Exceptional points and stopping of light

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- Hermitian vs. non-Hermitian degeneracies
- Exceptional points and PT symmetry
- PT symmetric wave guides
- Stopping light

Closed systems → Hermitian operators

$$H = H^\dagger, \quad H = \begin{pmatrix} E_1 & d_1 + id_2 \\ d_1 - id_2 & E_2 \end{pmatrix}$$

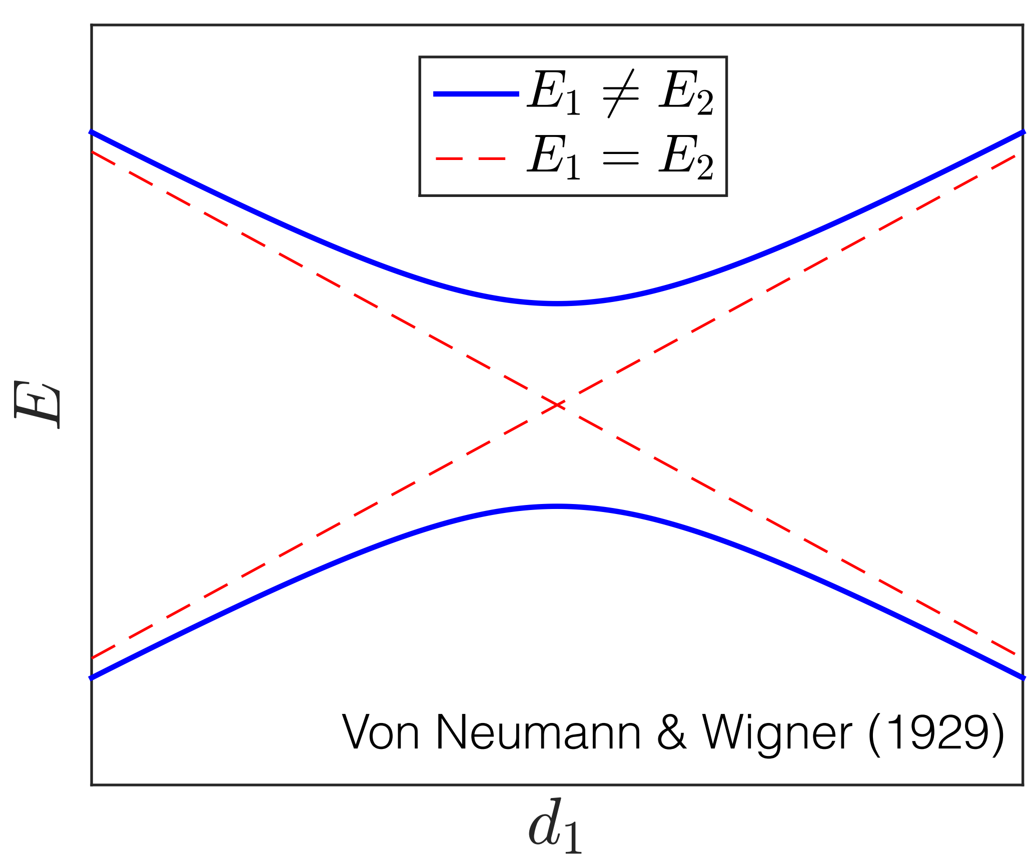
$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\langle\psi|\psi\rangle = 1$$

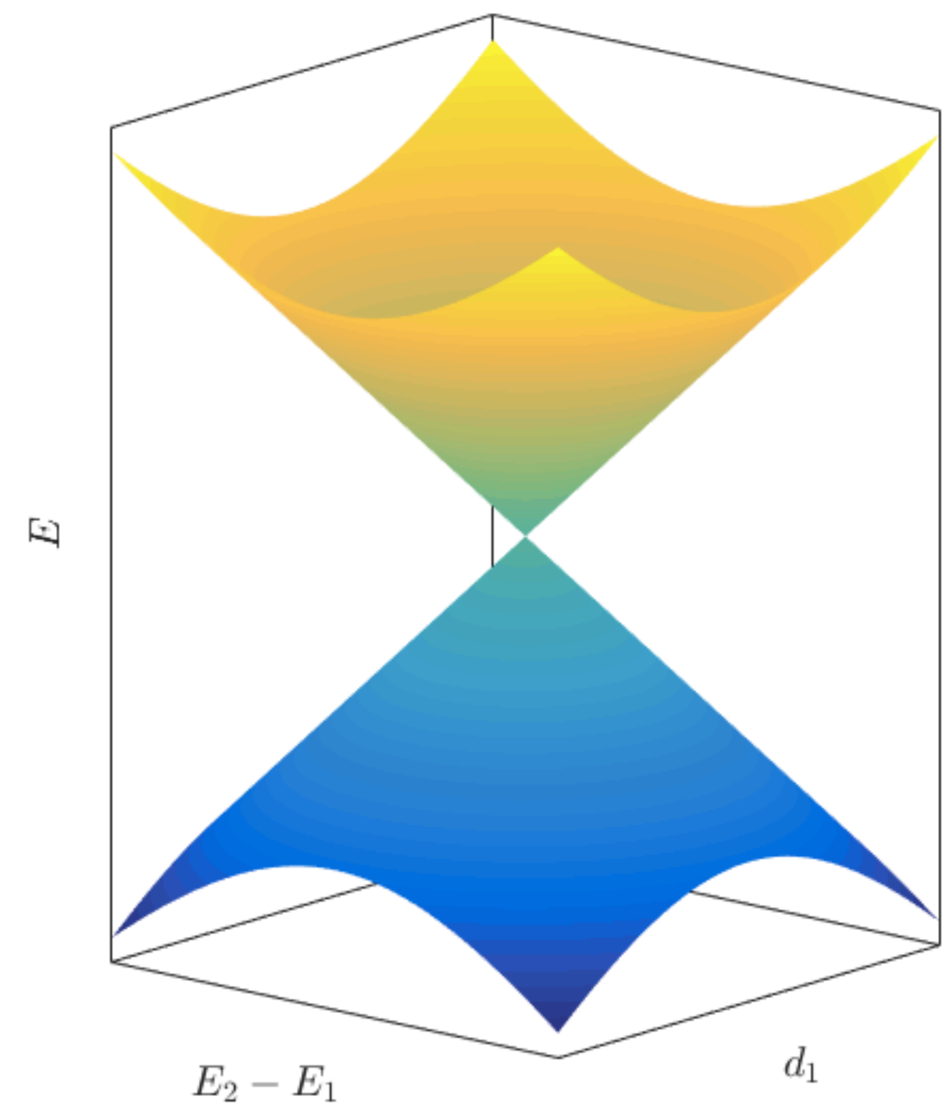
Eigenvalues: $$E = \frac{E_1 + E_2}{2} \pm \sqrt{\frac{(E_2 - E_1)^2}{4} + d_1^2 + d_2^2}$$

Eigenvalue (avoided) crossing:

$d_2 = 0$



conical intersection



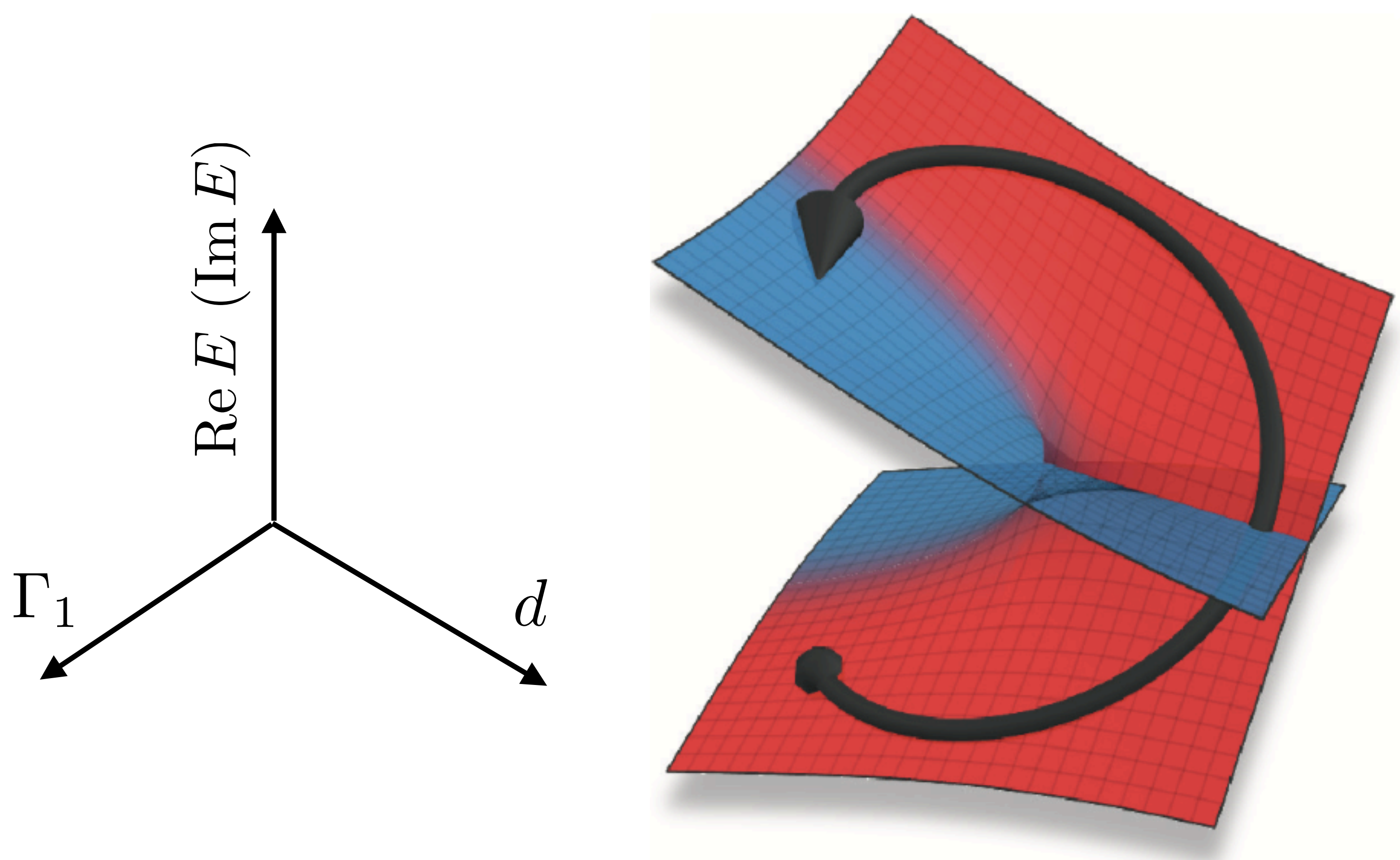
Open systems $\longrightarrow$ non-Hermitian operators

