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AEROELASTIC STABILITY OF A WING WITH BRACING STRUTS (KELDYSH PROBLEM)

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The problem of the influence of bracing struts of two types on the aeroelastic stability of a wing is studied. The formulation of the problem follows that considered by M. V. Keldysh [1]. The behavior of the eigenvalues is studied in the complex plane and the stability, flutter, and divergence domains are constructed.

The problem of the aeroelastic stability of a high-aspect-ratio unswept wing with bracing struts [1] is considered. The bracing strut is an absolutely rigid rod connecting the wing to the fuselage at a point P (Fig. 1). The presence of a bracing strut means that the point P is fixed and leads to additional boundary conditions imposed on functions of the shape of the vibrations.

In the case of a particular rectangular wing with a single bracing strut fastened at the centre of rigidity of the cross-section h (type A), from the calculations [1] it was concluded that "in the neighborhood of $h = 0.47\,l$ the critical velocity becomes imaginary and, consequently, when $h > 0.47\,l$ the vibration of a wing with a bracing strut become impossible" (l is the length of the wing), i.e., the wing becomes stable. A similar conclusion was also made for bracing of type B in which two struts fix the cross-section h: "when $h/l > 0.8$ the critical velocity does not exist for a wing with bracing of type B". These conclusions, based on the one-term approximation in accordance with the Bubnov-Galerkin method, are not confirmed when the number of the terms in the approximate solution is increased.

The aim of the present paper is to clarify the way in which the wing "becomes non-vibrating" and give a qualitative and quantitative description of this phenomenon. The problem of the aeroelastic stability of a wing reduces to studying the behavior of the eigenvalues λ in the complex plane for the linearized equations of motion of the wing as a function of the distance h from the base of the wing to the point of attachment of the strut and the flow velocity V. Accordingly, the values of the critical velocities of the vibrational (flutter) and static (divergence) modes of loss of stability are determined and the stability, flutter and divergence domains are constructed in the plane of the parameters V, h.

1. BASIC RELATIONS

We will consider the vibrations of a slender high-aspect-ratio wing braced by a strut at the point P in an air flow (Fig. 1). The wing is simulated by an elastic beam which is subject to torsion and bending and has a straight elastic axis y (the axis of rigidity) perpendicular to the fuselage. The wing deformation can be characterized by the deflection $z(y, \tau)$ and the angle of rotation $\theta(y, \tau)$ about the elastic axis, where τ is time. The linearized equations of motion of the wing in a flow have the form [2, 3]:

$$\frac{\partial^2}{\partial y^2}\left(EI\frac{\partial^2 z}{\partial y^2}\right) + m\frac{\partial^2 z}{\partial \tau^2} - m\sigma\frac{\partial^2 \theta}{\partial \tau^2} = L_a,$$
$$-\frac{\partial}{\partial y}\left(GJ\frac{\partial \theta}{\partial y}\right) - m\sigma\frac{\partial^2 z}{\partial \tau^2} + I_m\frac{\partial^2 \theta}{\partial \tau^2} = M_a \tag{1.1}$$

In these equations EI and GJ are the bending stiffness and the torsional rigidity, respectively, m and I_m are the mass and the moment of inertia about the elastic axis per unit of span, and σ is the distance from the center of rigidity to the center of gravity of the cross-section. The aerodynamic force L_a and moment M_a per unit of span are determined on the basis of the stationarity hypothesis [2, 3], in accordance with which at each instant the aerodynamic characteristics of the wing are replaced by the characteristics of the same wing moving at constant linear and angular velocities equal to the velocities of the real motion. The expressions for L_a and M_a can be written in the form:

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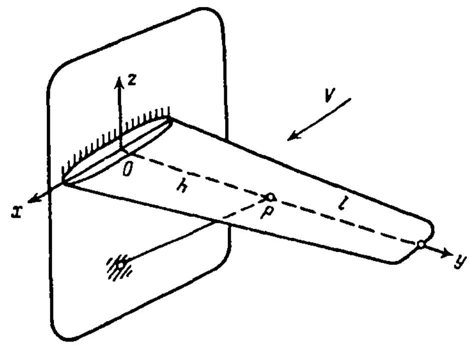

Fig. 1. Wing braced by a strut.

$$
\begin{aligned}
L_\alpha &= C_y^\alpha \rho V^2 t\left[\theta + \frac{t}{V}\left(\frac{3}{4} - \frac{x_0}{t}\right)\frac{\partial\theta}{\partial\tau} - \frac{1}{V}\frac{\partial z}{\partial\tau}\right]\\
M_\alpha &= C_m^\alpha \rho V^2 t^2\left[\theta + \frac{t}{V}\left(\frac{3}{4} - \frac{x_0}{t} - \frac{\pi}{16C_m^\alpha}\right)\frac{\partial\theta}{\partial\tau} - \frac{1}{V}\frac{\partial z}{\partial\tau}\right]
\end{aligned}
\tag{1.2}
$$

where t is the wing chord, x_0 is the distance from the leading edge to the elastic axis, and V is the flow velocity. For a slender infinite-span wing the theoretical values of the aerodynamic coefficients C_y^α and C_m^α are, respectively, $C_y^\alpha = \pi$ and $C_m^\alpha = \pi(x_0/t - 1/4)$ [2, 3].

