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On Stability of Polynomials Depending on Parameters

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Abstract—Characteristic polynomials with real coefficients that smoothly depend on a vector of real parameters are considered. A constructive approach is proposed that allows one to determine, in a first approximation, the stability domain or a domain with a bounded decrement in the neighborhood of a singular or regular point of its boundary from the information available at this point (the roots and coefficients of the polynomial as well as the first derivatives of the coefficients with respect to parameters). Examples are presented.

INTRODUCTION

The study of stability problems for many mechanical and control systems is reduced to the analysis of the roots of the appropriate characteristic equation. Asymptotic stability is achieved when all roots of the characteristic polynomial lie in the left complex half-plane. The polynomials possessing this property are said to be stable. The properties of stable polynomials have been intensively studied in view of their high practical importance. There exist a number of methods for determining the stability, such as the classical Routh–Hurwitz method and a method of *D*-decompositions [1–3]. These methods allow one to determine either the stability of a specific polynomial or, if the coefficients of the polynomial are known functions of parameters, the stability domain (the set of values of the parameters for which the polynomial is stable).

Nevertheless, one meets considerable computational difficulties when constructing the stability domains of complex multiparameter systems because of a high order of polynomials and large computational cost for the determination of the coefficients and roots of the polynomial with many values of the parameters. One meets difficulties of a different type when multiple roots arise, which lead to the nondifferentiability of the roots of the polynomial with respect to the parameters and give rise to singularities (points of nonsmoothness) on the boundary of the stability domain [4]. Therefore, it seems topical to develop further qualitative and quantitative methods for investigating the stability domains of the families of polynomials that depend on many parameters.

Qualitative aspects of the structure of a stability domain and its boundary in the space of all polynomials with the leading coefficient equal to unity were studied in [5, 6]. In these papers, the singularities of stability-domain boundaries that appear in the case of general position were classified and the tangent cones (linear approximations) to the stability domain at singular points on its boundary were described up to a nondegenerate change of parameters.

In the present paper, we develop a constructive method for determining an approximation of the stability domain in the neighborhood of a point on this boundary in the space $\mathbb{R}^n$ of parameters based on information available at this point (using the values of the roots, the coefficients, and the first-order derivatives of the coefficients of the polynomial with respect to the parameters). We consider the points of the stability-domain boundaries of arbitrary type (characterized by roots of arbitrary multiplicity). It is assumed that the leading coefficients of the polynomial may vanish. Then, we extend the results obtained to the case when a domain with a bounded decrement (degree of stability) is considered instead of the domain of stability. The efficiency of the proposed method is demonstrated by two examples from automatic control theory.

The results obtained are the development of works [7, 8], where the case of two- and three-parameter families of polynomials of general position were studied.

1. STABILITY DOMAIN OF POLYNOMIAL FAMILIES

Consider a polynomial of degree M,

$$P(\lambda, \mathbf{p}) = a_M(\mathbf{p})\lambda^M + \ldots + a_1(\mathbf{p})\lambda + a_0(\mathbf{p}), \quad (1.1)$$

with real coefficients that smoothly depend on the vector $\mathbf{p} = (p_1, \ldots, p_n)$ of real parameters (a family of polynomials). Suppose that, for $\mathbf{p} = \mathbf{p}_0$, the polynomial $P_0(\lambda) = P(\lambda, \mathbf{p}_0)$ has the form

$$P_0(\lambda) = a_m^0\lambda^m + \ldots + a_1^0\lambda + a_0^0, \quad m \le M, \quad (1.2)$$

where $a_j^0 = a_j(\mathbf{p}_0)$, $j = 0, \ldots, M$; $a_m^0 \neq 0$; and $a_{m+1}^0 = \ldots = a_M^{\nu} = 0$. The case $m < M$ is singular, since the leading coefficients of the polynomial vanish. Denote by $\lambda_1, \ldots, \lambda_k$

the distinct roots of polynomial (1.2) and by $m_1, \ldots, m_k$, their multiplicities. When $m < M$, we assume that $P_0(\lambda)$ has an infinite root $\lambda_0 = \infty$ of multiplicity $m_0 = M - m$.

The polynomial $P(\lambda, \mathbf{p})$ is called stable if all its roots have negative real parts. The set of values of the vector $\mathbf{p}$ of parameters for which $P(\lambda, \mathbf{p})$ is stable is called the stability domain.