$$H \neq H^{\dagger}, \quad H = \begin{pmatrix} E_1 - i\Gamma_1/2 & d \\ d & E_2 - i\Gamma_2/2 \end{pmatrix}$$

$$\langle \psi(t) | \psi(t) \rangle \neq const$$

Eigenvalues: $$E = \frac{E_1 + E_2}{2} - i\frac{\Gamma_1 + \Gamma_2}{4} \pm \sqrt{\left(\frac{E_2 - E_1}{2} - i\frac{\Gamma_2 - \Gamma_1}{4}\right)^2 + d^2}$$

Exceptional point (EP) or branch point or Jordan block: $$E_1 = E_2, \quad d = \pm\frac{|\Gamma_2 - \Gamma_1|}{4}$$

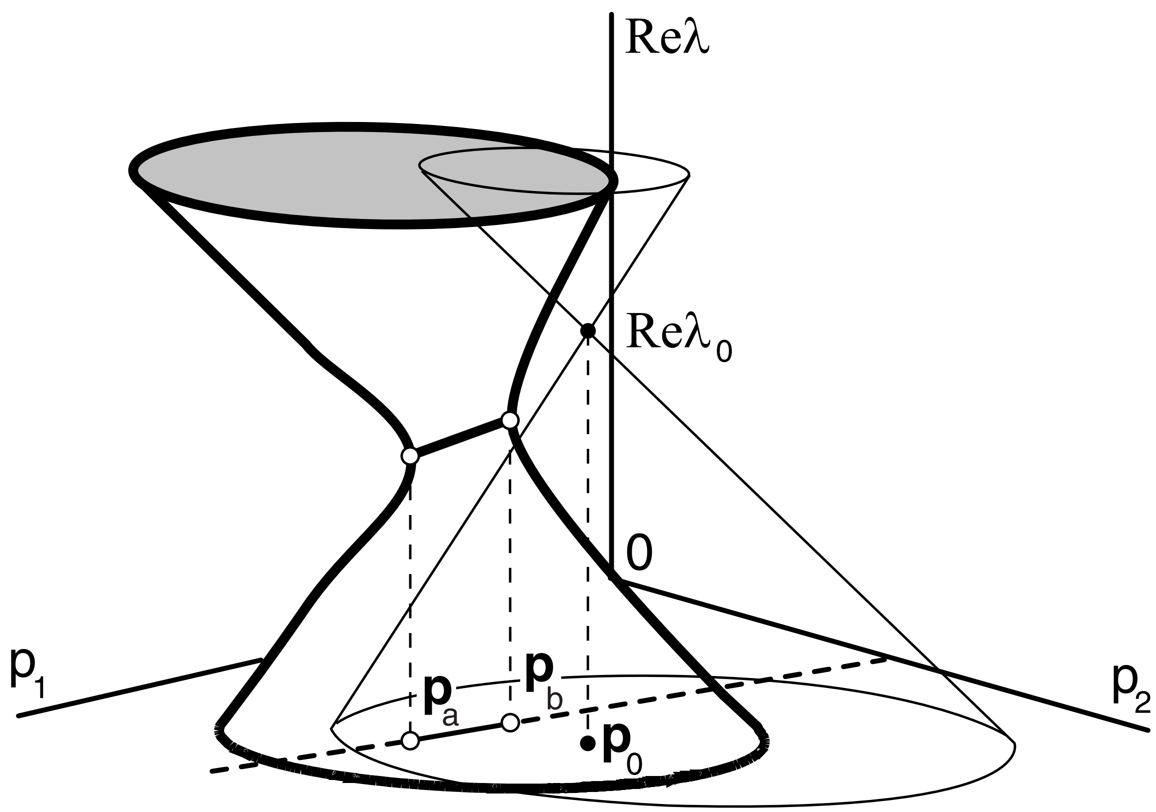


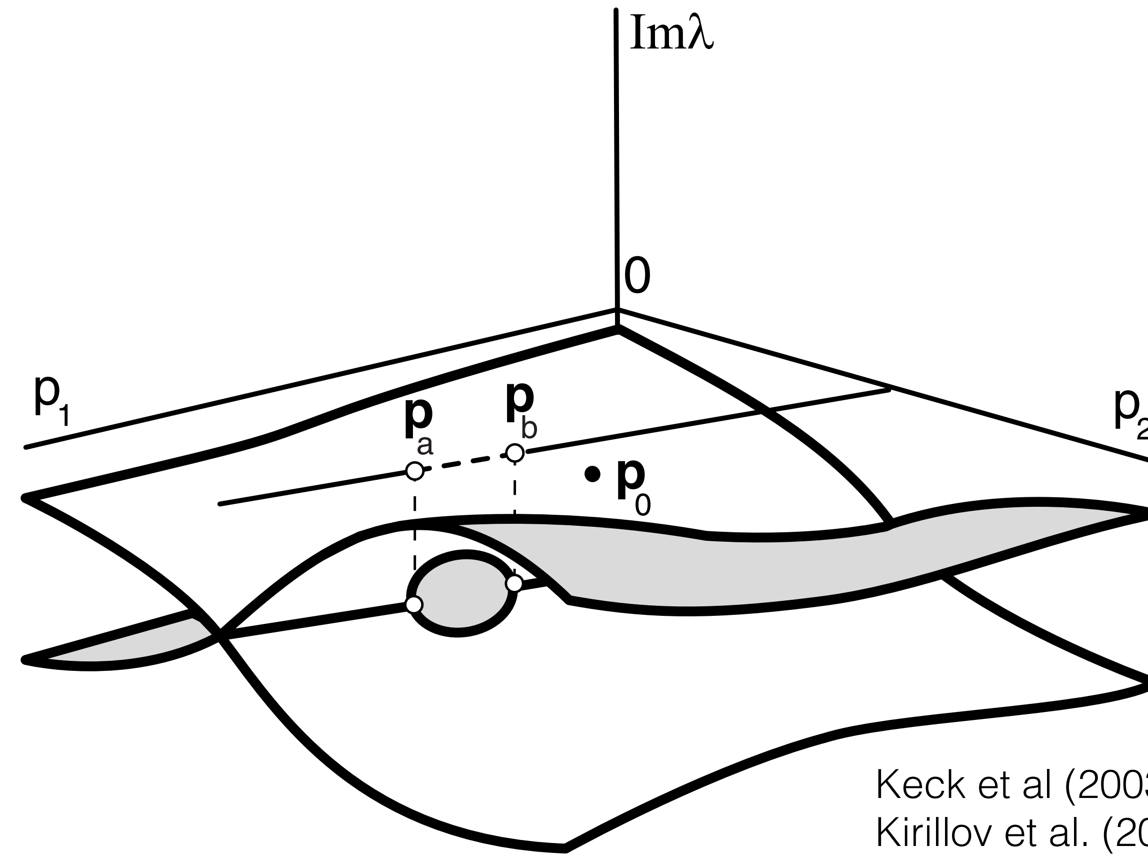
Double-sheet Riemann surface (branch point singularity):

$$\pm\sqrt{x + iy}$$

Coalescence of both eigenvalues and eigenvectors

Non-Hermitian perturbations of Hermitian operators





Keck et al (2003)
Kirillov et al. (2005)

Hermitian degeneracy: co-dimension 2 (real operators) or 3 (complex operators)

Non-Hermitian degeneracy: co-dimension 1 (real operators) or 2 (complex operators)

PT symmetry

$$\hat{H} = \begin{pmatrix} E_0 + i\Gamma/2 & d \\ d & E_0 - i\Gamma/2 \end{pmatrix}$$

|gain| = |**loss**|

Parity (P): $|1\rangle \leftrightarrow |2\rangle$

Time reversibility (T): $\Gamma \leftrightarrow -\Gamma$

$$\hat{H}^\dagger \hat{P} = \hat{P}\hat{H} \qquad \hat{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(pseudo-Hermitian)

Eigenvalues and eigenvectors:

$$E_\pm = E_0 \pm \sqrt{d^2 - \Gamma^2/4}, \qquad |\psi_\pm\rangle = \begin{pmatrix} 1 \\ \pm\sqrt{1-(\Gamma/2d)^2} - i\Gamma/2d \end{pmatrix}$$

