k_1 = k - \delta$ and $k_2 = k + \delta$ with $\delta \ll k$. Hence,

$$\omega_1 = \omega(k_1) = \omega(k-\delta) = \omega(k) - \frac{d\omega}{dk}\delta + o(\delta), \tag{7.9}$$

$$\omega_2 = \omega(k_2) = \omega(k+\delta) = \omega(k) + \frac{d\omega}{dk}\delta + o(\delta). \tag{7.10}$$

Substituting into (7.8), we have

$$\begin{aligned} u &= 2\cos\left[kx - \omega(k)t + o(\delta)\right]\cos\left[\delta x - \frac{d\omega}{dk}\delta t + o(\delta)\right] \\ &\approx 2\cos[kx - \omega(k)t]\cos\left[\delta\left(x - \frac{d\omega}{dk}t\right)\right]. \end{aligned} \tag{7.11}$$

Introducing

$$v_g(k) = \frac{d\omega}{dk} \tag{7.12}$$

called the *group speed*, we write

$$u \approx 2\cos\left[k(x - v_f t)\right]\cos\left[\delta(x - v_g t)\right]. \tag{7.13}$$

This solution in shown in Fig. 13: it represents a combination of two types of dynamics. At smaller scale, the rapid oscillations propagate with the phase speed v_f (this is a speed of each maximum and minimum). At larger scale, the envelope of the whole signal (a slow modulation of the wave amplitude) propagates with the group speed v_g. Manifestation of the dispersion in the system is the difference between the phase and group speeds. It is important to understand that the signal sent from a certain point propagate with the group speed, and this group speed may be very different from the phase speed (it even may have the opposite sign).

7.2 Wave packet

A more general expression for a wave packet solution (a wave with nearly constant frequency $\omega \approx \omega_0$ and wavenumber $k \approx k_0$, but slowly modulated amplitude) can be represented as

$$u(x,t) = \mathrm{Re}\int c(k)e^{i(kx-\omega(k)t)}dk. \tag{7.14}$$

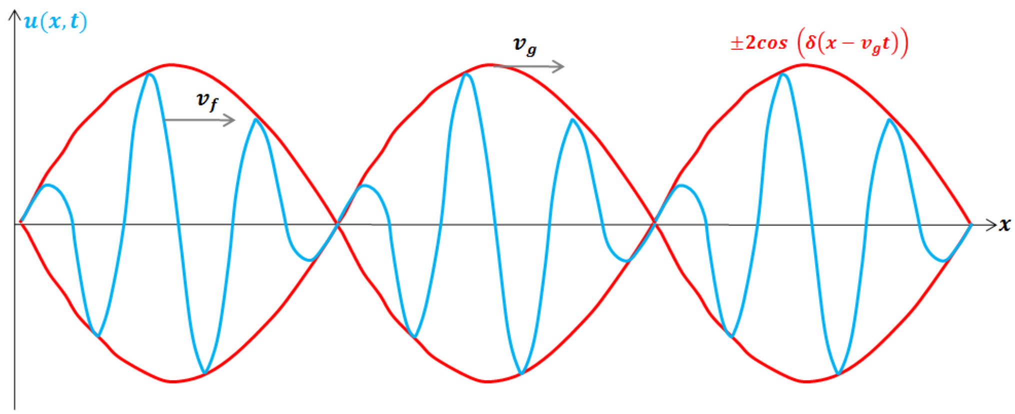

Figure 13: Dynamics for a combination of two sinusoidal waves.