Considering the case of a wing rigidly connected to the fuselage, we write the boundary conditions at $y=0$ (rigid connection) and $y=l$ (free edge)

$$
\begin{aligned}
y=0: &\quad z = \frac{\partial z}{\partial y} = \theta = 0\\
y=l: &\quad EI\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(EI\frac{\partial^2 z}{\partial y^2}\right) = GJ\frac{\partial\theta}{\partial y} = 0
\end{aligned}
\tag{1.3}
$$

For bracing of type A (wing braced by means of an absolutely rigid strut at a point P lying on the elastic axis) we have

$$
y=h: \quad z_- = z_+ = 0 \tag{1.4}
$$

The minus and plus subscripts denote the limiting values of the quantity as $y \to h - 0$ and $y \to h + 0$, respectively. In addition to (1.4), in the cross-section h the conditions of continuity of both the torsion angle and the derivative of the deflection, as well as the conditions of continuity of the torsion and bending moments, must be satisfied [1]:

$$
\begin{aligned}
&\theta_- = \theta_+, \quad \left(\frac{\partial z}{\partial y}\right)_- = \left(\frac{\partial z}{\partial y}\right)_+\\
&\left(GJ\frac{\partial\theta}{\partial y}\right)_- = \left(GJ\frac{\partial\theta}{\partial y}\right)_+, \quad \left(EI\frac{\partial^2 z}{\partial y^2}\right)_- = \left(EI\frac{\partial^2 z}{\partial y^2}\right)_+
\end{aligned}
\tag{1.5}
$$

For bracing of type B (two struts bracing the cross-section h and making it fixed during vibration) we have [1]:

$$
z_- = z_+ = 0, \quad \theta_- = \theta_+ = 0, \quad \left(\frac{\partial z}{\partial y}\right)_- = \left(\frac{\partial z}{\partial y}\right)_+, \quad \left(EI\frac{\partial^2 z}{\partial y^2}\right)_- = \left(EI\frac{\partial^2 z}{\partial y^2}\right)_+ \tag{1.6}
$$

The systems of equations (1.1)—(1.5) and (1.1)—(1.3), (1.6) represent linear homogeneous boundary-value problems for bracing of types A and B, respectively. We will seek the solution of the problems in the form:

$$
z(y, \tau) = f(y)e^{\lambda\tau}, \quad \theta(y, \tau) = \varphi(y)e^{\lambda\tau} \tag{1.7}
$$

where λ is an eigenvalue and $f(y)$ and $\varphi(y)$ are eigenfunctions. Substituting (1.7) in (1.1)—(1.2), we obtain the following system of ordinary differential equations in $f(y)$ and $\varphi(y)$:

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} f \\ \varphi \end{pmatrix} = 0 \tag{1.8}$$

$$\begin{gathered} L_{11} = \frac{d^2}{dy^2}\left(EI\frac{d^2}{dy^2}\right) + m\lambda^2 + \lambda C_y^\alpha \rho t V \\ L_{12} = -m\sigma\lambda^2 - C_y^\alpha \rho t V^2 - \lambda C_y^\alpha \rho t^2 V\left(\frac{3}{4} - \frac{x_0}{t}\right) \\ L_{21} = -m\sigma\lambda^2 + \lambda C_m^\alpha \rho t^2 V \\ L_{22} = -\frac{d}{dy}\left(GJ\frac{d}{dy}\right) + I_m\lambda^2 - C_m^\alpha \rho t^2 V^2 - \lambda C_m^\alpha \rho t^3 V\left(\frac{3}{4} - \frac{x_0}{t} - \frac{\pi}{16 C_m^\alpha}\right) \end{gathered} \tag{1.9}$$

Substituting (1.7) in (1.3)—(1.6), we obtain the boundary conditions both at the base and the endpoint of the wing

$$\begin{gathered} y=0: \quad f = \frac{df}{dy} = \varphi = 0 \\ y=l: \quad EI\frac{d^2f}{dy^2} = \frac{d}{dy}\left(EI\frac{d^2f}{dy^2}\right) = GJ\frac{d\varphi}{dy} = 0 \end{gathered} \tag{1.10}$$

and in the cross-section h ((1.11) and (1.12) for bracing of types A and B, respectively)

$$\begin{gathered} f_- = f_+ = 0, \quad \varphi_- = \varphi_+, \quad \left(\frac{df}{dy}\right)_- = \left(\frac{df}{dy}\right)_+ \\ \left(GJ\frac{d\varphi}{dy}\right)_- = \left(GJ\frac{d\varphi}{dy}\right)_+, \quad \left(EI\frac{d^2f}{dy^2}\right)_- = \left(EI\frac{d^2f}{dy^2}\right)_+ \end{gathered} \tag{1.11}$$

$$f_- = f_+ = 0, \quad \varphi_- = \varphi_+ = 0, \quad \left(\frac{df}{dy}\right)_- = \left(\frac{df}{dy}\right)_+, \quad \left(EI\frac{d^2f}{dy^2}\right)_- = \left(EI\frac{d^2f}{dy^2}\right)_+ \tag{1.12}$$

Generally speaking, by virtue of the nonconservativeness (non-self-adjointness) of the problem the eigenvalues λ are complex: $\lambda = \alpha + i\omega$. Depending on the flow velocity V the amplitudes of the solutions (1.7) may decrease with time ($\mathrm{Re}\lambda < 0$, stability), remain constant ($\mathrm{Re}\lambda = 0$, stability limit) or increase ($\mathrm{Re}\lambda > 0$, instability). Two basic types of loss of stability are generally distinguished: vibrational $\mathrm{Re}\lambda = 0$, $\mathrm{Im}\lambda = \omega \neq 0$ (flutter) and aperiodic $\lambda = 0$ (divergence). The critical velocity of the system V_c is equal to the smaller of the critical flutter and divergence velocities V_f and V_d.