Real spectrum condition: $|\Gamma| < 2d$

Exceptional point (EP) in a PT symmetric system

$$\hat{H} = \begin{pmatrix} E_0 + i\Gamma/2 & d \\ d & E_0 - i\Gamma/2 \end{pmatrix} \qquad \Gamma_{EP} = 2d$$

Degenerate eigenvalues and eigenvectors: $E_{EP} = E_0, \qquad |\psi_{EP}\rangle = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Near EP: $E_\pm(\lambda) \approx E_{EP} \pm a\sqrt{\lambda - \lambda_{EP}},$

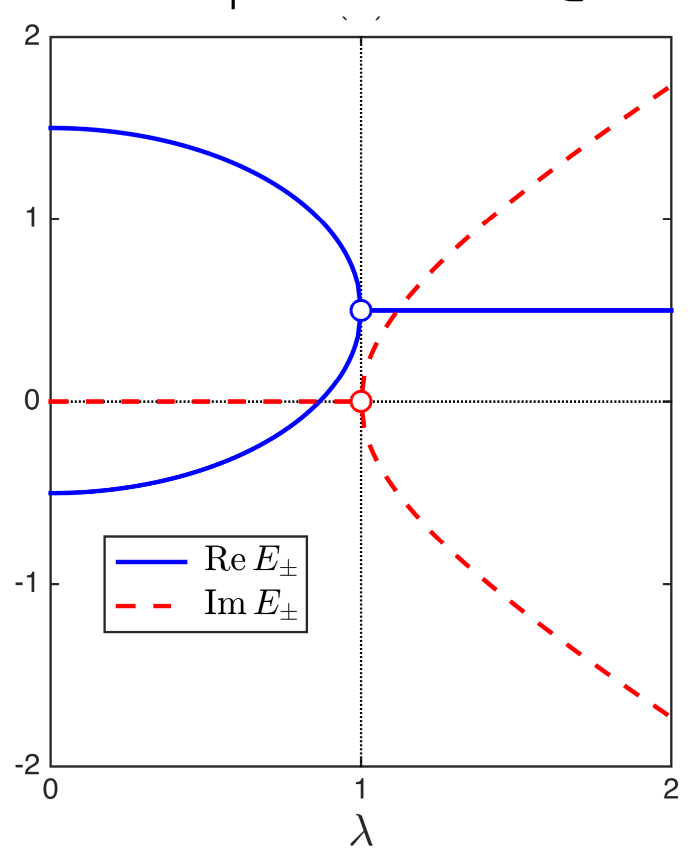
$$a = i\sqrt{2\lambda_{EP}}$$

Self-orthogonality condition:

$$(\psi_{EP}|\psi_{EP}) = \langle \psi^*_{EP}|\psi_{EP}\rangle = 0$$

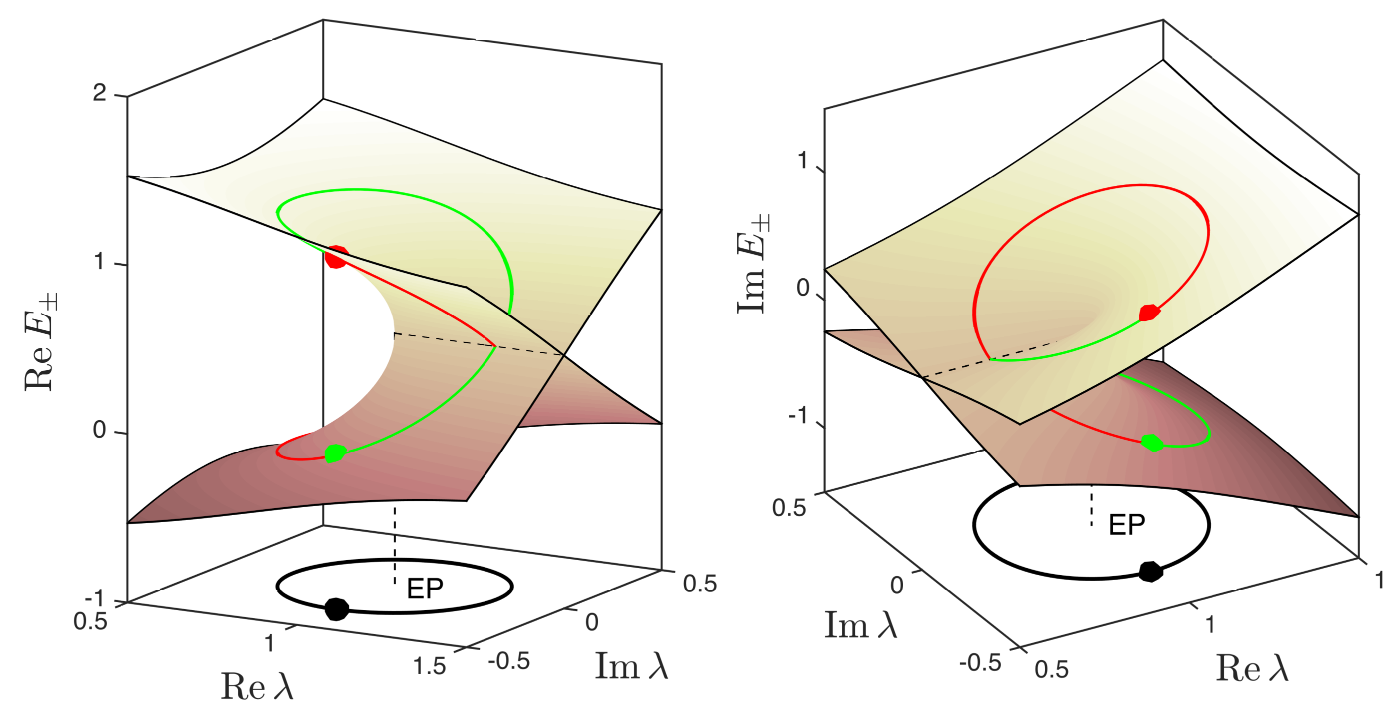
(c-product)

parameter $\lambda \in \mathbb{R}$



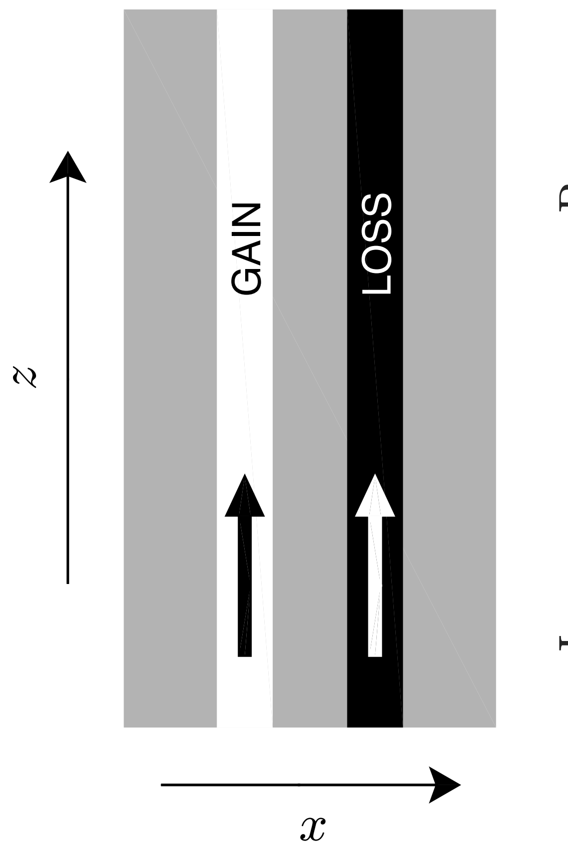
EP: co-dimension 1

$\lambda \in \mathbb{C}$



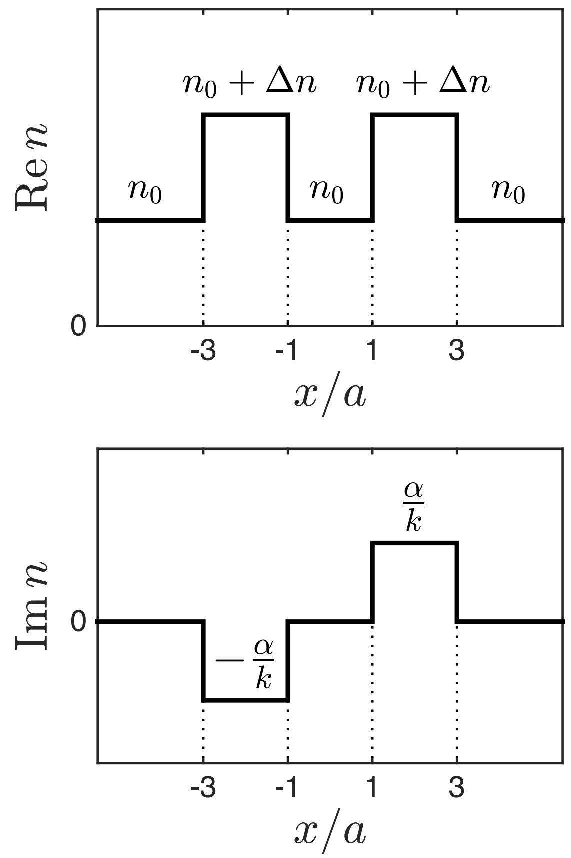
EP: co-dimension 2

PT symmetric waveguides



Klaiman et al. (2007)

refractive index



Transverse-electric (TE) modes:

$$\frac{\partial^2 \psi}{\partial x^2} + \left(\frac{n(x)^2 \omega^2}{c^2} - \beta^2 \right) \psi = 0,$$

$$E_y(x,z,t) = \psi(x) e^{i(\omega t - \beta z)}$$

β is the propagation constant and ω is the frequency

Equivalent form (SE):