Theorem 1. The family of polynomials $P(\lambda, \mathbf{p})$ in the neighborhood of the point $\mathbf{p} = \mathbf{p}_0$ can be represented as

$$P(\lambda, \mathbf{p}) = \alpha(\mathbf{p})P^0(\lambda, \mathbf{p})P^1(\lambda, \mathbf{p})\ldots P^k(\lambda, \mathbf{p}), \quad (1.3)$$

where

$$P^0(\lambda, \mathbf{p}) = \beta_0(\lambda_0)\lambda^{m_0} + \ldots + \beta_{m_0-1}(\lambda_0)\lambda + 1,$$

$$P^j(\lambda, \mathbf{p}) = (\lambda - \lambda_j)^{m_j} + \beta_{m_j-1}(\lambda_j) \quad (1.4)$$

$$\times(\lambda - \lambda_j)^{m_j-1} + \ldots + \beta_0(\lambda_j), \quad j = 1, \ldots, k.$$

The coefficients $\beta_l(\lambda_j)$ are smooth (and complex-valued for $\lambda_j \in \mathbb{C}$) functions of vector $\mathbf{p}$ such that $\beta_l(\lambda_j) = 0$ for $\mathbf{p} = \mathbf{p}_0$ and $\alpha(\mathbf{p})$ is a smooth nonvanishing function. If $\lambda_i = \bar{\lambda}_j$, then $P^i(\lambda, \mathbf{p}) = \overline{P^j(\bar{\lambda}, \mathbf{p})}$, where the bar denotes complex conjugation.

Proof. Introduce $M \times M$ matrices $\mathbf{A}(\mathbf{p})$ and $\mathbf{B}(\mathbf{p})$:

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -a_0(\mathbf{p}) & -a_1(\mathbf{p}) & \ldots & -a_{M-1}(\mathbf{p}) \end{pmatrix},$$

$$\mathbf{B}(\mathbf{p}) = \begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & a_M(\mathbf{p}) \end{pmatrix}.$$

It can be readily shown that $P(\lambda, \mathbf{p}) = (-1)^M \det(\mathbf{A}(\mathbf{p}) - \lambda\mathbf{B}(\mathbf{p}))$. In [9], it was shown that there exists a transformation $\mathbf{A}(\mathbf{p}) - \lambda\mathbf{B}(\mathbf{p}) = \mathbf{R}(\mathbf{p})(\mathbf{A}'(\mathbf{p}) - \lambda\mathbf{B}'(\mathbf{p}))\mathbf{Q}^{-1}(\mathbf{p})$, where $\mathbf{R}(\mathbf{p})$ and $\mathbf{Q}(\mathbf{p})$ are nonsingular complex $M \times M$ matrices smoothly depending on $\mathbf{p}$ in the neighborhood of $\mathbf{p} = \mathbf{p}_0$, such that the matrix $\mathbf{A}'(\mathbf{p}) - \lambda\mathbf{B}'(\mathbf{p})$ has a block-diagonal form $\mathbf{A}'(\mathbf{p}) - \lambda\mathbf{B}'(\mathbf{p}) = \mathrm{Diag}[\mathbf{A}'_0(\mathbf{p}) - \lambda\mathbf{B}'_0(\mathbf{p}), \ldots, \mathbf{A}'_k(\mathbf{p}) - \lambda\mathbf{B}'_k(\mathbf{p})]$. The $m_j \times m_j$ blocks $\mathbf{A}'_j(\mathbf{p}) - \lambda\mathbf{B}'_j(\mathbf{p})$ correspond to the roots λ_j; i.e., $\det(\mathbf{A}'_j(\mathbf{p}_0) - \lambda\mathbf{B}'_j(\mathbf{p}_0)) = (\lambda - \lambda_j)^{m_j}$, $j = 1, \ldots, k$, and $\det(\mathbf{A}'_0(\mathbf{p}_0) - \lambda\mathbf{B}'_0(\mathbf{p}_0)) = 1$. Denoting

$$P^0(\lambda, \mathbf{p}) = \det(\mathbf{A}'_0(\mathbf{p}) - \lambda\mathbf{B}'_0(\mathbf{p}))/\det\mathbf{A}'_0(\mathbf{p}),$$

$$P^j(\lambda, \mathbf{p}) = (-1)^{m_j}\det(\mathbf{A}'_j(\mathbf{p}) - \lambda\mathbf{B}'_j(\mathbf{p}))/\det\mathbf{B}'_j(\mathbf{p}),$$

$$j = 1, \ldots, k,$$

$$\alpha(\mathbf{p}) = (-1)^{m_0}\det\mathbf{R}(\mathbf{p})\det\mathbf{A}'_0(\mathbf{p})\det\mathbf{B}'_1(\mathbf{p})\ldots$$