For bracing of type A the divergence velocity can be directly determined from Eqs. (1.8)—(1.11) by setting $\lambda = 0$ in these equations. As a result, we arrive at the eigenvalue problem:

$$\frac{d}{dy}\left(GJ\frac{d\varphi}{dy}\right) + C_m^\alpha \rho V_d^2 t^2 \varphi = 0 \tag{1.13}$$

$$\varphi(0) = 0, \quad \left(GJ\frac{d\varphi}{dy}\right)_{y=l} = 0 \tag{1.14}$$

$$y=h: \quad \varphi_- = \varphi_+, \quad \left(GJ\frac{d\varphi}{dy}\right)_- = \left(GJ\frac{d\varphi}{dy}\right)_+ \tag{1.15}$$

The function $GJ(y)$ is assumed to be continuously differentiable. The critical divergence velocity is the minimum eigenvalue of the problem (1.13)—(1.15). Using the variational formulation, we find

$$V_d^2 = \min_\varphi \int_0^l GJ\left(\frac{d\varphi}{dy}\right)^2 dy \Bigg/ \int_0^l \rho C_m^\alpha t^2 \varphi^2 dy \tag{1.16}$$

where the comparison function $\varphi(y)$ is continuously differentiable and satisfies the first (kinematic) of the boundary conditions (1.14) (the second of the boundary conditions (1.14) and conditions (1.15) are natural for functional (1.16)). From (1.16) it follows that V_d^2 is independent of the location of the strut h. This is natural since by virtue of the above assumption the bracing point on the wing lies on the elastic axis and, consequently, does not affect the wing torsion.

2. METHOD OF SOLUTION

For solving the eigenvalue problems (1.8)—(1.11) and (1.8)—(1.10), (1.12) we will use the Bubnov-Galerkin method [4]. For this purpose we choose two systems of linearly independent coordinate functions $f_1, ..., f_n$ and $\varphi_1, ..., \varphi_n$ which must be, respectively, four and two times continuously differentiable on the intervals $(0, h)$ and (h, l) and satisfy the boundary conditions (1.10) and (1.11) in the first case and (1.10) and (1.12) in the second. We represent the eigenfunctions f and φ of the system (1.8) in the form of the following linear combinations of the coordinate functions with the unknown coefficients α_j and β_j, respectively:

$$f(y)=\sum_{j=1}^{n}\alpha_j f_j(y), \qquad \varphi(y)=\sum_{j=1}^{n}\beta_j\varphi_j(y) \tag{2.1}$$

Substituting these expansions in Eqs. (1.8), multiplying the left sides of these equations by f_i and φ_i, respectively, and integrating with respect to y from 0 to l, we obtain $2n$ linear homogeneous equations in the coefficients α_j and β_j, $j=1, ..., n$

$$\begin{aligned}&\sum_{j=1}^{n}\alpha_j\int_0^l L_{11}[f_j]f_i\,dy+\sum_{j=1}^{n}\beta_j\int_0^l L_{12}[\varphi_j]f_i\,dy=0, \qquad i=1,...,n\\ &\sum_{j=1}^{n}\alpha_j\int_0^l L_{21}[f_j]\varphi_i\,dy+\sum_{j=1}^{n}\beta_j\int_0^l L_{22}[\varphi_j]\varphi_i\,dy=0\end{aligned} \tag{2.2}$$

where the differential operators L_{ij} are defined in (1.9). Integrating (2.2) by parts with allowance for the boundary conditions (1.10), (1.11) or (1.10), (1.12), in both cases we obtain the algebraic eigenvalue problem

$$[\lambda^2\mathbf{M}+\lambda V\mathbf{C}+\mathbf{K}+V^2\mathbf{D}]\xi=0 \tag{2.3}$$

where ξ is a column vector consisting of the unknown coefficients $\xi^T=(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n)$, and **M**, **C**, **K**, and **D** are the following $2n\times 2n$ matrices:

$$\mathbf{M}=\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix}, \quad \mathbf{C}=\begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}, \quad \mathbf{K}=\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}, \quad \mathbf{D}=\begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}$$

$$\mathbf{M}_{11}=\left[\int_0^1 m f_i f_j\,ds\right], \quad \mathbf{M}_{12}=\left[-\int_0^1 m\sigma f_i\varphi_j\,ds\right]$$

$$\mathbf{M}_{21}=\left[\int_0^1 m\sigma\varphi_i f_j\,ds\right], \quad \mathbf{M}_{22}=\left[\int_0^1 I_m\varphi_i\varphi_j\,ds\right]$$

