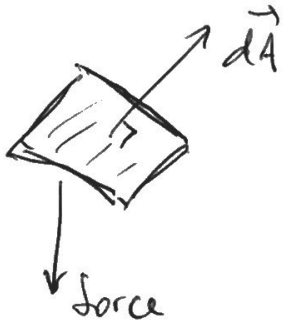
Dissipative effects in fluid flow

When flow is not homogeneous (the gradients ∇v, ∇T, $\nabla \rho$ are not identically zeros $\Rightarrow$ v, T, ρ are not const) $\Rightarrow$ the system is not exactly in thermodynamic equilibrium, and there are dissipative (irreversible) processes. (Obs: exception - rotation as a "solid" body).

We consider "sufficiently slow" motion, such that dissipative processes are small and can be studied in the first-order approximation w.r.t. ∇v, ∇T, $\nabla \rho$.

Viscous flow:



Force in a general case:
(the Cauchy stress theorem based on conservation of momenta - linear & angular)

$$\boxed{(\text{force})_i = \sigma_{ij}\, dA_j}, \qquad i = 1, 2, 3$$

Here we use coordinates $r = (x_1, x_2, x_3)$ and assume summation w.r.t. repeated indices: $\sigma_{ij} dA_j := \sum_{j=1}^{3} \sigma_{ij} dA_j$.

By the Cauchy theorem, σ_{ij} is a symmetric tensor called the stress tensor. We define

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + \sigma'_{ij}}$$

where $p = p(T, \sigma)$ is the pressure given by the equilibrium thermodynamics, and σ'_{ij} is the remaining symmetric part, called viscous tensor.

In equilibrium, $\underline{\sigma'_{ij} = 0}$. Also, fluid is <u>isotropic</u>.

Hence, σ'_{ij} is proportional to $\partial v_i / \partial x_j$ as a tensor (maintaining the form of relation under rotations).

<u>Obs</u>: There are no <u>linear</u> tensor terms proportional to ∇T or $\nabla \rho$.

Velocity gradient: $$\frac{\partial v_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\substack{\text{antisymmetric} \\ \to \text{ vorticity (rotation)}}}$$

$\sigma'_{ij} = 0$ for a rotating fluid (exact thermod. equilibrium)

$\Rightarrow$ σ'_{ij} depends only on the symmetric part, which yields two independent tensor terms:

$$\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad \text{and} \quad \underbrace{\left(\frac{\partial v_k}{\partial x_k}\right)}_{\operatorname{div} v} \delta_{ij} \qquad \left(\begin{array}{c}\text{see tensor} \\ \text{analysis}\end{array}\right)$$

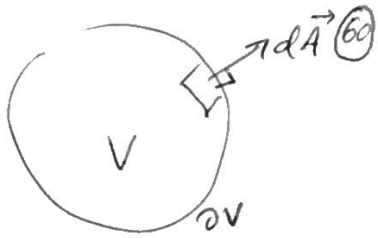
We choose $\cancel{\sigma'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\left(\frac{\partial v_k}{\partial x_k}\right)\right) + \zeta}$

$$\sigma'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \left(\zeta - \frac{2}{3}\eta\right)\left(\frac{\partial v_k}{\partial x_k}\right)\delta_{ij}$$

where η and ζ are viscous coefficients.

<u>Obs</u>: η and ζ depend on thermodynamic variables, e.g. $\eta = \eta(T, \rho)$ or $\eta(T, P)$.

<u>Obs</u> ζ is called the second viscosity (defines dissipation due to volume changes)



2d Newton's law:

$$\int_V \rho \frac{dv_i}{dt}\, dV = \oint_{\partial V} \sigma_{ij}\, dA_j \quad , \quad i = 1,2,3$$

Stokes formula: $\displaystyle\oint_{\partial V} \sigma_{ij}\, \partial A_j = \int_V \frac{\partial \sigma_{ij}}{\partial x_j}\, dV$

$$\int_V \left(\rho \frac{dv_i}{dt} - \frac{\partial \sigma_{ij}}{\partial x_j} \right) dV = 0 \quad \text{for any volume } V.$$

$$\Rightarrow \quad \rho \underbrace{\frac{dv_i}{dt}}_{\text{material derivative}} - \underbrace{\frac{\partial \sigma_{ij}}{\partial x_j}}_{\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij} = \dots} = 0$$

$\Longrightarrow$ after long derivation (exercise)

$$\boxed{\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \eta \Delta v + \left(\zeta + \frac{\eta}{3} \right) \operatorname{grad} \operatorname{div} v}$$

Navier-Stokes equations.

Obs Can add body forces, e.g. gravity: $+\rho \vec{g}$.