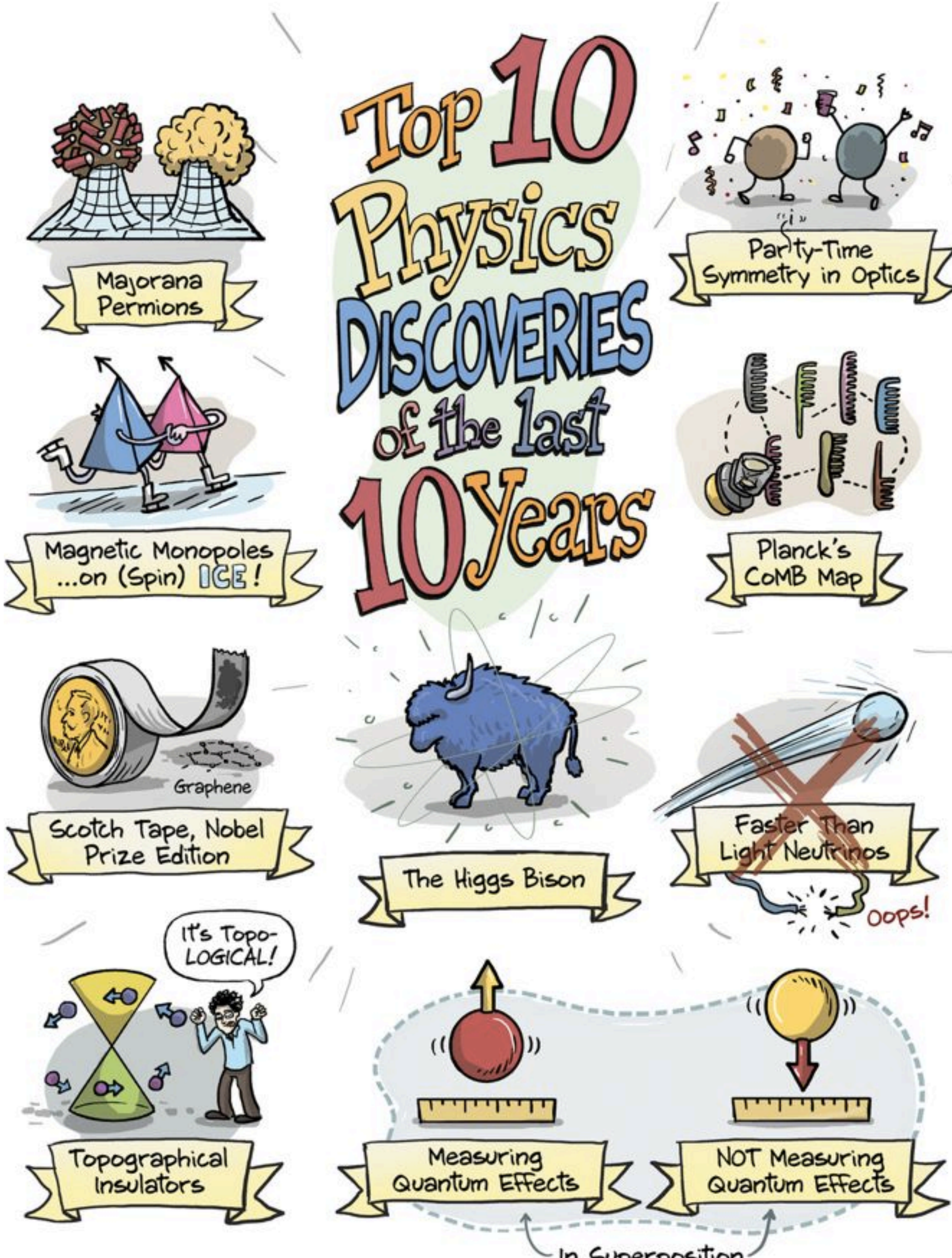
$$\left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x) = E \psi(x)$$