$$\begin{aligned}&\mathbf{C}_{11}=\left[C_y^{\alpha}\rho\int_0^1 t f_i f_j\,ds\right], \quad \mathbf{C}_{12}=\left[-C_y^{\alpha}\rho\int_0^1 t^2\left(\frac{3}{4}-\frac{x_0}{t}\right)f_i\varphi_j\,ds\right]\\ &\mathbf{C}_{21}=\left[\rho\int_0^1 C_m^{\alpha}t^2\varphi_i f_j\,ds\right], \quad \mathbf{C}_{22}=\left[-\rho\int_0^1 C_m^{\alpha}t^3\left(\frac{3}{4}-\frac{x_0}{t}-\frac{\pi}{16C_m^{\alpha}}\right)\varphi_i\varphi_j\,ds\right]\end{aligned} \tag{2.4}$$

$$K_{11}=\left[\frac{1}{l^4}\int_0^1 EIf_i''f_j''\,ds\right],\qquad K_{12}=K_{21}=[0]$$

$$K_{22}=\left[\frac{1}{l^2}\int_0^1 GJ\,\varphi_i'\varphi_j'\,ds\right],\qquad D_{11}=D_{21}=[0]$$

$$D_{12}=\left[-C_y^\alpha\rho\int_0^1 tf_i\,\varphi_j\,ds\right],\qquad D_{22}=\left[-\rho\int_0^1 C_m^\alpha t^2\varphi_i\varphi_j\,ds\right],\qquad i,j=1,\ldots,n$$

Here, for the sake of convenience, we have used the dimensionless variable $s=y/l$ and the primes denote differentiation with respect to s.

By virtue of the homogeneity of the problem (2.3) a nonzero solution for ξ exists only on condition that:

$$\det[-\lambda^2 M + \lambda VC + K + V^2 D]=0$$

This equation serves to determine the eigenvalues λ as functions of the flow velocity V.

Using the change of variables $\zeta=\lambda\xi$ and doubling the dimensionality, we can reduce the quadratic eigenvalue problem (2.3) to the equivalent linear problem

$$Au=\lambda u$$

$$u=\begin{bmatrix}\xi\\ \zeta\end{bmatrix},\qquad A=\begin{bmatrix}O & E\\ -M^{-1}K-V^2M^{-1}D & -VM^{-1}C\end{bmatrix}\tag{2.5}$$

where O and E are the zero and unit $2n\times 2n$ matrices. Form (2.5) is convenient for using standard calculation software.

In the above transformations we assumed that the inverse matrix M^{-1} exists, i.e., $\det M\neq 0$. We will prove that $\det M > 0$ for the two systems of linearly independent coordinate functions f_i and φ_i, $i=1,\ldots,n$.

For this purpose we will consider the space of the vector functions $\psi(s)=(u(s), v(s))$, where the functions $u(s)$ and $v(s)$ are continuous on the closed interval [0, 1], and introduce the scalar product in accordance with the formula

$$(\psi_1,\psi_2)=\int_0^1\begin{pmatrix}u_1\\ v_1\end{pmatrix}^T\begin{pmatrix}m & -m\sigma\\ -m\sigma & I_m\end{pmatrix}\begin{pmatrix}u_2\\ v_2\end{pmatrix}ds=\int_0^1(mu_1u_2 - m\sigma(u_1v_2+v_1u_2)+I_mv_1v_2)\,ds\tag{2.6}$$

It is obvious that $(\psi_1,\psi_2)=(\psi_2,\psi_1)$. We will demonstrate that $(\psi,\psi)>0$ for any nonzero $u(s)$ and $v(s)$. From (2.6) we have

$$(\psi,\psi)=\int_0^1(mu^2-2m\sigma uv+I_mv^2)\,ds\tag{2.7}$$

In (2.7) the integrand represents a positive definite quadratic form since the Silvester conditions lead to the inequalities $m>0$, $m(I_m-\sigma^2m)=mI_c>0$, where I_c is the moment of inertia about the centre of mass per unit of span. These conditions are satisfied for functions $m(s)$ and $I_c(s)$ continuous and positive on the closed interval [0, 1].

In the space introduced we will consider the system of $2n$ vector functions $\psi_1=(f_1,0)$, $\psi_2=(f_2,0)$, ..., $\psi_n=(f_n,0)$, $\psi_{n+1}=(0,\varphi_1)$, ..., $\psi_{2n}=(0,\varphi_n)$.

Using (2.6), we can readily verify that the Gram matrix [5] of this system is the matrix M in (2.3)

$$M=\begin{bmatrix}(\psi_1,\psi_1) & (\psi_1,\psi_2) & \cdots & (\psi_1,\psi_{2n})\\ \vdots & \vdots & \vdots & \vdots\\ (\psi_{2n},\psi_1) & (\psi_{2n},\psi_2) & \cdots & (\psi_{2n},\psi_{2n})\end{bmatrix}$$

The determinant of the Gram matrix is positive for a linearly independent system [5]. Therefore, for the linearly independent coordinate functions f_i and φ_i, $i=1,\ldots,n$, we have $\det M>0$ and, consequently, the inverse matrix M^{-1} exists.

3. CHOICE OF THE COORDINATE FUNCTIONS

As the coordinate functions f_i and φ_i, $i=1, ..., n$ it is convenient to take the inherent shapes of pure bending and torsional vibration of a wing, located in a vacuum, having a constant spanwise cross-section and braced by a strut of the corresponding type in the cross-section h.