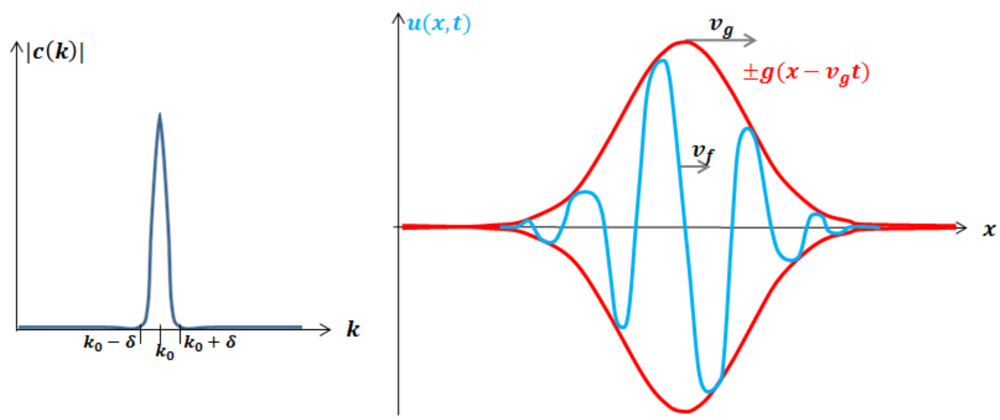

Figure 14: Dynamics of a general wave packet.

Where $c(k)$ is a complex function concentrated in a small δ-neighborhood of k_0, see Fig. 14. This expression is a linear combination of modes, all of which have nearly the same frequencies and wavenumbers. In this case we have $k = k_0 + \delta$ and

$$\omega(k) = \omega(k_0 + \delta) \approx \omega(k_0) + \frac{d\omega}{dk}\delta = \omega_0 + v_g\delta \tag{7.15}$$

for small δ.

The assumption that $c(k)$ is supported in a small neighborhood of k_0 allows using (7.15) in the integral (7.14). At $t = 0$, this integral becomes

$$t = 0: \quad u(x) = \text{Re}\int c(k_0+\delta)e^{i(k_0+\delta)x}d\delta = \text{Re}\left[e^{ik_0x}\int c(k_0+\delta)e^{i\delta x}d\delta\right]. \tag{7.16}$$

Let us define real functions $g(x)$ and $\varphi(x)$ (absolute value and phase) as

$$g(x)e^{i\varphi(x)} = \int c(k_0+\delta)e^{i\delta x}d\delta. \tag{7.17}$$

Then we write (7.16) as

$$t = 0: \quad u(x) = g(x) \cos[k_0 x + \varphi(x)]. \tag{7.18}$$

It is important to see that the integral (7.17) depends on x through the combination δx with small δ. This means that the functions $g(x)$ and $\varphi(x)$ depend on x very slowly, compared to the fast oscillation due to the term $k_0 x$ in (7.18). The function $g(x)$ represents the wave envelope and $\varphi(x)$ determines a slow phase shift, see see Fig. 14.

A similar derivation can be done for an arbitrary time t. In this case, using (7.15) in (7.14), we obtain

$$u \approx \mathrm{Re}\left[e^{i(k_0 x - \omega_0 t)} \int c(k_0 + \delta) e^{i\delta(x - v_g t)} d\delta\right]. \tag{7.19}$$

Using the definition (7.17), we write

$$u \approx g(x - v_g t) \cos[k_0(x - v_f t) + \varphi(x - v_g t)]. \tag{7.20}$$

One can see that the solution has the sinusoidal form locally, traveling with the phase speed v_f. However, the shape (envelope) of the wave is given by $g(x - v_g t)$ and it travels with the different group speed v_g, Fig. 14.

8 Kelvin ship wake

Surface waves in deep water (when wave length ℓ is small compared to the depth H) is an example of strongly dispersive media. The dispersion relation has the form

$$\omega = \sqrt{gk}, \quad \ell = \frac{2\pi}{k} \ll H, \tag{8.1}$$

where g is the acceleration of gravity. Recall that $\omega \approx k\sqrt{gH}$ for shallow water (long waves, $\ell \gg H$), as we mentioned in Section 1.3. From (8.1) we obtain

$$v_f = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \quad v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}}. \tag{8.2}$$

This provides the relation

$$v_f = 2v_g, \tag{8.3}$$

showing that in deep water the phase (wave crests) propagate twice faster than the wave envelope.