Parity (P): $x \to -x$

Time reversibility (T): dissipation $\leftrightarrow$ loss

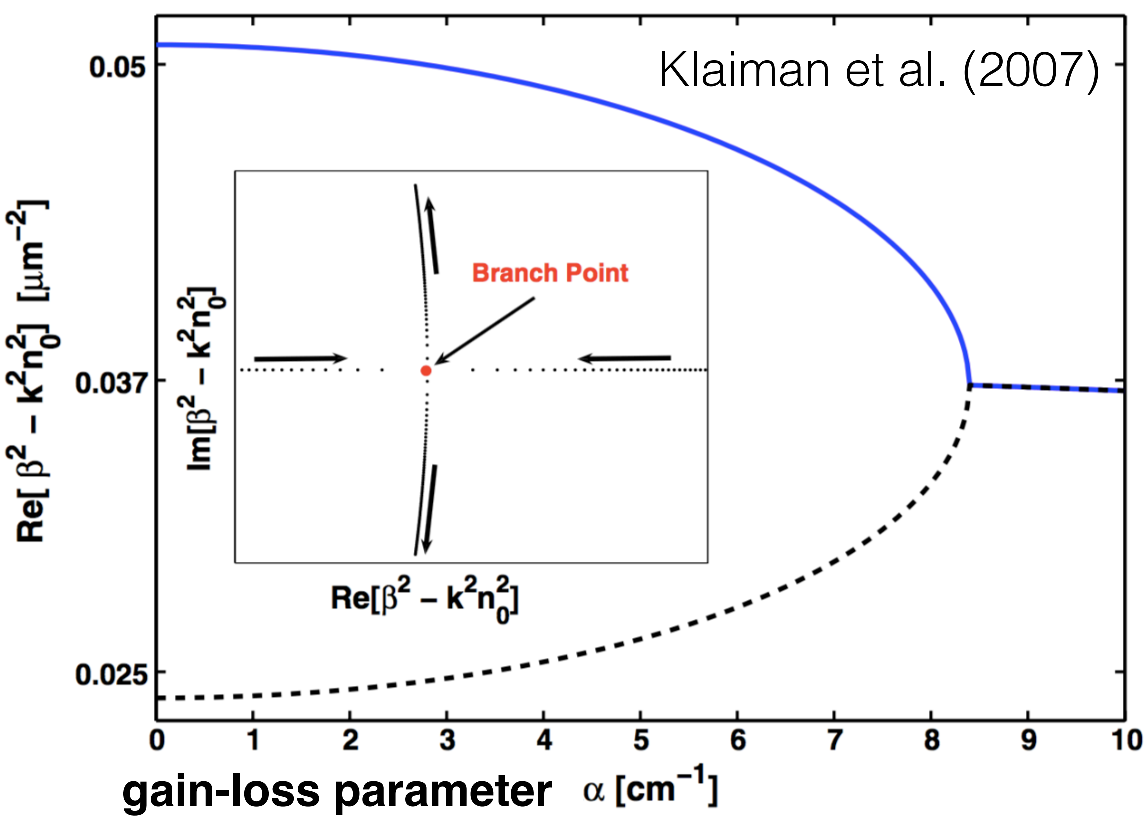
Potential $V(x) = -k^2 n^2(x)/2$

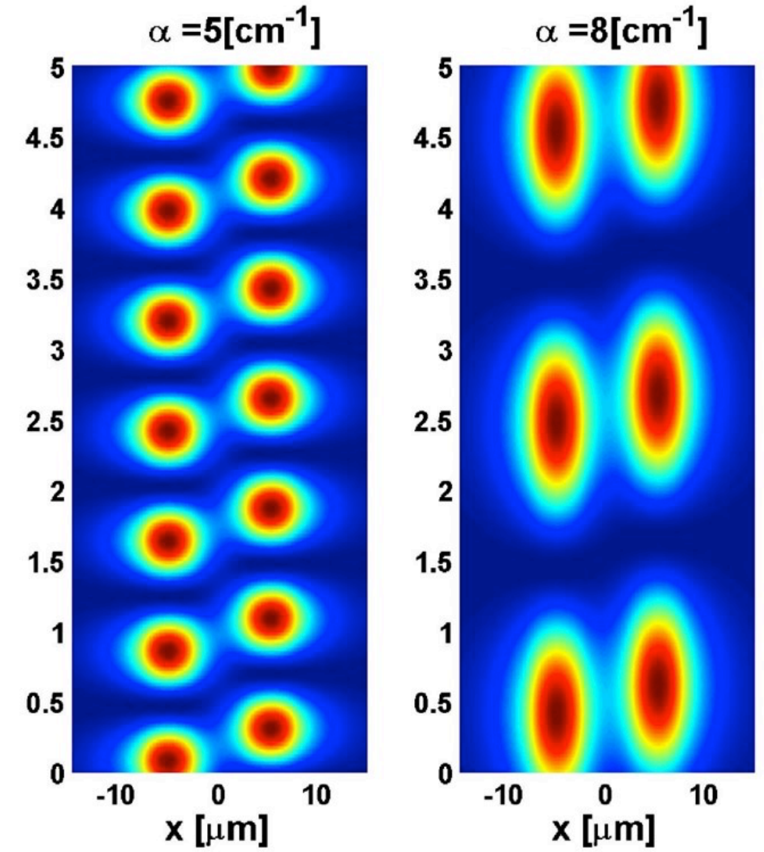
Energy $E = -\beta^2/2$



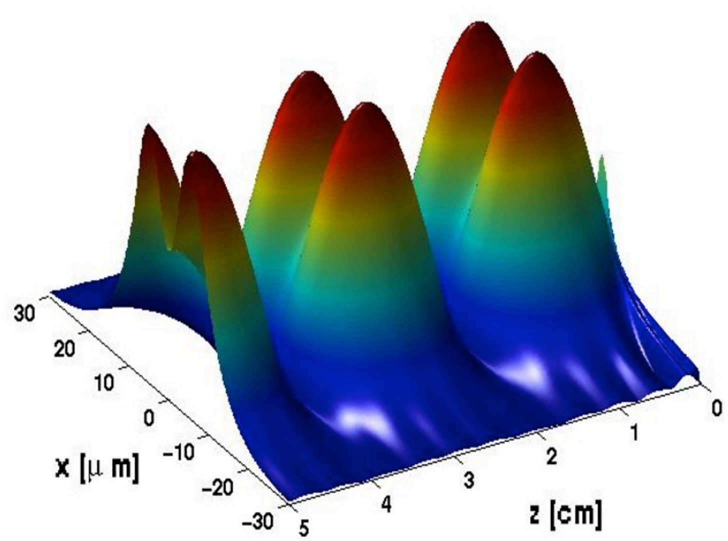
Nature Physics 2015

Light propagation in PT symmetric waveguide

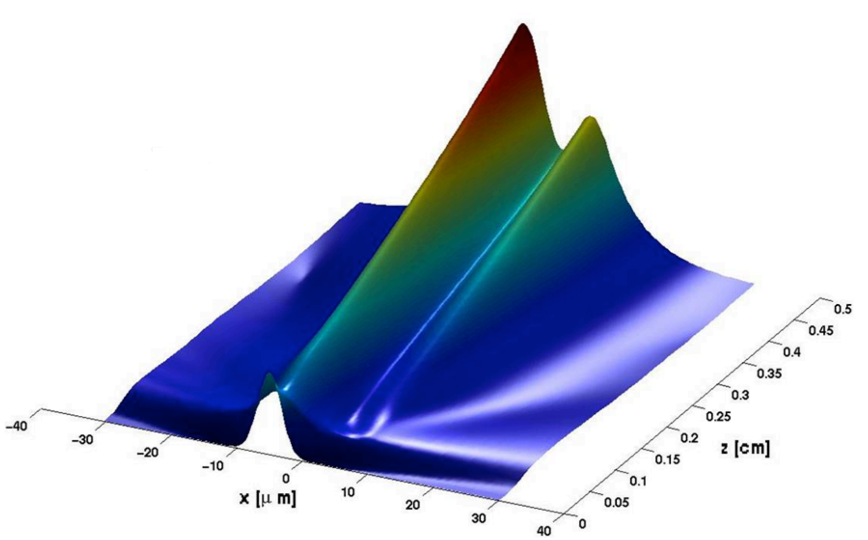




$(a)\ \ \alpha = 8\,\mathrm{cm}^{-1} < \alpha_{EP}$



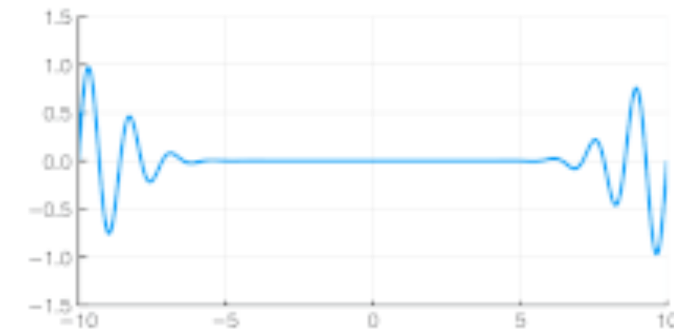
$(b)\ \ \alpha = 9\,\mathrm{cm}^{-1} > \alpha_{EP}$



Intensity of light
for two equally
populated modes

Group velocity of light in the medium

Wikipedia



Group velocity: $v_g = \dfrac{c}{n + \omega \frac{\partial n}{\partial \omega}}$ **Phase velocity:** $v_p = \dfrac{c}{n}$

$$(\text{frequency } \omega, \ \text{refractive index } n)$$

Small group speed requires large derivatives of the refractive index.

Electromagnetically induced transparency (EIT):
17 m/s in a cloud of ultracold atoms of sodium (Hau et al. Nature 1999)

Nowadays:

- group velocity around 1 m/s,
- slow light at room temperatures in solids
- storage of a pulse in atomic state,
- stopping with a counter propagating pulse,

see the review by Baba 2008 in Nature Phys.

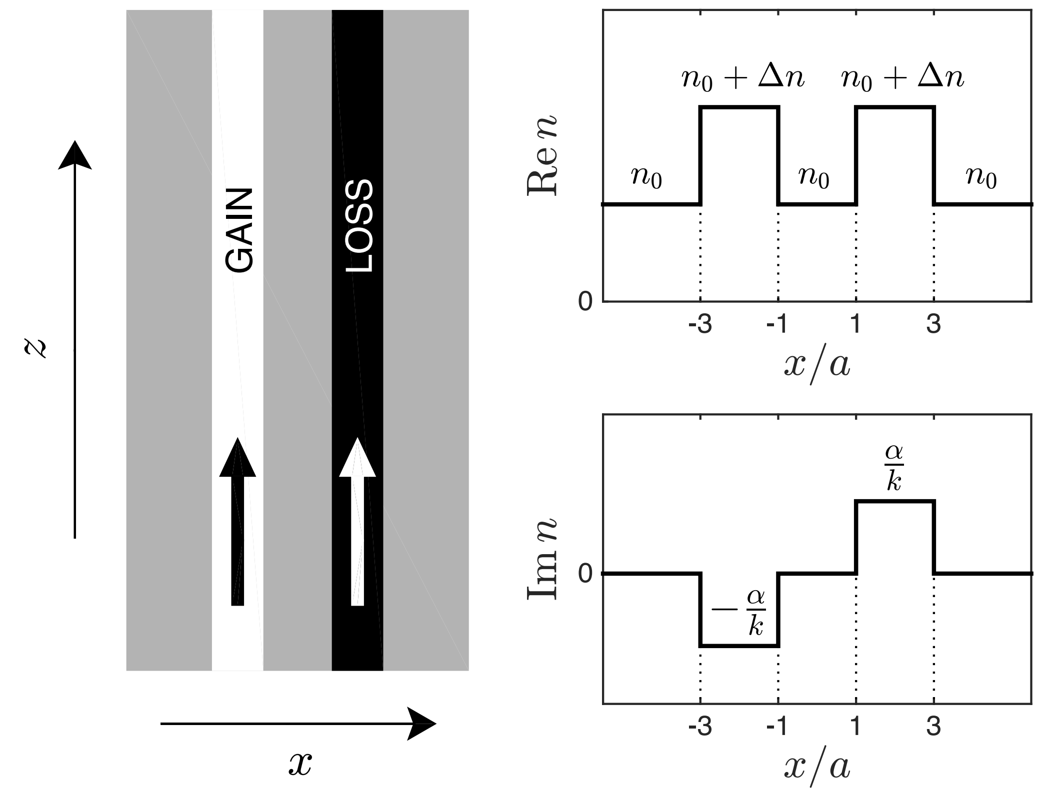
Full stop of light implies:

- zero group velocity at nonzero phase velocity
- **infinite** derivative of the refractive index

Group velocity in a waveguide

Group velocity: $v_g = \left(\frac{d\beta}{d\omega}\right)^{-1}$

Phase velocity: $v_p = \frac{\omega}{\beta}$



Eigenvalue problem:

$$\frac{\partial^2 \psi}{\partial x^2} + \left(\frac{n(x)^2\omega^2}{c^2} - \beta^2\right)\psi = 0,$$

Differentiated with respect to the frequency:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{n^2\omega^2}{c^2} - \beta^2\right)\frac{\partial \psi}{\partial \omega} + \left(\frac{\partial(n^2\omega^2/c^2)}{\partial \omega} - \frac{\partial \beta^2}{\partial \omega}\right)\psi = 0$$

After the scalar product with the eigenfunction:

$$v_g = (d\beta/d\omega)^{-1} = \frac{2c^2\beta \int \psi^2 dx}{\int [\partial(n^2\omega^2)/\partial\omega]\psi^2 dx}$$

Condition for the vanishing group velocity

Group velocity: $$v_g = (d\beta/d\omega)^{-1} = \frac{2c^2\beta \int \psi^2 dx}{\int [\partial(n^2\omega^2)/\partial\omega]\psi^2 dx}$$

PT symmetry property: $\psi(x) = \psi^*(-x)$

Hence, $\int \psi^2 dx$ is real but not necessarily positive!

Vanishing group velocity condition: $\boxed{\int \psi^2 dx = 0}$ $\Longleftrightarrow$ Self-orthogonality condition for an EP

The group speed in a PT-symmetric WG vanishes at an exceptional point.