For this purpose we will consider the following problem determining the shapes of pure bending vibration

$$f^{(4)}(s) - k^4 f(s) = 0, \quad s \in [0, 1]$$
$$f(0) = f'(0) = 0, \quad f''(1) = f'''(1) = 0$$
$$s = h/l: \quad f_- = f_+ = 0, \quad f_-' = f_+', \quad f_-'' = f_+''$$

The eigenfunctions of this problem have the form [1]:

$$f_i(s) = \begin{cases} C_i^1 \cos a_i + C_i^2 \sin a_i - C_i^1 \cosh a_i - C_i^2 \sinh a_i, & s \in [0, \eta] \\ D_i^1 \cos b_i + D_i^2 \sin b_i + D_i^1 \cosh b_i + D_i^2 \sinh b_i, & s \in [\eta, 1] \end{cases} \tag{3.1}$$
$$a_i = k_i s, \quad b_i = k_i(s-1)$$

where we have introduced the dimensionless variable $\eta = h/l$.

The quantities $k_i > 0$, $i=1, 2, ...$ can be found from the characteristic equation

$$\begin{gathered}(-\cosh\nu \sin\nu + \sinh\nu \cos\nu)[1 + \cos\zeta \cosh\zeta] + \\ [-\cosh\zeta \sin\zeta + \sinh\zeta \cos\zeta](1 - \cosh\nu \cos\nu) = 0 \\ \nu = k_i \eta, \quad \zeta = k_i(\eta - 1)\end{gathered} \tag{3.2}$$

From (3.1) the coefficients C_i^1, C_i^2, D_i^1, D_i^2, $i=1, 2, ...$ can be expressed in terms of k_i and η as follows:

$$\begin{gathered}C_i^1 = 1, \quad C_i^2 = -\frac{\cosh\nu - \cos\nu}{\sinh\nu - \sin\nu} \\ D_i^1 = \frac{\sinh\zeta + \sin\zeta}{1 + \cos\zeta\cosh\zeta}\frac{1 - \cos\nu\cosh\nu}{\sinh\nu - \sin\nu} \\ D_i^2 = -\frac{\cosh\zeta + \cos\zeta}{1 + \cos\zeta\cosh\zeta}\frac{1 - \cos\nu\cosh\nu}{\sinh\nu - \sin\nu}\end{gathered} \tag{3.3}$$

As the coordinate functions φ_i, we choose the inherent shapes of pure torsional vibration of a wing, located in a vacuum, having a constant cross-section and braced by a strut of type A [1]

$$\varphi_i^a(s) = \sin[\pi(i - 1/2)s], \quad s \in [0, 1], \quad i = 1, 2,... \tag{3.4}$$

or by a strut of type B

$$\begin{gathered}\varphi_{2n-1}^b(s) = \sin\left(\frac{\pi n s}{\eta}\right), \quad 0 \le s \le \eta \\ \varphi_{2n}^b(s) = \sin\left(\frac{\pi(n - 1/2)(s - \eta)}{1 - \eta}\right), \quad \eta \le s \le 1, \quad n = 1, 2,...\end{gathered} \tag{3.5}$$

where the functions $\varphi_{2n-1}^b(s)$ and $\varphi_{2n}^b(s)$ are equal to zero when $\eta \le s \le 1$ and $0 \le s \le \eta$, respectively.

The systems of functions (3.1), (3.4), and (3.5) are orthogonal and satisfy the relations

$$\int_0^1 f_i f_j\, ds = 0, \quad \int_0^1 \varphi_i^a \varphi_j^a\, ds = 0, \quad \int_0^1 \varphi_i^b \varphi_j^b\, ds = 0, \quad \int_0^1 f_i'' f_j''\, ds = 0$$

$$\int_0^1 (\varphi_i^a)'(\varphi_j^a)'\, ds = 0, \quad \int_0^1 (\varphi_i^b)'(\varphi_j^b)'\, ds = 0$$

$$\int_0^1 f_i'' f_i''\, ds = k_i^4 \int_0^1 f_i^2\, ds, \quad \int_0^1 (\varphi_i^a)'(\varphi_i^a)'\, ds = \frac{1}{2}\pi^2\left(i - \frac{1}{2}\right)^2 \tag{3.6}$$

$$\int_0^1 (\varphi_i^b)'(\varphi_i^b)'\, ds = \begin{cases} \dfrac{(\pi n)^2}{2\eta}, & i = 2n - 1 \\ \dfrac{\pi^2(n - ½)^2}{2(1 - \eta)}, & i = 2n \end{cases}$$

$$n = 1, 2, \ldots, \quad i, j = 1, 2, \ldots, \quad i \neq j$$

The functions (3.1), (3.4) and (3.1), (3.5) are complete systems of the coordinate functions for bracing of types A and B, respectively.

For relatively simple distributions $EI(y)$, $GJ(y)$, $m(y)$, $I_m(y)$, $t(y)$, $\sigma(y)$, and $x_0(y)$ (for example, linear distributions) the integrals (2.4) can be calculated analytically using (3.1)—(3.6). This increases the calculation speed and accuracy.