$$\det\mathbf{B}'_k(\mathbf{p})/\det\mathbf{Q}(\mathbf{p}),$$

we obtain

$$P(\lambda, \mathbf{p}) = (-1)^M\det(\mathbf{A}(\mathbf{p}) - \lambda\mathbf{B}(\mathbf{p}))$$

$$= (-1)^M\det(\mathbf{R}(\mathbf{p})(\mathbf{A}'(\mathbf{p}) - \lambda\mathbf{B}'(\mathbf{p}))\mathbf{Q}^{-1}(\mathbf{p}))$$

$$= (-1)^M\det\mathbf{R}(\mathbf{p})\det(\mathbf{A}'_0(\mathbf{p})$$

$$-\lambda\mathbf{B}'_0(\mathbf{p}))\ldots\det(\mathbf{A}'_k(\mathbf{p}) - \lambda\mathbf{B}'_k(\mathbf{p}))\det\mathbf{Q}^{-1}(\mathbf{p})$$

$$= \alpha(\mathbf{p})P^0(\lambda, \mathbf{p})\ldots P^k(\lambda, \mathbf{p}).$$

The polynomials $P^j(\lambda, \mathbf{p})$, $j = 0, \ldots, k$, satisfy the hypotheses of the theorem. The equation $P^i(\lambda, \mathbf{p}) = \overline{P^j(\bar{\lambda}, \mathbf{p})}$ for $\lambda_i = \bar{\lambda}_j$ follows from the properties of the blocks $\mathbf{A}'_i(\mathbf{p}) = \overline{\mathbf{A}'_j(\mathbf{p})}$ and $\mathbf{B}'_i(\mathbf{p}) = \overline{\mathbf{B}'_j(\mathbf{p})}$ [9]. The theorem is proved.

When the coefficients of the polynomial analytically depend on the parameters, Theorem 1 follows from the Weierstrass preparation theorem [10, 11].

Corollary 1. Suppose that $m = M$ and the real parts of all roots of the polynomial $P_0(\lambda)$ are negative ($\mathrm{Re}\lambda_j < 0$, $j = 1, \ldots, k$). Then, the polynomial $P(\lambda, \mathbf{p})$ is stable in a neighborhood of the point $\mathbf{p} = \mathbf{p}_0$.

Corollary 2. Suppose that the real part of a certain root of the polynomial $P_0(\lambda)$ is positive ($\mathrm{Re}\lambda_j > 0$). Then, the polynomial $P(\lambda, \mathbf{p})$ is unstable in a neighborhood of the point $\mathbf{p} = \mathbf{p}_0$.

Corollaries 1 and 2 distinguish the interior points of stability and instability domains of $P(\lambda, \mathbf{p})$. Thus, the presence of infinite, zero, and pure imaginary roots of the polynomial $P_0(\lambda)$ is of interest for the local analysis of stability, provided that the remaining roots have negative real parts. In this case, the point $\mathbf{p} = \mathbf{p}_0$ belongs to the stability-domain boundary.

2. TANGENT CONES TO THE STABILITY DOMAIN

Denote by $b_l(\lambda_j)$, $j = 1, \ldots, M$, the coefficients of the polynomial $P(\lambda + \lambda_j, \mathbf{p})$ that smoothly depend on $\mathbf{p}$; i.e.,

$$\begin{aligned} P(\lambda + \lambda_j, \mathbf{p}) &= b_M(\lambda_j)\lambda^M + \ldots + b_0(\lambda_j) \\ &= a_M(\lambda + \lambda_j)^M + \ldots + a_1(\lambda + \lambda_j) + a_0. \end{aligned} \tag{2.1}$$

Differentiating (2.1) l times with respect to λ and setting $\lambda = 0$, we obtain

$$b_l(\lambda_j) = \frac{1}{l!}\frac{\partial^l P}{\partial \lambda^l}\bigg|_{\lambda = \lambda_j} = \sum_{t=0}^{M-l} C^l_{M-t} a_{M-t} \lambda_j^{M-l-t}, \quad C^l_k = \frac{k!}{l!(k-l)!}. \tag{2.2}$$

For the infinite root $\lambda_0 = \infty$, define

$$b_l(\lambda_0) = a_{M-l}, \quad l = 0, \ldots, M. \tag{2.3}$$

When $\lambda_j \in \mathbb{R}$, we have $b_l(\lambda_j) \in \mathbb{R}$ for all l. Since λ_j is a root of the polynomial $P_0(\lambda)$of multiplicity m_j, then $b_0(\lambda_j) = \ldots = b_{m_j - 1}(\lambda_j) = 0$ and $b_{m_j}(\lambda_j) \neq 0$ for $\mathbf{p} = \mathbf{p}_0$.