In this section, we describe the geometry of a wave generated by a ship that moves with constant speed v along a straight line, Fig. 15. Our description will be based on two simple hypotheses. First, we assume that the wave patters is stationary in the reference frame of the

Figure 15: Ship wake. Photo from `http://www.wikiwaves.org/Ship_Kelvin_Wake`.

ship, i.e., a person on a ship sees a stationary wave profile all around. Second, we assume that the wave pattern is self-similar. This means that the angle of wave crests and wave lengths are the same along any straight line starting at the ship, see Fig. 15.

The wave patter at large distance from the ship can be understood from the dispersion relation. The wave crests propagate with the phase velocity $v_f(k)$. This speed is constant at all points of the straight line passing through the boat at an angle ψ, due to self-similarity. At the same time, the whole wave structure (envelope) corresponding to waves with given wavenumber k propagates with a twice smaller group velocity v_g. By self-similarity, the waves of given k must remain on the same line at angle ψ. These properties are summarized in Fig. 16, by showing the dynamics of wave crests and nearby times t and $t + dt$. Here we denote by θ the angle of wave crests along these lines. By the self-similarity assumption, θ is constant along the line and does not depend on time, but it depends on ψ.

The triangle in Fig. 16 has the sides of length

$$AC = vdt, \quad AB = v_f dt, \quad AD = v_g dt. \tag{8.4}$$

Due to right angle at point B, the trigonometric relation yields

$$v_f = v \cos\theta. \tag{8.5}$$

Furthermore, for the triangle ADC, we can use the law of sines:

$$\frac{\sin(\pi - \theta - \psi)}{AC} = \frac{\sin\psi}{AD}. \tag{8.6}$$

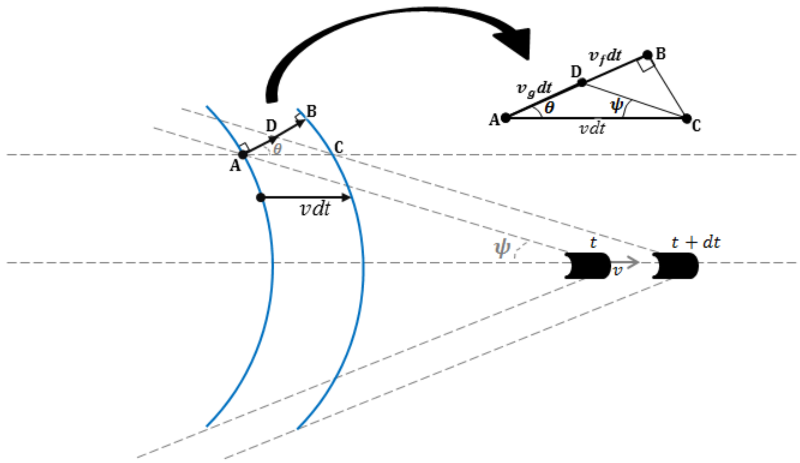

Figure 16: Triangle for the propagation of wave crests (shown in blue). The phase speed v_f determines the crest speed. The group speed v_g corresponds to the motion of a pattern (envelope), i.e., to the motion of a region with a specific wavenumber.

We can express AD and AC from (8.7), (8.5) and (8.3) as

$$AC = vdt, \quad AD = v_g dt = \frac{v_f}{2}\, dt = \frac{v\cos\theta}{2}\, dt. \tag{8.7}$$

Substituting into (8.6), the ship speed v cancels and we obtain

$$\sin(\pi - \psi - \theta) = \frac{2\sin\psi}{\cos\theta}. \tag{8.8}$$

After elementary manipulations we get

$$\cos\psi\sin\theta + \sin\psi\cos\theta = \frac{2\sin\psi}{\cos\theta}. \tag{8.9}$$

Multiplying by $\cos\theta/\cos\psi$, yields

$$\cos\theta\sin\theta + \cos^2\theta\tan\psi = 2\tan\psi. \tag{8.10}$$

Finally, resolving with respect to $\tan\psi$, we have

$$\tan\psi = f(\theta), \quad f(\theta) = \frac{\cos\theta\sin\theta}{2-\cos^2\theta} = \frac{\cos\theta\sin\theta}{1+\sin^2\theta}. \tag{8.11}$$