4. RESULTS OF NUMERICAL CALCULATIONS FOR A WING WITH BRACING OF TYPE A

Using the method described above, we studied numerically the aeroelastic stability of a rectangular wing with bracing of type A as a function of the parameters V and η (nondimensionalized h). As in [1], in our calculations the wing parameters were borrowed from [6]:

$$l = 0.55\ \text{m}; \quad t = 0.18\ \text{m}; \quad x_0 = 0.071\ \text{m}; \quad \sigma = 0.017\ \text{m}; \quad EI = 1.481\ \text{kg}\cdot\text{m}^2$$

$$GJ = 0.25\ \text{kg}\cdot\text{m}^2; \quad ml = 0.0254\,\text{kg}\cdot\text{s}^2\cdot\text{m}^{-1}; \quad I_m l = 0.000059\ \text{kg}\cdot\text{m}\cdot\text{s}^2 \tag{4.1}$$

$$C_y^a = 1.36; \quad C_m^a = 0.143; \quad \rho = 0.117\ \text{kg}\cdot\text{s}^2\cdot\text{m}^{-4}$$

The calculations were carried out on the interval of velocities V from 0 to 155 m/s. In the Bubnov-Galerkin method the number of coordinate functions was taken equal to 5. This corresponds to a dimensionality of the algebraic eigenvalue problem (2.5) equal to 20. Increasing the number of coordinate functions to 10 showed that the maximum relative error in calculating the eigenvalues for the first six modes amounted to 0.15%.

The modes were numbered according to increasing positive imaginary parts of the eigenvalues $\lambda = \alpha + i\omega$ at $V=0$. Since the matrix operator A in the problem (2.5) is real, the complex conjugate quantity $\bar{\lambda} = \alpha - i\omega$ is also an eigenvalue. Thus, a pair of complex conjugate quantities λ and $\bar{\lambda}$ corresponds to a single vibration mode.

Behavior of the eigenvalues in the complex plane. In Fig. 2 we have reproduced the trajectories of the eigenvalues λ in the complex plane as functions of the velocity V for certain characteristic values of h. In Fig. 2 the arrows correspond to increase in V. The calculations showed that for any h the unreproduced branches corresponding to higher modes lie in the left half-plane, i.e., are stable. In view of the symmetry, only the upper half-plane $\text{Im}\lambda \geq 0$ is presented. The numbers on the imaginary axis mark the values of the critical velocity of the corresponding mode.

For small η loss of stability takes place in the first, second, and fourth modes (Fig. 2*a*). The first mode is divergent: two complex conjugate eigenvalues approach each other, collide, and diverge in opposite directions along the real axis. One of these eigenvalues crosses the imaginary axis at the velocity $V_{d1} = 61.3$ m/s. For multiparameter linear vibrational systems the behavior of the eigenvalues in the complex plane was studied in [7, 8].

The second and fourth modes are of the flutter type: when $\eta = 0.1$ the eigenvalues of these modes cross the imaginary axis at the points $\text{Im}\lambda \neq 0$ at the values $V_{f2} = 28$ and $V_{f4} = 128$ m/s, respectively. Thus, the second vibration mode is critical and $V_c = 28$ m/s. The branch of the second mode repeatedly intersects the imaginary mode, i.e., this mode becomes stable starting from the velocity value $V = 137$ m/s.

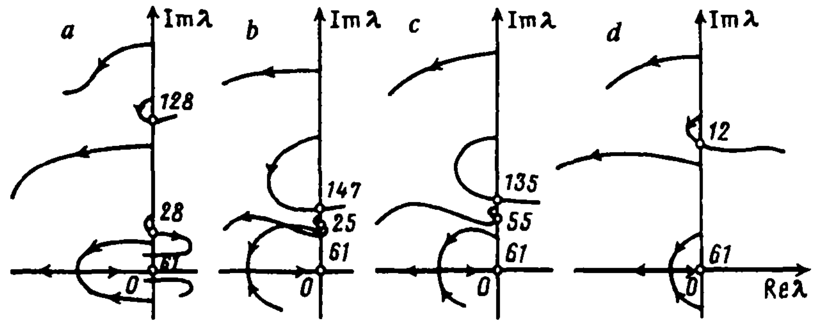

Fig. 2. Behavior of the eigenvalues in the complex plane as a function of the flow velocity V for bracing of type A for $\eta = 0.1$, 0.4, 0.471, and 0.8 (*a–d*, respectively).

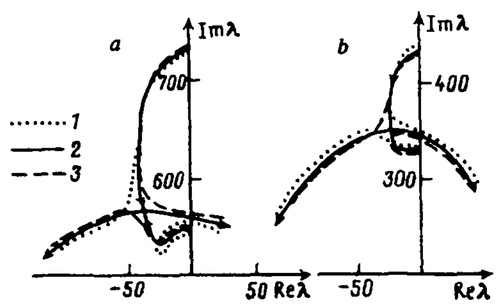

Fig. 3

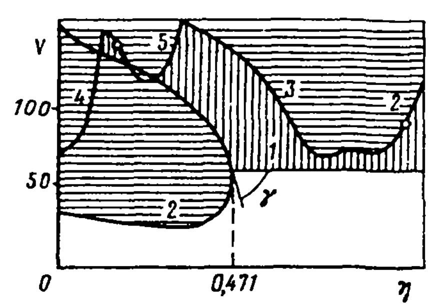

Fig. 4

Fig. 3. Crossover of the eigenvalue branches: for $\eta = 0.145$ (*a*) and $\eta = 0.935$ (*b*). Curves *1* and *3* correspond to the branches of the interacting modes for values of η less and greater by 0.005, respectively.

Fig. 4. Stability, flutter, and divergence domains for bracing of type A.