Obs Here $p = p(T, \rho)$ is defined by equilibrium thermodynamic relations.

<u>Boundary conditions</u>: no-slip condition on a rigid wall: $v = 0$ for $r \in$ rigid boundary.

Particular cases:

(1) Neglecting dependence on temperature, $P = p(\rho)$ and taking $\eta = \text{const}$, $\zeta = \text{const}$, we obtain the closed system of equations for $\rho(r,t)$ and $v(r,t)$: continuity eq. + N.S. eq.

(2) Neglecting variations of density, $\rho \approx \text{const}$.

Continuity eq. $\Longrightarrow$ $\boxed{\operatorname{div} v = 0}$ incompressibility condition.

N.S. eq $\Longrightarrow$ $\boxed{\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{\nabla p}{\rho} + \nu \Delta v}$

incompressible Navier-Stokes

where $\nu := \frac{\eta}{\rho}$ is called the kinematic viscosity (η is the dynamic viscosity)

Neglecting dependence of ν on temperature, (N.S. eq + incompr. condition) is the closed system of equations. Here pressure p is uniquely defined for the given v and boundary conditions (no need for thermodynamics!).

In general, we need one more equation that determines the evolution of thermodynamic (second) variable, e.g., temperature $T(r,t)$.

Energy is conserved, so we can model dissipative processes by extra terms in the energy conservation law

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2}+\rho e\right)+\operatorname{div}\Gamma=0.$$

Here $\Gamma=(\Gamma_1,\Gamma_2,\Gamma_3)$ is the energy flux given by

$$\Gamma_i=\underbrace{\rho v_i\left(\frac{v^2}{2}+h\right)}_{\substack{\text{energy flux}\\ \text{in ideal flow}}}+\underbrace{q_i}_{\substack{\text{dissipative}\\ \text{contribution}}}$$

In the first-order approximation, $q=(q_1,q_2,q_3)$ is proportional to gradients of velocity v and gradients of thermodynamical variables (we will choose T and P).

(1) terms proportional to $\partial v_i/\partial x_j$:

Consider $T\equiv$ const $\Rightarrow$ there is no heat transfer

$\Rightarrow$ energy flux is due to mechanical work only.

This means that q is due to work of viscous forces.

$$\begin{array}{ccc} \text{change of energy} & = & \text{work} \\ \| & & \| \\ \underbrace{\text{energy flux}}_{\|}\cdot dt & & \underbrace{\text{force}}\cdot\underbrace{\text{displacement}} \\ q_j dA_j\cdot dt & = & -\sigma'_{ij}\,dA_j\cdot v_i\,dt. \end{array}$$

After dropping $dA_j dt$, we obtain

$$q_j = -\sigma'_{ij} v_i \quad \text{or} \quad \boxed{\mathbf{q} = \sigma' \mathbf{v}} \quad \text{(matrix form)}$$

(viscous stress tensor)

This is the energy flux due to viscosity.

(2) terms proportional to ∇T. Due to isotropy of fluid $\Rightarrow$ $\mathbf{q} = -\lambda \nabla T$ with some coefficient, called the thermal conductivity.

Obs λ is in general a function of local thermodynamic variables, e.g., $\lambda = \lambda(\rho, T)$.

(3) there cannot be terms proportional to ∇P, ρ as we will prove later below.

We obtained the <u>energy equation</u> in the form:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho e\right) = -\operatorname{div}\left[\rho \mathbf{v}\left(\frac{v^2}{2} + h\right) - \sigma' \mathbf{v} - \lambda \nabla T\right],$$

where energy density e, enthalpy density h and thermal conductivity λ are functions of local thermodynamic variables (T, ρ) - local equilibrium condition.

Expression for the viscous stress tensor σ' was given above (see derivation of N.S. eq.).

Entropy equation

We will use : $h = e + \frac{p}{\rho}$

thermodynamics

$$\begin{cases} de = T\,ds + \frac{p}{\rho^2}\,d\rho \Rightarrow \frac{\partial e}{\partial t} = T\frac{\partial s}{\partial t} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} \\ dh = T\,ds + \frac{1}{\rho}\,dp \Rightarrow \nabla h = T\nabla s + \frac{1}{\rho}\nabla p \Rightarrow \nabla p = \rho\nabla h - \rho T\nabla s. \end{cases}$$

fluid dynamics

$$\begin{cases} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho v) = 0 \quad \text{continuity eq.} \\ \rho\left(\frac{\partial v}{\partial t} + v\cdot\nabla v\right) = -\nabla p + \nabla\sigma', \quad \sigma'_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \left(\zeta - \frac{2\eta}{3}\right)\frac{\partial v_k}{\partial x_k}\delta_{ij} \\ \frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \rho e\right) = -\mathrm{div}\left[\rho v\left(\frac{v^2}{2} + h\right) - \sigma' v - \lambda\nabla T\right] \end{cases}$$