Introduce the differential operator

$$\nabla = \left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right),$$

where the derivatives are taken for $\mathbf{p} = \mathbf{p}_0$, and denote by $(\mathbf{a}, \mathbf{b}) = a_1 b_1 + \ldots + a_n b_n$ the scalar product in $\mathbb{R}^n$ (when $\mathbf{a} \in \mathbb{C}^n$, we set $(\mathbf{a}, \mathbf{b}) = (\mathrm{Re}\,\mathbf{a}, \mathbf{b}) + i(\mathrm{Im}\,\mathbf{a}, \mathbf{b})$). With regard to (2.2) and (2.3), the expression for $\nabla b_l(\lambda_j)$ is represented as

$$\nabla b_l(\lambda_j) = \sum_{t=0}^{M-l} C^l_{M-t} \lambda_j^{M-l-t} \nabla a_{M-t}, \quad j \neq 0; \qquad \nabla b_l(\lambda_0) = \nabla a_{M-l}. \tag{2.4}$$

Define the n-dimensional real vectors $\mathbf{f}_l(\lambda_j)$ and $\mathbf{g}_l(\lambda_j)$ corresponding to the roots λ_j as

$$\begin{aligned} &\mathbf{f}_l(\lambda_j) + i\,\mathbf{g}_l(\lambda_j) = \nabla b_l(\lambda_j), \quad l = 0, \ldots, m_j - 3, \\ &\mathbf{f}_{m_j-2}(\lambda_j) + i\,\mathbf{g}_{m_j-2}(\lambda_j) = \nabla b_{m_j-2}(\lambda_j)/b^0_{m_j}(\lambda_j), \\ &\mathbf{f}_{m_j-1}(\lambda_j) = \mathrm{Re}[(b^0_{m_j}(\lambda_j)\nabla b_{m_j-1}(\lambda_j) \\ &\qquad - b^0_{m_j+1}(\lambda_j)\nabla b_{m_j-2}(\lambda_j))/(b^0_{m_j}(\lambda_j))^2], \end{aligned} \tag{2.5}$$

where $b^0_l(\lambda_j)$ is the value of $b_l(\lambda_j)$ for $\mathbf{p} = \mathbf{p}_0$ and i is the imaginary unit. Note that $\mathbf{f}_l(\lambda_j) = \mathbf{f}_l(\lambda_t)$ and $\mathbf{g}_l(\lambda_j) = -\mathbf{g}_l(\lambda_t)$ for $\lambda_j = \overline{\lambda_t}$.

As we noted above, the values of the vector of parameters $\mathbf{p} = \mathbf{p}_0$ for which the polynomial $\mathbf{p}_0(\lambda)$ has infinite, zero, and (or) pure imaginary roots are of interest, provided that the real parts of other roots are negative. Let us enumerate the roots λ_j so that $\lambda_1, \ldots, \lambda_{2s}$ are distinct nonzero pure imaginary roots and, additionally, the roots $\lambda_{j+s} = \overline{\lambda_s}$, $m_{j+s} = m_j$, $j = 1, \ldots, s$, are followed by the zero root $\lambda_{2s+1} = 0$ if it exists. For the remaining roots, we have $\mathrm{Re}\lambda_j < 0$, $j = 2s + 2, \ldots, k$.

Let a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\varepsilon \geq 0$, emanate from the point $\mathbf{p}(0) = \mathbf{p}_0$ in the direction of $\mathbf{e} = d\mathbf{p}/d\varepsilon|_{\varepsilon=0}$. The set of direction vectors $\mathbf{e}$ of the curves $\mathbf{p} = \mathbf{p}(\varepsilon)$ that lie in the stability domain for $\varepsilon > 0$ is called a tangent cone to the stability domain at the point $\mathbf{p}_0$ [5]. The tangent cone is the first-order approximation to the stability domain in the neighborhood of the point considered. Thus, the local analysis of the stability domain reduces to the determination of the stability conditions for the polynomial $P(\lambda, \mathbf{p})$ along the above curve.