When η reaches the value 0.145, the fourth and fifth modes interact. The interaction is characterized by a change in the unstable mode: the fourth mode becomes stable, and the fifth unstable. The flutter velocity of the fifth mode, increasing with η, becomes greater than 155 m/s, i.e., leaves the calculation range. In this case the flutter appears in the third mode, $V_{f3} = 147$ m/s when $\eta = 0.4$ (Fig. 2*b*).

From Figs. 2*a*, 2*b*, and 2*c* it can be seen that the branch of the second (critical) mode makes a loop which is displaced to the left with increase in η. When $\eta = 0.471$ the loop is tangent to the imaginary axis at $V_{f2} = 55.7$ m/s (Fig. 2*c*) and at $\eta > 0.471$ the second mode becomes stable. This leads to a jump in the critical velocity which becomes equal to the divergence velocity $V_{d1} = 61.3$ m/s for $\eta > 0.471$. With further increase in η the branch of the second mode continues to be extended to the left and straightens, whereas the branch of the third mode is displaced to the right ($V_{f3} = 72$ m/s when $\eta = 0.8$) (Fig. 2*d*). When $\eta = 0.935$ the second and third modes begin to interact. As a result, the third mode becomes stable and the second unstable.

Thus, regardless of the behavior of the higher modes, the second (flutter type) mode is critical for any $\eta < 0.471$, while for $\eta > 0.471$ the critical mode is the first (divergent type) with the critical velocity $V_c = 61.3$ m/s.

Change in unstable mode. As noted above, at the values $\eta = 0.145$ and 0.935 the fourth-fifth and, respectively, second-third modes interact, which leads to a change in the unstable mode. This event takes place in the neighborhood of the double point λ_0 in the complex plane. In [7, 8] the theory of this phenomenon was developed for vibrational systems depending on several parameters. In these studies it was shown, in particular, that in the neighborhood of the double point the interaction of the eigenvalues can be described by hyperbolas with orthogonal asymptotes.

In Figs. 3*a* and 3*b* the behavior of the modes for $\eta = 0.145$ and 0.935, respectively, is shown by curves *2*. The arrows correspond to increase in V. The eigenvalues approach each other in the complex plane, collide along a straight line at

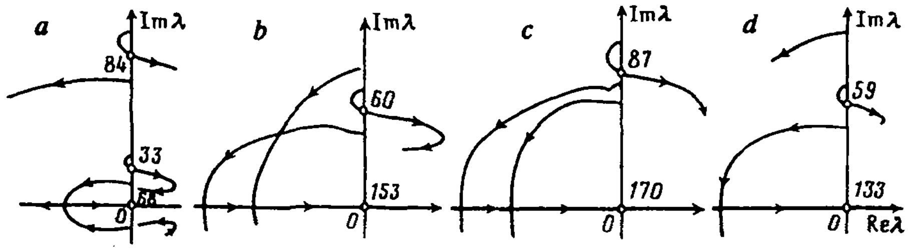

Fig. 5. Behavior of the eigenvalues in the complex plane as a function of the flow velocity V for bracing of type B for $\eta = 0.1$, 0.6, 0.72, and 0.92 (*a–d*, respectively).

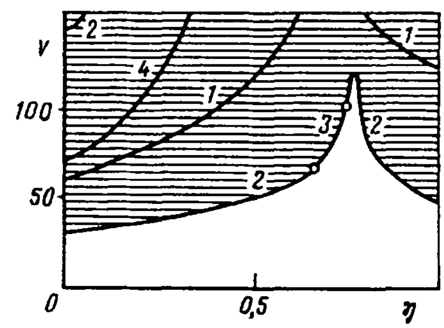

Fig. 6. Stability and flutter domains for bracing of type B.

the point of double λ_0, and then diverge at right angles to the line of approach. The unstable mode changes as a result of the "crossover" of the branches in the neighborhood of the double point λ_0.

Location of the stability, flutter, and divergence domains in the parameter plane and their characteristic features. In Fig. 4 we have reproduced the location of the stability, flutter, and divergence domains in the plane of the parameters V, η. The flutter domain is determined by the presence of at least one complex eigenvalue $\lambda = \alpha + i\omega$, $\alpha > 0$, $\omega \neq 0$, and the divergence domain by the presence of only real unstable roots $\lambda = \alpha > 0$. In Fig. 4 the flutter and divergence domains are hatched with horizontal and vertical lines, respectively. The numbers indicate the number of the mode that becomes unstable when the corresponding boundary is crossed. The mode is unstable on the side of the boundary on which the number is located.

The instability domain is the union of the flutter and divergence domains. It is well known that the stability limit of the common two-parameter family of matrices consists of smooth arcs intersecting transversely at their endpoints. These singularities correspond to corner points on the boundary of the stability domain with an angle $\gamma < \pi$ [9]. In the problem considered such a singularity is realized in Fig. 4 at the point lying on the boundary between the flutter and divergence domains at the vertex of the acute angle γ.

The phenomenon of change in the unstable mode described above takes place in the neighborhood of the double point λ_0 and, consequently, if $\mathrm{Re}\lambda_0 \neq 0$ does not lead to singularities on the boundaries of the stability, flutter and divergence domains.