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \rho e\right) = \left(\frac{v^2}{2} + e\right)\frac{\partial \rho}{\partial t} + \rho v\cdot\frac{\partial v}{\partial t} + \rho T\frac{\partial s}{\partial t} + \frac{p}{\rho}\frac{\partial \rho}{\partial t} =$$

$\Longrightarrow$ long derivation using all above eqs. (exercise)

$$\boxed{\underbrace{\rho T\frac{ds}{dt}}_{\text{heat received}} = \underbrace{\sigma'_{ij}\frac{\partial v_i}{\partial x_j}}_{\text{viscosity}} + \underbrace{\mathrm{div}(\lambda\nabla T)}_{\text{thermal conduction.}}}$$

$$\frac{\partial}{\partial t}(\rho s) = s\frac{\partial \rho}{\partial t} + \rho\frac{\partial s}{\partial t} \underset{\text{(exercise)}}{=} \mathrm{div}(\ldots) + \lambda\frac{(\nabla T)^2}{T^2} +$$

$$+ \frac{\eta}{2T}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right)^2 + \frac{\zeta}{T}(\mathrm{div}\,v)^2$$

$$\frac{\partial}{\partial t}\underbrace{\int \rho s\,dV}_{\text{total entropy (in whole space)}} = \underbrace{\cancel{\int \mathrm{div}(\ldots)\,dV}}_{\text{(vanishes at } \infty)} + \lambda\int\frac{(\nabla T)^2}{T^2}dV + \frac{\eta}{2T}\int(\ldots)^2 dV + \frac{\zeta}{T}\int(\ldots)^2 dV$$

2^d law of thermodynamics: $\frac{dS}{dt} \geq 0$.

$\Rightarrow$ $\boxed{\lambda \geq 0, \quad \eta \geq 0, \quad \zeta \geq 0}$ all dissipative coefficients are <u>non-negative</u>.

<u>Obs</u> If one adds $+\gamma \nabla p$ to the energy flux, this yields the term proportional to $+\gamma \int \nabla p \cdot \nabla T \, dV$ in the total ~~energy~~ entropy change. Since, it is <u>not sign-definite</u>, from the 2^d law of thermod. $\Rightarrow$ <u>$\gamma = 0$</u>.

<u>Full system of equations:</u>

<u>5 dependent variables</u>: velocity field $v(r,t)$ and thermodynamic fields, e.g., $\rho(r,t)$ and $T(r,t)$.

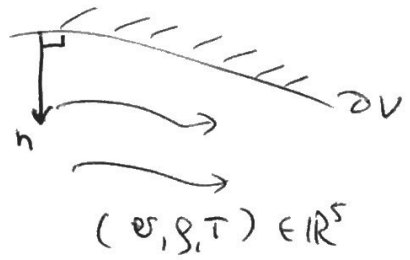
<u>5 equations</u>: continuity eq, N.S. eqs, energy conservation eq.

+ thermodynamic relations $e(\rho, T)$ and $h = e + \frac{p}{\rho}$.

+ dissipative coefficients $\lambda(\rho,T)$, $\eta(\rho,T)$ and $\zeta(\rho,T)$.

<u>Boundary conditions</u> (example)

For a <u>rigid isolating</u> wall:



$r \in \partial V$: (1) $v = 0$ (no-slip)

(2) $n \cdot \nabla T = 0$ (no heat flux through the boundary)

<u>Obs</u>. Recall that heat flux is: $-\lambda \nabla T$.

Particular case (incompressible flow)

Main assumptions:

(1) Flow is slow with $v \sim \frac{L}{t_*} \ll c$, where L and t_* are characteristic length and time of the flow, $c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}$ (adiabatic derivative) is the <u>sound speed</u>.

<u>Obs</u>: $\left(\frac{\partial P}{\partial \rho}\right)_s = c^2 \Rightarrow$ ("integrating" w.r.t. ρ) $P \sim \rho c^2$

From the N.S. equation: $\underbrace{\rho v \cdot \nabla v}_{\text{convective term}} \sim \nabla p$

$\Rightarrow$ (by order of magnitude) $\frac{\rho v^2}{L} \sim \frac{\delta P}{L} \Rightarrow \delta P \sim \underbrace{\rho v^2}_{\text{(second-order small!)}} \ll P \sim \rho c^2$