Theorem 2. In order that the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, with the direction $\mathbf{e} = d\mathbf{p}/d\varepsilon|_{\varepsilon=0}$ lies in the stability domain for $\varepsilon > 0$, it is necessary that

$$\begin{aligned} &(\mathbf{f}_l(\lambda_j), \mathbf{e}) = (\mathbf{g}_l(\lambda_j), \mathbf{e}) = 0, \quad l = 0, \ldots, m_j - 3, \\ &(\mathbf{f}_{m_j-2}(\lambda_j), \mathbf{e}) \geq 0, \\ &(\mathbf{g}_{m_j-2}(\lambda_j), \mathbf{e}) = 0, \quad (\mathbf{f}_{m_j-1}(\lambda_j), \mathbf{e}) \geq 0 \end{aligned} \tag{2.6}$$

for infinite, zero, and pure imaginary roots λ_j, $j = 0, 1, \ldots, s, 2s + 1$.

Theorem 3. If the system of vectors

$$V_0 \cup V_1 \cup \ldots \cup V_s \cup V_{2s+1}, \tag{2.7}$$

where

$$\begin{aligned} &V_j = \{\mathbf{f}_l(\lambda_j), \mathbf{g}_t(\lambda_j);\ l = 0, \ldots, m_j - 1, \\ &\quad t = 0, \ldots, m_j - 2\}, \quad j = 1, \ldots, s, \\ &V_r = \{\mathbf{f}_l(\lambda_r);\ l = 0, \ldots, m_r - 1\}, \quad r = 0, 2s+1, \end{aligned} \tag{2.8}$$

is linearly independent, then the tangent cone to the stability domain at the point $\mathbf{p} = \mathbf{p}_0$ consists of vectors $\mathbf{e}$ satisfying conditions (2.6) for infinite, zero, and pure imaginary roots λ_j, $j = 0, 1, \ldots, s, 2s + 1$.

Theorem 4. Theorems 2 and 3 remain valid under the replacement of the stability condition $\mathrm{Re}\lambda < 0$ by the condition $\mathrm{Re}\lambda < -\delta_0$ for all roots λ; the latter condition implies the boundedness of the decrement (the degree of stability). In this case, the roots λ_j, $j = 0, \ldots, k$, of the polynomial $P_0(\lambda)$ must be enumerated so that $\lambda_0 = \infty$; $\mathrm{Re}\lambda_l = -\delta_0$, $\mathrm{Im}\lambda_l \neq 0$, $l = 1, \ldots, 2s$ ($\lambda_{r+s} = \overline{\lambda_r}$, $r = 1, \ldots, s$); $\lambda_{2s+1} = -\delta_0$ and $\mathrm{Re}\lambda_t < -\delta_0$, $t = 2s + 2, \ldots, k$.

When the coefficient in (2.6) falls outside the domain of definition, the corresponding quantity is assumed to be zero (for example, when $m_j = 1$, we set $\mathbf{f}_{m_j-2}(\lambda_j) = \mathbf{f}_{-1}(\lambda_j) = 0$). When $j = 0$, $2s + 1$, we have

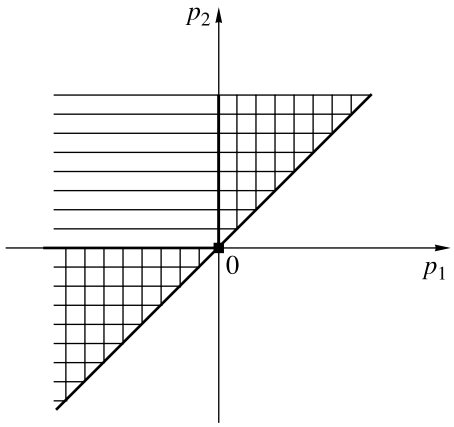

Fig. 1. Example of the stability domain in the degenerate case.

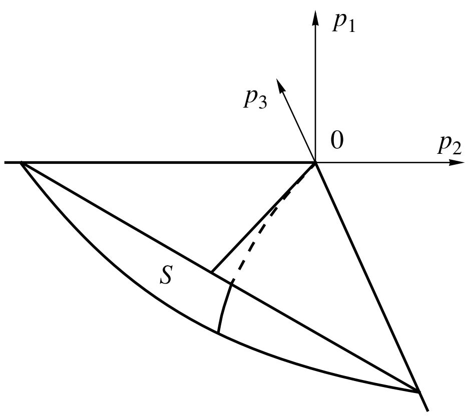

Fig. 2. Stability domain in the neighborhood of the point with a threefold zero root (“break of edge”).

$b_l(\lambda_j) \in \mathbb{R}$, and, hence, conditions (2.6) involving the vectors $\mathbf{g}_l(\lambda_j)$ are fulfilled automatically.