Alternative explanation:

Since $\beta - \beta_{EP} \propto \sqrt{\omega - \omega_{EP}}$ we have $d\beta/d\omega = \infty$ and $v_g = (d\beta/d\omega)^{-1} = 0$

(does not rely on the PT symmetry)

Simplified 2 x 2 model

$$\begin{pmatrix} \beta_w^2 - i\tilde{\alpha}k & \delta \\ \delta & \beta_w^2 + i\tilde{\alpha}k \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \beta^2 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$\beta_\omega = n_\omega k, \quad k = \dfrac{\omega}{c}$ the real propagation constant in each separate WG

δ the coupling constant

With no gain and loss: $\alpha = 0$ $\quad \beta^2 = \beta_\omega^2 \pm \delta$ $\quad (\psi_1, \psi_2) = (\pm 1, 1)$

With gain and loss: $\alpha \neq 0$ $\quad \beta^2 = \beta_\omega^2 \pm \sqrt{\delta^2 - \alpha^2 k^2}$

Increasing α the propagation constants come closer and eventually coalesce at the EP:

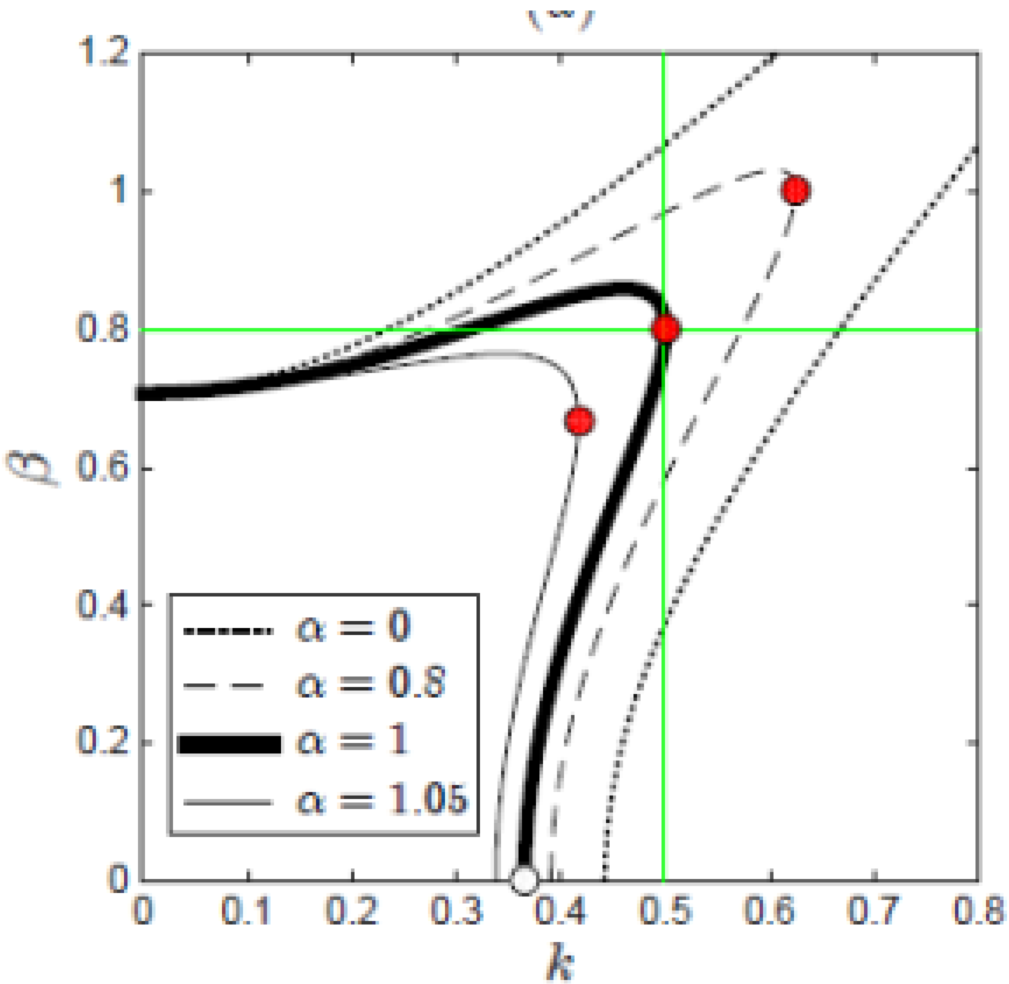
$$\alpha_{EP} = \frac{\delta}{k} \qquad (\psi_1, \psi_2) = (1, i)$$

For $\alpha < \alpha_{EP}$ the propagation constants β are real for the PT – symmetric coupled WGs

Spectrum of the simplified 2 x 2 model

$for \quad \alpha = 1$

$k_{EP} = 0.5 \ , \ \beta_{EP} = 0.8$



$$\begin{pmatrix} \beta_w^2 - i\widetilde{\alpha}k & \delta \\ \delta & \beta_w^2 + i\widetilde{\alpha}k \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \beta^2 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\beta_\omega = n_\omega k, \quad k = \frac{\omega}{c}$$

$$\frac{n_w^2}{c^2}\frac{\partial^2 \Phi_1}{\partial t^2} - \frac{\widetilde{\alpha}}{c}\frac{\partial \Phi_1}{\partial t} - \delta\,\Phi_2 - \frac{\partial^2 \Phi_1}{\partial z^2} = 0,$$

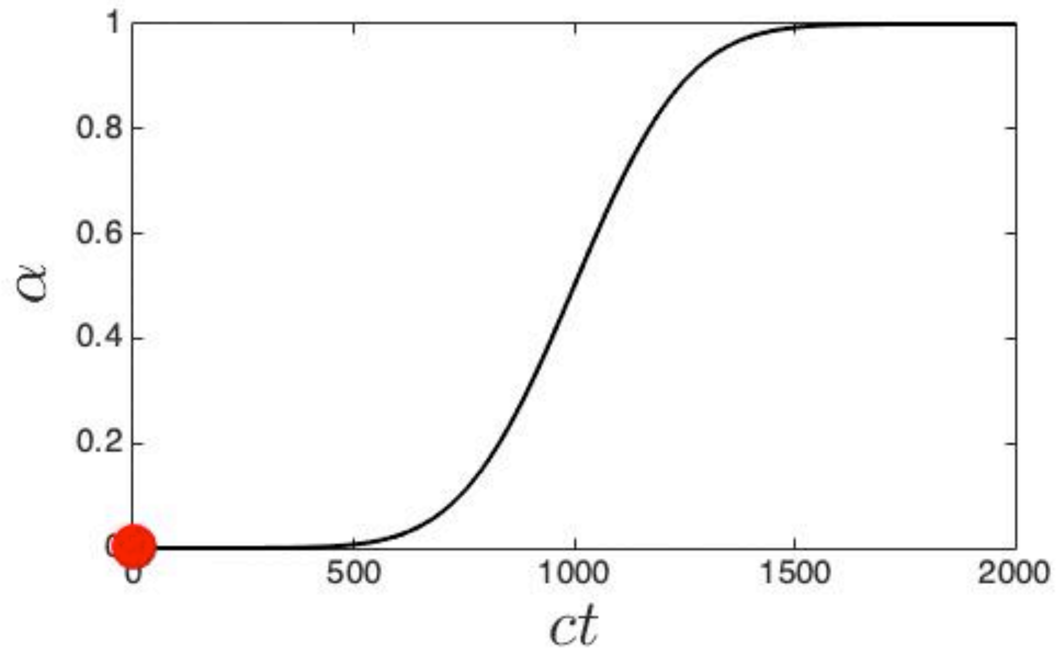
$$\frac{n_w^2}{c^2}\frac{\partial^2 \Phi_2}{\partial t^2} + \frac{\widetilde{\alpha}}{c}\frac{\partial \Phi_2}{\partial t} - \delta\,\Phi_1 - \frac{\partial^2 \Phi_2}{\partial z^2} = 0.$$

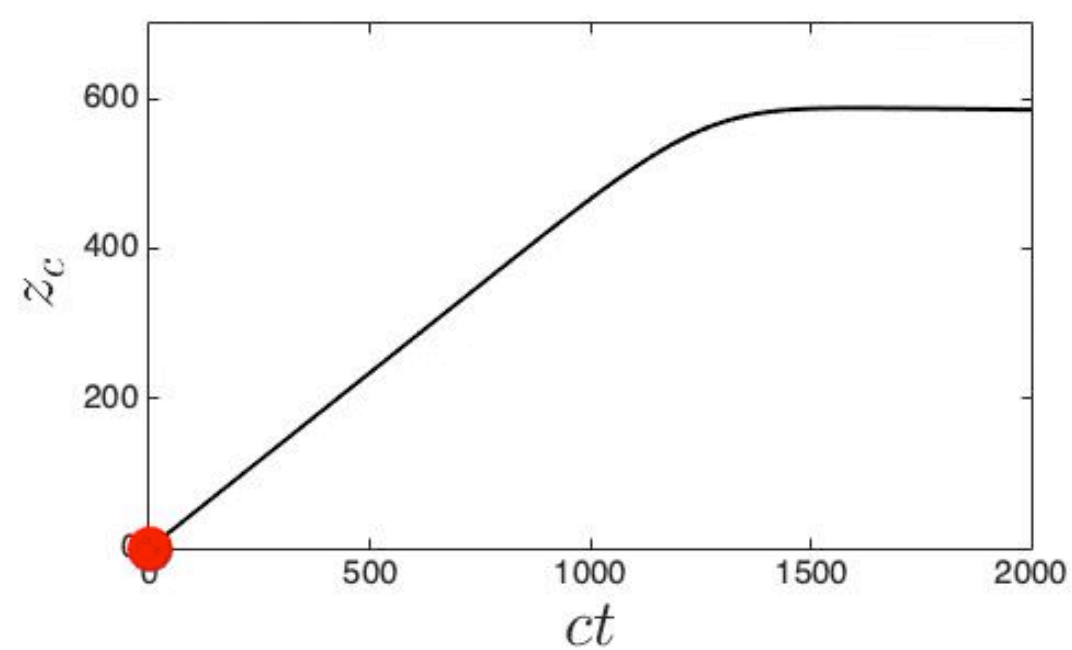
The full-stop of a Gaussian pulse can be accomplished by an adiabatic increase of the gain/loss parameter.