The function $f(\theta)$ is plotted in Fig. 17. Solution of equation (8.11) exist for $|\theta| < \theta_*$ with $\theta_* = \arcsin\frac{1}{\sqrt{3}} \approx 35.26°$ and the limiting value $\psi_* = \arctan 2^{-3/2} \approx 19.47°$. This means that the ship wake is confined within an angle $-\psi_* \leq \psi \leq \psi_*$. In this interval (except for the limiting pointe), equation (8.11) has two solutions, θ_1 and θ_2, which correspond to two families

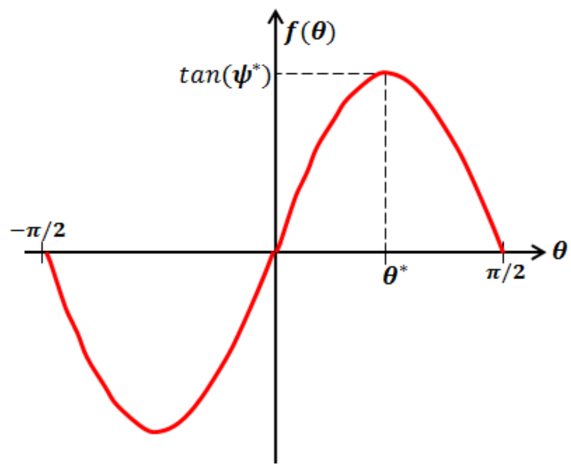

Figure 17: Function $f(\theta)$ in the equation $\tan\psi = f(\theta)$.

of waves. For each family, the corresponding solution determines the angle of the wave crest. Distribution of these angles is sketched on the left side of Fig. 18; recall that these angles do not change along each line passing through the ship due to the self-similarity. The wave crests are the lines, which have the specified angle at each point. This defines two types of wave profiles shown in red and blue in the Fig. 18. These two families of wave behind the ship can now be recognized in the photo of Fig. 15.

9 KdV equation

In the previous sections, we considered various corrections for the wave equation. These are the dissipation, nonlinearity and dispersion. Till now we studied these effects separately. In this section, we consider a combined effect of nonlinearity and dispersion, which leads to the new phenomenon: a solitary wave.

Let us start with equation (6.6) for the wave moving in the direction of positive x, where

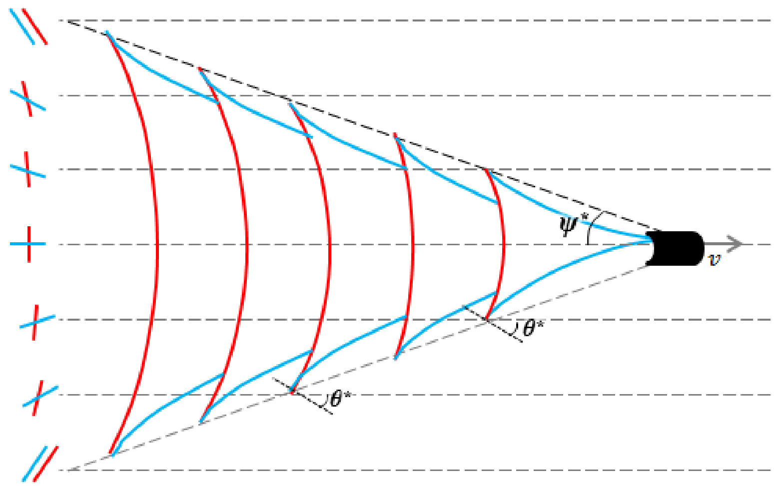

Figure 18: Analytic geometry of waves in the ship wake.

the leading nonlinear term is already included:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \varepsilon u\,\frac{\partial u}{\partial x} = 0. \tag{9.1}$$

It remains to determine how to add the effect of dispersion. As we showed in Section 7, dispersion is induced by linear terms with higher-order derivatives. For long waves, the leading dispersive terms are those with the smallest number of derivatives. For selecting such terms correctly, we need the symmetry argument. We already mentioned that equation for a single wave (6.3) is not parity-symmetric, neither it is time-reversible, because the change of sign for x or t interchanges the two equations in (6.2): the wave changes the propagation direction. However, the combined parity-time symmetry is preserved: by changing the signs of both space and time variables ($x \mapsto -x$, $t \mapsto -t$), the wave propagation is restored and, hence, the equation must remain intact.