Theorems 2 and 3 allow one to determine, in a first approximation, the stability domain in the neighborhood of the point $\mathbf{p} = \mathbf{p}_0$ from the information available at this point (the first derivatives of the coefficients of the polynomial $P(\lambda, \mathbf{p})$, the parameters calculated for $\mathbf{p} = \mathbf{p}_0$, as well as the values of the coefficients and the roots of the polynomial $P_0(\lambda)$). Theorem 4 provides similar information concerning the domain in which the decrement $\delta = -\mathrm{Re}\lambda$ is greater than a given value δ_0.

Remark 1. In Theorem 3, the condition of linear independence of vectors is essential. For example, the necessary conditions of Theorem 2 for the family of polynomials $P(\lambda, \mathbf{p}) = \lambda^2 + (p_2 - p_1)\lambda + p_1p_2$, $\mathbf{p} = (p_1, p_2)$, in a neighborhood of the point $\mathbf{p}_0 = 0$ are rewritten as $(\mathbf{f}_0(\lambda_1), \mathbf{e}) \geq 0$, $(\mathbf{f}_1(\lambda_1), \mathbf{e}) \geq 0$, where $\lambda_1 = 0$, $m_1 = 2$, and the vectors $\mathbf{f}_0(\lambda_1) = (0, 0)$ and $\mathbf{f}_1(\lambda_1) = (-1, 1)$ are linearly dependent. These conditions single out the half-plane $p_2 - p_1 \geq 0$ on the plane of parameters; in Fig. 1, this half-plane is hatched by horizontal lines. However, the stability domain of $P(\lambda, \mathbf{p})$ is given by $p_2 - p_1 > 0$, $p_1p_2 > 0$ (hatched by vertical lines in Fig. 1). The tangent cone to the stability domain at the point $\mathbf{p}_0 = 0$ is determined by the inequalities $\{\mathbf{e} = (e_1, e_2)$: $e_2 - e_1 \geq 0, e_1 \geq 0\} \cup \{\mathbf{e}$: $e_2 - e_1 \geq 0, e_2 \leq 0\}$.

Remark 2. In order to satisfy the condition of linear independence in Theorem 3, it is necessary that the number of parameters n be no less than the number of vectors in (2.7), $N = m_0 + 2(m_1 + \ldots + m_s) + m_{2s+1} - s$. In the case of general position, when considering arbitrary n-parameter families of polynomials $P(\lambda, \mathbf{p})$, the appearance, for a certain value of the vector of parameters $\mathbf{p} = \mathbf{p}_0$, of the polynomial $P_0(\lambda) = P(\lambda, \mathbf{p})$ of a given type (the type is determined by the multiplicities of the roots $\lambda_0, \lambda_1, \ldots, \lambda_s, \lambda_{2s+1}$) is possible only for $n \geq N$ [5, 6]. Hence, when considering polynomial families of the general form, the condition of linear independence of the system of vectors in Theorem 3 is fulfilled in the case of general position.

Remark 3. The problem of limit stability, i.e., the stability of the polynomial $P(\lambda, \varepsilon) = P_0(\lambda) + \varepsilon P_1(\lambda)$ for small values of the parameter $\varepsilon > 0$, is close to the problem of local analysis of the stability domain. The limit stability can be achieved only in the case when the multiplicities of the roots of $P_0(\lambda)$ lying on the imaginary axis are no greater than two [12, 13]. The situation is different in the multiparameter case: a polynomial $P_0(\lambda)$ with imaginary roots of arbitrary multiplicity may correspond to a point on the boundary of the stability domain. In this case, if there exists a zero or infinite root of multiplicity greater than two or an imaginary nonzero root with multiplicity greater than unity, the stability domain approaches the point $\mathbf{p} = \mathbf{p}_0$ in a narrow tongue; i.e., a tangent cone is degenerate (there is an equality among conditions (2.6)). For example, the stability domain of the polynomial $P(\lambda, \mathbf{p}) = \lambda^3 - p_3\lambda^2 - p_2\lambda - p_1$ is given by $p_1 + p_2p_3 > 0$, $p_j < 0$, $j = 1, 2, 3$ (Fig. 2), while its boundary at the point $\mathbf{p}_0 = 0$ corresponding to a three-fold zero root has a singularity of the “break-of-edge” type [4, 8].

Thus, the presence of several parameters results in new effects. This is associated with the fact that the analysis of the cases of general position is the most informative and important [4]. Therefore, when there are multiple roots, one should carry out the stability analysis in the space of parameters of sufficiently large dimension.