On the discontinuity of the critical velocity. A discontinuity of the critical velocity (as a function of η) appears when in the plane of the parameters V, η the instability domain has a tangent orthogonal to the y axis and is convex at the point of tangency. This case is realized in the problem in question (Fig. 4). The discontinuity of V_c corresponds to $\eta_* = 0.471$. The left and right limits of V_c are equal to 55.7 and 61.3 m/s, respectively. In the complex plane the discontinuity of V_c corresponds to the fact that a branch of the critical mode (in this case the second mode) makes a loop which, as η increases, is extended into the left half-plane $\mathrm{Re}\lambda < 0$, being tangent to the imaginary axis for $\eta = \eta_*$ (Figs. 2*b* and 2*c*).

This mechanism of discontinuity of V_c is common to two-parameter problems. We note that the existence of a stability domain at $V > V_c$ is associated with the discontinuity of the critical velocity. In the problem in question this domain exists on the interval $0.470 < \eta < 0.471$.

Comparison with the results of study [1]. Comparing the results obtained by M. V. Keldysh [1, Fig. 4] with those of the present study (Fig. 4), we see that the values of V_c differ only slightly on the interval $0 \leq \eta \leq 0.471$ on which the wing loses stability according to the second flutter mode. However, when $\eta > 0.471$ the critical velocity does not disappear, as in [1], but becomes discontinuous and equal to the divergence velocity. As shown in section 1, when the strut is attached to the elastic axis of the wing, the critical divergence velocity is independent of η. For a rectangular wing with constant characteristics along the span the nonzero solutions of the divergence problem (1.13)—(1.15) have the form $\varphi_i = C_i \sin(\pi s(i - 1/2))$, $i = 1, 2, \ldots$ The first eigenfunction φ_1 corresponds to a value of the critical divergence velocity equal to

$$V_d = \frac{\pi}{2lt}\sqrt{\frac{GJ}{C_m^{\alpha}\rho}} = 61.3 \text{ m/s}$$

In [1] instability of the divergence type was not studied. This led to the conclusion that the wing is stable when $\eta > 0.47$. If we assume that this conclusion concerning the "impossibility of vibration" relates only to instability of the flutter type, then even in this case the possibility of flutter in higher modes was not taken into account due to the use of only a single coordinate function in expansions (2.1).

In the problem in question loss of stability due to flutter takes place in the third and second modes on the intervals $0.471 < \eta < 0.935$ and $0.935 < \eta < 1$, respectively (Fig. 4). Since for certain η V_β only slightly exceeds the critical velocity, this quantity needs to be investigated.

For $\eta > 0.471$ mounting a bracing strut on the elastic axis increases the critical velocity, as compared with the wing without a strut, approximately by a factor of two: from 30.3 to 61.3 m/s.

5. RESULTS OF NUMERICAL CALCULATIONS FOR A WING WITH BRACING OF TYPE *B*

The calculations for a rectilinear wing with bracing of type *B* with parameters (4.1) were carried out in accordance with the methods described in sections 2 and 3. The number of functions in expansions (2.1) was taken equal to 4. By increase in the number of coordinate functions to 10 it was shown that the maximum relative error in calculating the eigenvalues for the first 3—4 modes amounts to 0.1%.

As in the case of bracing of type *A*, the critical divergence velocity is estimated analytically. This quantity can be found independently for the portions of the wing to the left and right of the bracing and is equal to $V_l = \pi(GJ/(C_m^{\alpha}\rho))^{1/2}/(\eta lt)$ and $V_r = \pi(GJ/(C_m^{\alpha}\rho))^{1/2}/(2l(1 - \eta)t)$, respectively. In the V, η plane the velocities V_l and V_r are hyperbolas with the asymptotes $\eta = 0$ and $\eta = 1$.

Figure 5 illustrates the behavior of the eigenvalues λ in the complex plane as a function of the velocity V for various η. For small and large η ($0 < \eta \le 0.65$ and $0.74 < \eta \le 1$) the second, flutter mode is critical. For any $0 \le \eta \le 1$ the first mode remains divergent. For small (large) values of η the first mode corresponds to the right (respectively, left) part of the wing with respect to the bracing. When $\eta = 0.65$ and 0.74 the second and third modes interact. As a result, on the interval $0.65 < \eta < 0.74$ the third mode becomes critical and flutter type and the second mode becomes divergent.

Figure 6 illustrates the location of the stability and instability domains in the plane of the parameters V and η. For any location of the bracing $0 \le \eta \le 1$ the wing becomes unstable due to flutter. At $\eta = 0.65$ and 0.74 the interaction of the modes does not lead to singularities on the boundary of the stability domain.

As η increases from zero, the critical flutter velocity increases monotonically and reaches a maximum $V_\beta = 119$ m/s for $\eta = 0.76$. This value is higher than the critical velocity for the wing without a strut almost by a factor of four. Thereafter, the critical velocity decreases (Fig. 6).

A comparison with the calculations carried out by M. V. Keldysh [1, Fig. 8] indicates that the results are in satisfactory agreement for $0 \le \eta \le 0.7$; however, for $\eta > 0.7$ the results do not coincide and the conclusion to the effect that "for $\eta > 0.8$ the wing becomes non-vibrating" [1] is not confirmed.

Summary. As a result of a parametric investigation of the problem it is shown that for bracing of type *A* instability of the flutter type gives way to the static form - divergence, as the distance of the point of attachment of the strut from the wing base increases. In this case the critical velocity becomes discontinuous and jumps to a higher value. In the case of bracing of type *B* the critical velocity turns out to be finite and continuous and has a well expressed maximum which significantly exceeds the critical velocity for the wing without a strut.

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