It is now easy to see that correction terms for equation (9.1) must contain odd number of total derivatives. In this case equation is invariant under the change $x \mapsto -x$, $t \mapsto -t$, because all terms change sign simultaneously. This criterion selects the leading dispersive terms in the form

$$\frac{\partial^3 u}{\partial x^3}, \quad \frac{\partial^3 u}{\partial x^2 \partial t}, \quad \frac{\partial^3 u}{\partial x \partial t^2}, \quad \frac{\partial^3 u}{\partial t^3}. \tag{9.2}$$

As earlier, see for example (6.5), it is easy to show that all the terms in (9.2) are equivalent in the leading-order approximation. Thus, for capturing a general effect of small dispersion, it is sufficient to consider the single extra term $\partial^3 u/\partial x^3$ in (9.1). This yields

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \varepsilon_1 u\,\frac{\partial u}{\partial x} + \varepsilon_2\frac{\partial^3 u}{\partial x^3} = 0, \tag{9.3}$$

where both small nonlinearity (described by a small coefficient ε_1) and small dispersion (described by a small coefficient ε_2) are taken into account.

Let us consider the change of coordinates

$$\tilde{x} = x - at, \tag{9.4}$$

which corresponds to a reference frame $\tilde{x}$ moving with speed a. Then equation (9.3) reduces to

$$\frac{\partial u}{\partial t} + \varepsilon_1 u \frac{\partial u}{\partial \tilde{x}} + \varepsilon_2 \frac{\partial^3 u}{\partial \tilde{x}^3} = 0. \tag{9.5}$$

Next we rescale the space and state variables as

$$\tilde{x} = \sqrt[3]{\varepsilon_2}\,\xi, \quad u = \frac{6\sqrt[3]{\varepsilon_2}}{\varepsilon_1}\,w. \tag{9.6}$$

In the new variables, equation (9.5) becomes

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial \xi^3} + 6w\frac{\partial w}{\partial \xi} = 0, \tag{9.7}$$

and it is called the *Kortweg – de Vries (KdV) equation*.

10 Soliton

In this section, we describe a traveling-wave solution of the KdV equation, which is called the soliton. Recall that the wave equation has traveling wave solutions of arbitrary shape but fixed speed a. On the contrary, the KdV equation has traveling wave solutions of different speeds, but the wave shape is not any more arbitrary.

We consider the solution in the form

$$w(\xi, t) = W(\xi - vt), \tag{10.1}$$

representing a wave traveling with some unknown speed v. We can define a traveling variable $\eta = \xi - vt$, in which case the solution is given simply by the function $W(\eta)$. For the derivatives, the chain rule yields

$$\frac{\partial w}{\partial t} = W'(\eta)\frac{\partial \eta}{\partial t} = -vW'(\eta), \quad \frac{\partial w}{\partial \xi} = W'(\eta), \quad \frac{\partial^3 w}{\partial \xi^3} = W'''(\eta), \tag{10.2}$$

where the prime stands for a derivative with respect to η. Substituting into (9.7), we have

$$\left(-vW + W'' + 3W^2\right)' = 0. \tag{10.3}$$

Integration with respect to η gives

$$W'' = -3W^2 + vW + \gamma \tag{10.4}$$

with the integration constant γ. A solitary wave must satisfy the conditions

$$W \to 0 \quad \text{as} \quad \eta \to \pm\infty. \tag{10.5}$$

This means that $W = 0$ is an equilibrium in system (10.4), specifying the integration constant $\gamma = 0$. The resulting equation reads

$$W'' = -3W^2 + vW. \tag{10.6}$$

Equation (10.6) can be written as

$$W'' = -\frac{dU}{dW}, \quad U(W) = W^3 - \frac{v}{2}W^2. \tag{10.7}$$

One can mention a direct analogy of this equation and the equation for a material point on a line $x \in \mathbb{R}$ under the force with potential energy $U(x)$. Such a system is governed by the equation

$$m\ddot{x} = -\frac{dU}{dx}, \tag{10.8}$$

and equation (10.7) follows after the substitution $x \mapsto W$ and $m = 1$.