3. PROOF OF THEOREMS 2–4

The idea of the proof consists in the factorization of the polynomial $P(\lambda, \mathbf{p})$ into multipliers $P^j(\lambda, \mathbf{p})$ using Theorem 1. After that, for each multiplier, we find the conditions of stability along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$; then, we rewrite these conditions in terms of the vectors $\mathbf{f}_l(\lambda_j)$

and $\mathbf{g}_l(\lambda_j)$. These steps of the proof are supported by the following three lemmas.

Lemma 1. Suppose that the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, with the direction $\mathbf{e}$ lies, for $\varepsilon > 0$, in the stability domain of the polynomial $P^j(\lambda, \mathbf{p})$ (1.4), $j \in \{0, 1, \ldots, s, 2s + 1\}$. Then,

$$(\nabla\beta_0(\lambda_j), \mathbf{e}) = \ldots = (\nabla\beta_{m_j-3}(\lambda_j), \mathbf{e}) = 0,$$
$$(\mathrm{Re}\,\nabla\beta_{m_j-2}(\lambda_j), \mathbf{e}) \ge 0, \ (\mathrm{Im}\,\nabla\beta_{m_j-2}(\lambda_j), \mathbf{e}) = 0, \tag{3.1}$$
$$(\mathrm{Re}\,\nabla\beta_{m_j-1}(\lambda_j), \mathbf{e}) \ge 0.$$

Proof. By the substitution $\mu = \lambda - \lambda_j$ (when $j = 0$, by the substitution $\mu = 1/\lambda$ and multiplication by μ^{m_0}), we reduce the polynomial $P^j(\lambda, \mathbf{p})$ to the form

$$\mu^{m_j} + \beta_{m_j-1}(\lambda_j)\mu^{m_j-1} + \ldots + \beta_0(\lambda_j). \tag{3.2}$$

When $j \in \{0, 1, \ldots, s, 2s + 1\}$, the stability of polynomial (3.2) is equivalent to the stability of $P^j(\lambda, \mathbf{p})$. Then, conditions (3.1) follow from the expression for the tangent cone to the stability domain of polynomial (3.2) that was obtained in [14]. The lemma is proved.

Lemma 2. Let the system of vectors

$$V'_0 \cup V'_1 \cup \ldots \cup V'_s \cup V'_{2s+1}, \tag{3.3}$$

where

$$V'_j = \{\mathrm{Re}\,\nabla\beta_l(\lambda_j), \mathrm{Im}\,\nabla\beta_t(\lambda_j);\ l = 0, \ldots, m_j - 1,$$
$$t = 0, \ldots, m_j - 2\}, \quad j = 1, \ldots, s, \tag{3.4}$$
$$V'_r = \{\mathrm{Re}\,\nabla\beta_l(\lambda_r);\ l = 0, \ldots, m_r - 1\}, \ r = 0, 2s + 1,$$

be linearly independent. Then, for any vector $\mathbf{e}$ that satisfies conditions (3.1) for $j = 0, 1, \ldots, s, 2s + 1$, there exists a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, $d\mathbf{p}/d\varepsilon = \mathbf{e}$, such that the polynomials $P^0(\lambda, \mathbf{p}(\varepsilon))$, $P^1(\lambda, \mathbf{p}(\varepsilon))$, …, $P^s(\lambda, \mathbf{p}(\varepsilon))$, $P^{2s+1}(\lambda, \mathbf{p}(\varepsilon))$ are stable for $\varepsilon > 0$.

Proof. Consider the polynomials

$$P^0 = (1 + \mu^2\lambda)^{m_0-2}(1 + (b_1^0\mu + \mu^2)\lambda + (b_2^0\mu + \mu^2)\lambda^2),$$
$$P^j = (\tilde{\lambda} + \mu^2)^{m_j-2} \times (\tilde{\lambda}^2 + (ia_j + b_1^j\mu + \mu^2)\tilde{\lambda} + b_2^j\mu + \mu^2),$$
$$\tilde{\lambda} = \lambda - \lambda_j, \quad j = 1, \ldots, s, \tag{3.5}$$
$$P^{2s+1} = (\lambda + \mu^2)^{m_{2s+1}-2} \times (\lambda^2 + (b_1^{2s+1}\mu + \mu^2)\lambda + b_2^{2s+1}\mu + \mu^2).$$