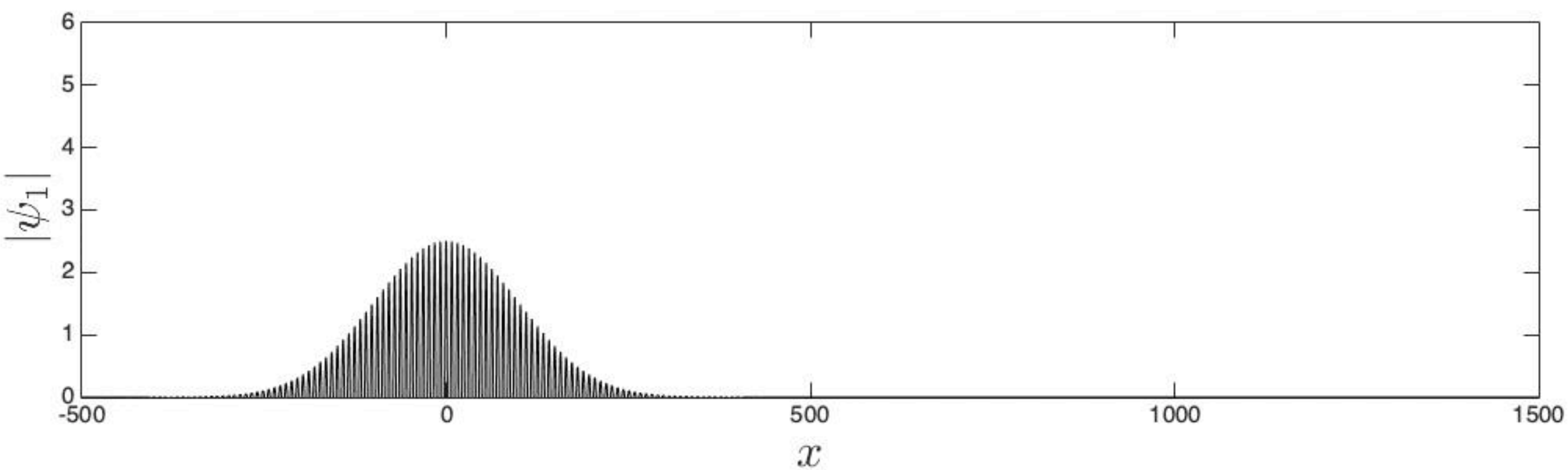
Dynamics of the simplified 2 x 2 model

Initial gaussian pulse corresponding to the antisymmetric mode with no gain and loss:

$$(\phi_1,\phi_2)=(-1,1)\,A\int \exp\left(-\frac{(\beta-\beta_0)^2}{2\sigma^2}+i\beta z\right)d\beta$$