Figure 19 shows the shape of potential $U(W)$ and the phase portrait for the equation (10.7) in the case (a) $v > 0$ and (b) $v < 0$. From the phase portrait it is clear that the wave cannot have negative speed v: in this case only a trivial solution $W \equiv 0$ satisfies the conditons (10.5). The solitary wave satisfying conditions (10.5) must be a separatrix in the case $v > 0$: an orbit starting and finishing at the unstable equilibrium $W = 0$. In particular, we see that $W > 0$ at all points of the wave profile.

Explicit solution can be obtained using the relation

$$\frac{(W')^2}{2} + U(W) = 0, \tag{10.9}$$

which plays the role analogous to the energy for system (10.8). Equation (10.9) can be checked by evaluating the derivative with (10.7). Solving for W' yields

$$\frac{dW}{d\eta} = \pm\sqrt{vW^2 - 2W^3} = \pm\sqrt{v}W\sqrt{1 - 2W/v}. \tag{10.10}$$

Writing this equation as

$$\sqrt{v}\,d\eta = \pm\frac{dW}{W\sqrt{1 - 2W/v}} \tag{10.11}$$

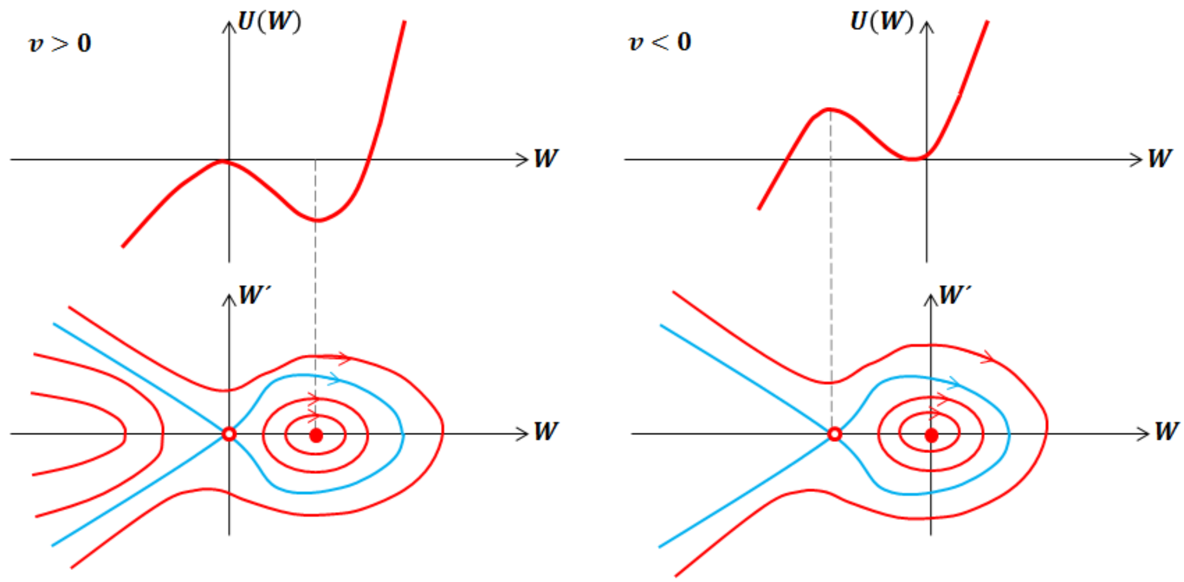

Figure 19: Potential $U(W)$ and phase portrait for equation (10.6). (a) The case $v > 0$, when the system has an unstable equilibrium for $W = 0$. The soliton solution is defined by a separatrix (homoclinic orbit) starting and finishing at the origin (right loop of the blue curve). Periodic orbits correspond to periodic-wave solutions. (b) The case $v < 0$: there are no nontrivial solutions starting and finishing at the origin.