~~Hence~~, Hence, relative variations of pressure are small:

$$P = P_0 + \delta P, \quad P_0 \equiv \text{const}, \quad \delta P \ll P_0.$$

For the temperature $T = T_0 + \delta T$ with $T_0 \equiv$ const and δT small, we have

$$\rho(T,P) \approx \underset{\rho(T_0,P_0)}{\underset{\parallel}{\rho_0}} + \left(\frac{\partial \rho}{\partial T}\right)_{T_0,P_0} \delta T + \left(\frac{\partial \rho}{\partial P}\right)_{T_0,P_0} \delta P$$

Neglecting the second-order term $\delta P \sim \rho \underline{\underline{v^2}}$, we have

$$\rho \approx \rho_0 + \left(\frac{\partial \rho}{\partial T}\right)_{T_0,P_0} \delta T = \rho_0 (1 - \alpha \delta T).$$

where $\alpha = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_{T_0,P_0}$ is the coefficient of thermal expansion.

(2) Thermal expansion is weak if $\delta T \ll 1/\alpha$.

In this case $\delta\rho = -\rho_0 \alpha \delta T \ll \rho_0$, i.e.,

the density is approximately constant.

<u>Resulting equations of motion:</u>

(A) Continuity eq.: $\underbrace{\frac{\partial \rho}{\partial t}}_{\sim \frac{\delta\rho}{t_*} \sim \frac{v\delta\rho}{L}} + \underbrace{v\cdot\nabla\rho}_{\sim \frac{v\delta\rho}{L}} + \underbrace{\rho\, \mathrm{div}\, v}_{\sim \rho_0 \frac{v}{L}} = 0$

$\ll$

$\Rightarrow$ in the first-order approximation: $\boxed{\mathrm{div}\, v = 0}$

incompressibility cond.

(B) Navier-Stokes eqs. (with <u>gravity</u>)

$$\rho\left(\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right) = -\nabla p + \eta \Delta v + \rho \vec{g}$$

The bouyancy term: $\rho\vec{g} = \rho_0\vec{g} + \delta\rho\vec{g} = \underbrace{\rho_0\vec{g}}_{} - \rho_0\alpha\delta T\vec{g}$

<u>this const. term can be included into P.</u>

Hence, in the first-order approximation (dividing by $\rho \approx \rho_0$):

$$\boxed{\frac{\partial v}{\partial t} + v\cdot\nabla v = -\frac{\nabla p}{\rho_0} + \underbrace{\nu}_{\eta/\rho_0}\Delta v - \alpha\Delta T\vec{g}} \quad \text{with } \Delta T = T - T_0.$$

This is the <u>Boussinesq approximation</u> for bouyancy.

(c) Heat equation:

$$\rho T \frac{ds}{dt} = \sigma'_{ij} \frac{\partial v_i}{\partial x_j} + \operatorname{div}(\lambda \nabla T)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla$ and $\sigma'_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$. Recall: $\frac{\partial v_k}{\partial x_k} = 0$

In the first-order approximation:

$$P = P_0 + \underbrace{\delta P}_{\sim v^2 \text{ (second-order)} \to \text{neglect}} \qquad T = T_0 + \underbrace{\delta T}_{\text{first-order} \to \text{keep.}}$$

Term 1: $\rho T \frac{ds}{dt} \approx \rho_0 T_0 \left(\frac{\partial s}{\partial T} \right)_{T_0, P_0} \frac{dT}{dt} = \rho_0 c_p \frac{dT}{dt}$

(Recall that $c_p := T \left(\frac{\partial S}{\partial T} \right)_P$ is the <u>heat capacity</u>)

term 2: $\sigma'_{ij} \frac{\partial v_i}{\partial x_j} \overset{\left(\substack{i \leftrightarrow j \\ \text{symmetry}}\right)}{=} \frac{1}{2} \sigma'_{ij} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \frac{\eta}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2$

term 3: $\operatorname{div}(\lambda \nabla T) \approx \lambda \Delta T$ with λ taken at T_0, P_0.

Resulting <u>heat equation</u> (dividing by $\rho_0 c_p$)

$$\boxed{\frac{\partial T}{\partial t} + v \cdot \nabla T = \chi \Delta T + \frac{\nu}{2 c_p} \underbrace{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2}_{\text{(summed w.r.t } i \text{ and } j)}}$$

where $\nu = \eta / \rho_0$ and

$\chi = \frac{\lambda}{\rho c_p}$ is called thermometric conductivity.

The <u>full system</u>: $\boxed{\operatorname{div} v = 0}$ + $\boxed{\text{Boussinesq eq}}$ + $\boxed{\text{heat eq}}$ determine the fields $v(r,t)$, $P(r,t)$ and $T(r,t)$, where all coefficients (ν, α, χ, c_p) are constants. <u>Obs</u>: $v \equiv 0 \Rightarrow \boxed{\frac{\partial T}{\partial t} = \chi \Delta T}$ stand. heat eq.