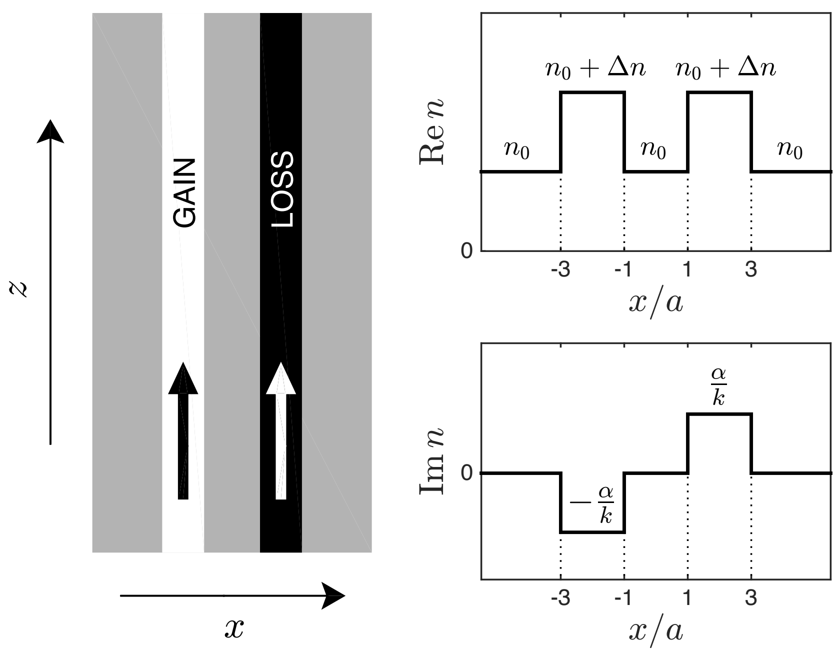




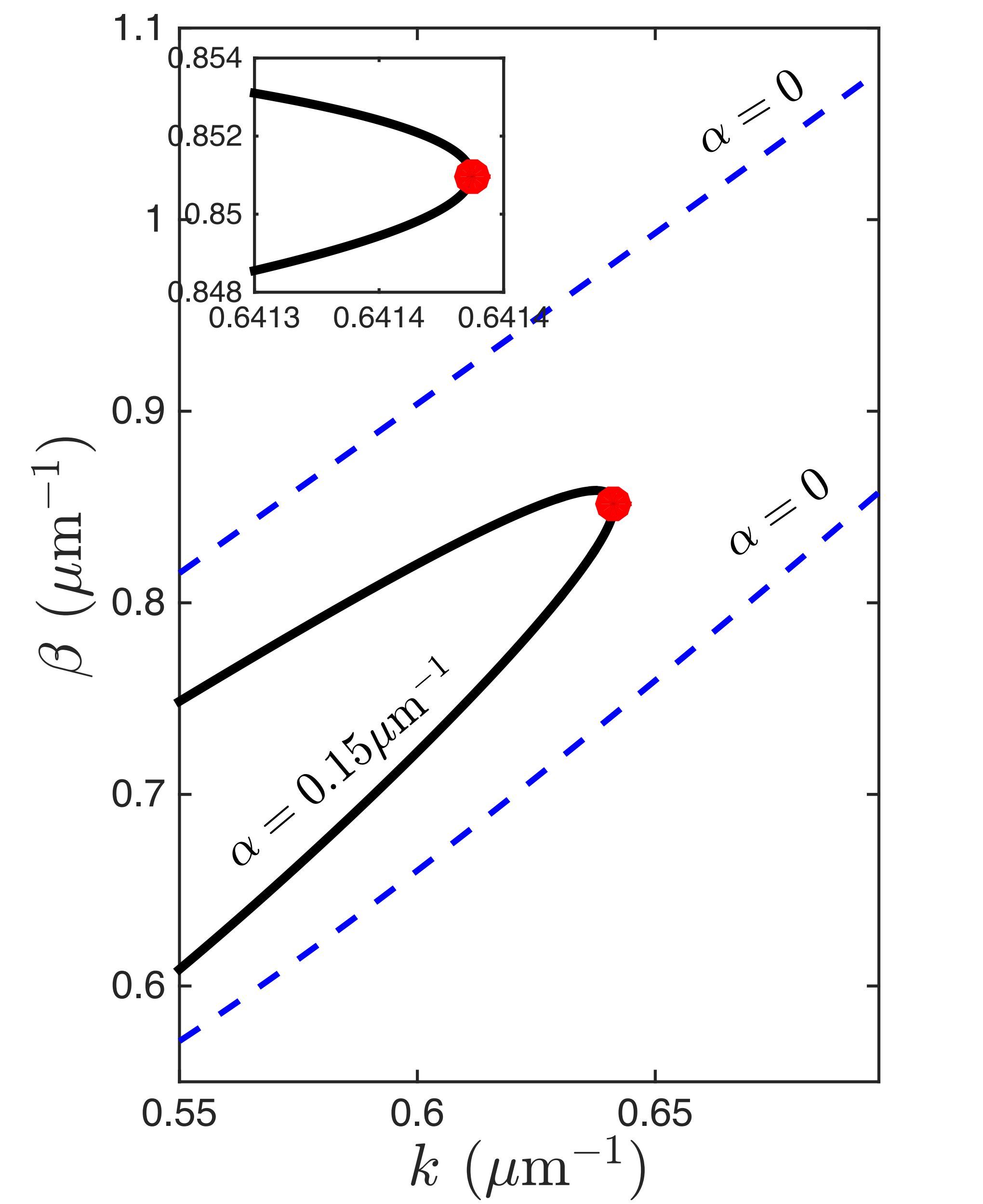


Full model: spectrum and modes

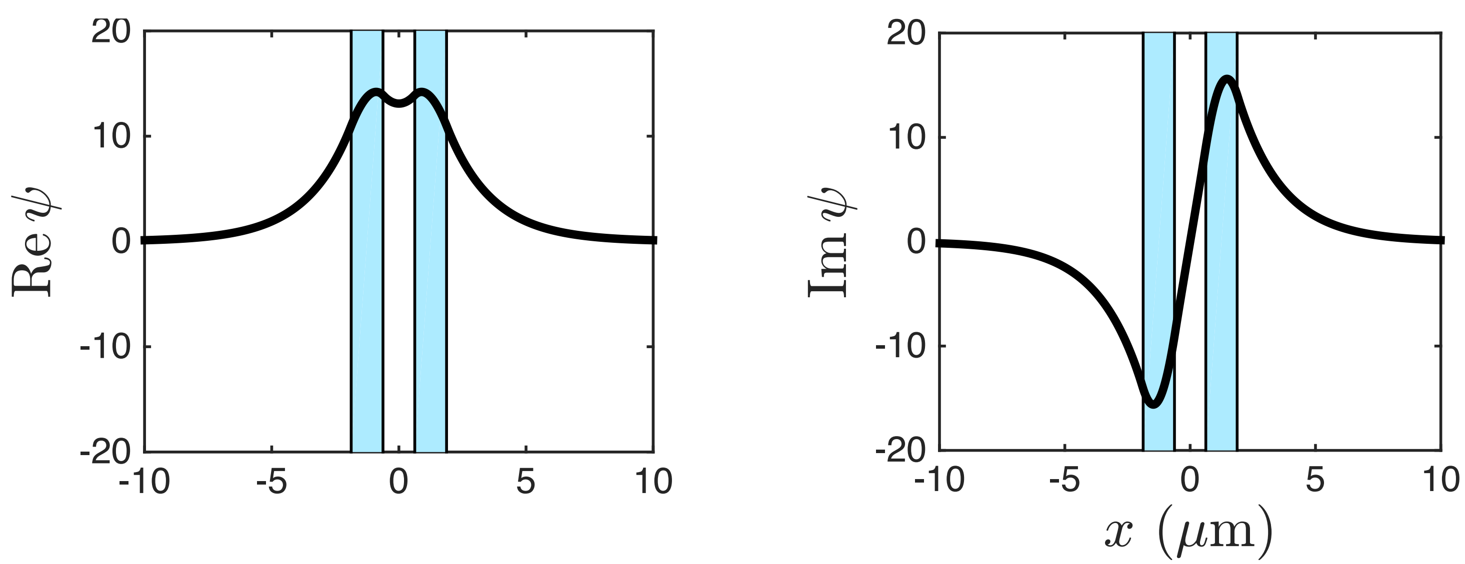
$$n_0 = 1,\ \ \Delta n = 1,\ D = a = 2.5\mu m$$



$$\beta_{EP} = 0.851\mu m^{-1} \ \ ,\ \ k_{EP} = 0.6414\mu m^{-1}$$



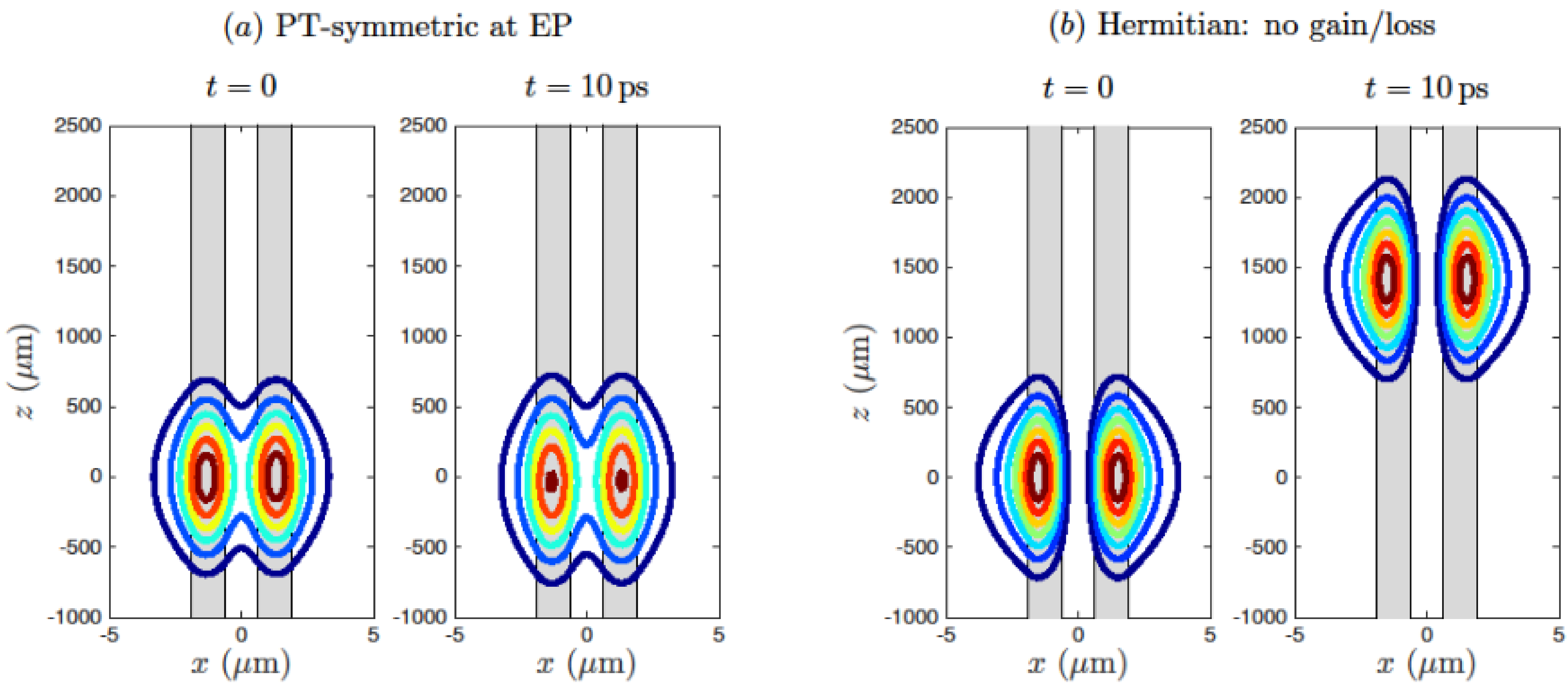
Mode at EP (real and imaginary parts):



Full model: dynamics

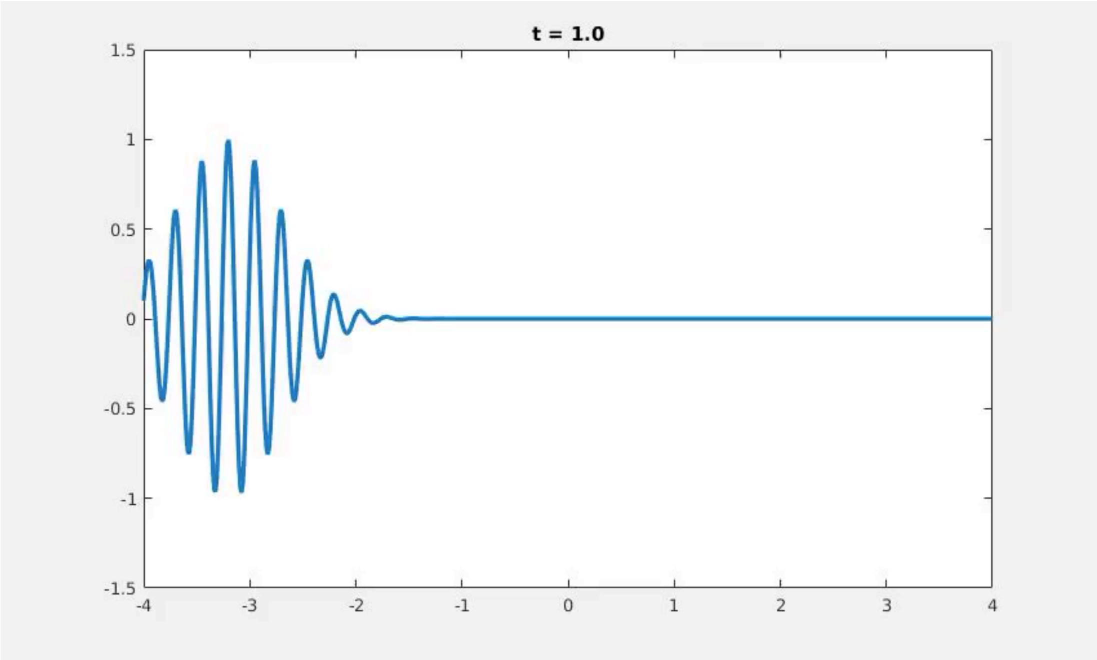
$$\phi(x,z,t) = A\int \exp\left(-\frac{(\beta-\beta_0)}{2\sigma^2} + i\beta z - i\omega t\right)\psi(x)\,d\beta$$

$$\beta_0 = \beta_{EP} = 0.851\,\mu m^{-1},\ \ \sigma = 0.002\,\mu m^{-1}$$
$$0.845 \le \beta \le 0.857\,\mu m^{-1}$$



Conclusions:

- Light stops (group velocity vanished) in a waveguides if and only if the the propagating mode is at the EP. This results from the singular structure of the branch point singularity - infinite derivative.
- Stop and release of a light pulse can be done by varying the gain/loss parameter in time.
- This effect can be implemented in PT-symmetric coupled waveguides. Advantages: avoiding power losses for propagating modes, robust protocol bringing the system to the EP.
- Practical implications: flexibility for controlling the EP position (non-resonant mechanism) with potential applications for short optical pulses.



Questions:

- Practical realization
- Stability issues
- Applicability to other systems (sound, fluids etc.)

Reference: PRL120, 013901 (2018)

Thank you!

alexei.impa.br