Figure 20: Soliton for the KdV equation.

and integrating, we have

$$\sqrt{v}(\eta-\eta_0) = \pm \int \frac{dW}{W\sqrt{1-2W/v}}. \tag{10.12}$$

Using a change of variables $z = 2W/v$ we write

$$\sqrt{v}(\eta-\eta_0) = \pm \int \frac{dz}{z\sqrt{1-z}} = \mp 2 \operatorname{arccosh} \frac{1}{\sqrt{z}}. \tag{10.13}$$

Expressing z from this relation yields

$$z = \cosh^{-2}\left[\frac{\sqrt{v}}{2}(\eta-\eta_0)\right]. \tag{10.14}$$

Using the expression $W = vz/2$, we have

$$w(\xi,t) = W(\xi - vt) = \frac{v}{2}\cosh^{-2}\left[\frac{\sqrt{v}}{2}(\xi - vt - \eta_0)\right]. \tag{10.15}$$

This solution represents a wave called the *soliton*. It moves with a constant speed $v > 0$, see Fig. 20. Profile of this wave have the same shape for all speeds, with speed-dependent wave length and amplitude.

Note that the phase portrait in Fig. 19 contains another type of bounded solutions: periodic orbits. These orbits correspond to periodic waves presented in Fig. 21. The shape of a wave depends on the constant value (the "energy")

$$\frac{(W')^2}{2} + U(W) = E = \mathit{const} < 0. \tag{10.16}$$

When E is close to zero, a periodic wave resembles a soliton, periodically repeated after some intervals. When E is getting close to the minimum of $U(W)$, a wave has almost sinusoidal shape as a solutions near the stable equilibrium (center).

As an application, one may think of waves on a sea surface. Near the beach, the dispersion term gets weak (recall that shallow water equation has weak dispersion). At the same time, if the wave is not high, nonlinear terms are small. This brings us to the realm of the KdV equation:

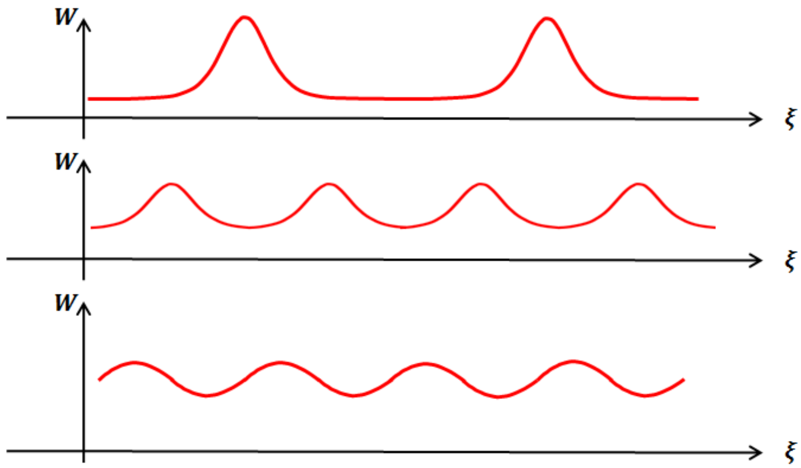

Figure 21: Periodic (cnoidal) waves in the KdV equation.

one can clearly see the periodic wave structure in the interval between the deep sea and the beach resembling the profiles in Fig 21. When the wave get too close to the beach, nonlinear terms get larger, while dispersion becomes less unimportant. This leads to inclination of the wave in the direction of the beach, as described by the inviscid Burgers equation in Section 6.2. Finally, when the wave hight gets comparable to the sea depth, nonlinearity is not small any more and a sophisticated process of wave breaking follows.