Polynomials (3.5) are stable for arbitrary real a_j, $b_1^t \ge 0$, $b_2^t \ge 0$, and $\mu > 0$. Let us fix the variables $b_1^t \ge 0$, $b_2^t \ge 0$, $t = 0, 1, \ldots, s, 2s + 1$, and compare the expressions for polynomials (3.5) and (1.4). We obtain a system of $m' = m_0 + 2(m_1 + \ldots + m_s) + m_{2s+1}$ equations for the coefficients of polynomials in $\mathbf{p}$, $\mathbf{a} = (a_1, \ldots, a_s)$, and μ:

$$B^1_{jl}(\mathbf{p}, \mathbf{a}, \mu) = \mathrm{Re}(\beta_l(\lambda_j)(\mathbf{p}) - \beta_{jl}(\mathbf{a}, \mu)) = 0,$$
$$j = 0, 1, \ldots, s, 2s + 1, \quad l = 0, \ldots, m_j - 1; \tag{3.6}$$
$$B^2_{rt}(\mathbf{p}, \mathbf{a}, \mu) = \mathrm{Im}(\beta_t(\lambda_r)(\mathbf{p}) - \beta_{rt}(\mathbf{a}, \mu)) = 0,$$
$$r = 1, \ldots, s, \quad t = 0, \ldots, m_t - 1.$$

In view of (1.4), (3.5), and (3.6), the differentials of the functions B^1_{jl} and B^2_{rt} at the point $\mathbf{p} = \mathbf{p}_0$, $\mathbf{a} = 0$, $\mu = 0$ are expressed as

$$dB^1_{jl} = (\mathrm{Re}\,\nabla\beta_l(\lambda_j), d\mathbf{p}) - (\delta_{(m_j-1)l}b_1^j + \delta_{(m_j-2)l}b_2^j)d\mu = 0, \tag{3.7}$$
$$dB^2_{rt} = (\mathrm{Im}\,\nabla\beta_t(\lambda_r), d\mathbf{p}) - \delta_{(m_j-1)t}da_r = 0,$$

where δ_{jl} is the Kronecker delta. Using the linear independence of vectors (3.3) and a specific form of the terms containing da_t in (3.7), we can show that the Jacobian matrix of system (3.6) has the maximum rank m' at the point $\mathbf{p} = \mathbf{p}_0$, $\mathbf{a} = 0$, $\mu = 0$. Hence, equations (3.6) in a neighborhood of the point $\mathbf{p} = \mathbf{p}_0$, $\mathbf{a} = 0$, $\mu = 0$ determine a smooth surface (of codimension r) whose tangent plane is given by equations (3.7).

Consider an arbitrary tangent vector of the form $d\mathbf{p} = \mathbf{e}d\varepsilon$, $d\mathbf{a} = \mathbf{d}d\varepsilon$, $d\mu = d\varepsilon$ that satisfies (3.7) and a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{a} = \mathbf{a}(\varepsilon)$, $\mu = \mu(\varepsilon)$ in the space $(\mathbf{p}, \mathbf{a}, \mu)$ that is emanated in the direction of the above vector along the surface (3.6). By the construction, the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ lies in the stability domain of the polynomials $P^j(\lambda, \mathbf{p})$ (3.5) for $0 > \varepsilon > \varepsilon'$, where $(0, \varepsilon')$ is a segment of positiveness of the function $\mu = \mu(\varepsilon)$. Note that, if we make the substitutions $d\mathbf{p} = \mathbf{e}d\varepsilon$, $d\mathbf{a} = \mathbf{d}d\varepsilon$, and $d\mu = d\varepsilon$ and take into account the arbitrariness of $\mathbf{d}$, $b_1^j \ge 0$, and $b_2^j \ge 0$, we obtain that equations (3.7) are equivalent to conditions (3.1). Thus, we constructed the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\varepsilon > 0$, that lies in the stability domain of the polynomials $P^j(\lambda, \mathbf{p})$ and has an arbitrary direction $\mathbf{e}$ that satisfies conditions (3.1) for $j = 0, 1, \ldots, s, 2s + 1$. The lemma is proved.

Lemma 3. Let $P(\lambda, \mathbf{p}) = Q(\lambda, \mathbf{p})R(\lambda, \mathbf{p})$, where $Q(\lambda, \mathbf{p})$ and $R(\lambda, \mathbf{p})$ are polynomials of degree q and r $(M = q + r)$ with complex coefficients that smoothly depend on $\mathbf{p}$:

$$Q(\lambda, \mathbf{p}) = c_q(\lambda - \lambda_j)^q + \ldots + c_0,$$
$$R(\lambda, \mathbf{p}) = d_r(\lambda - \lambda_j)^r + \ldots + d_0, \quad j \ne 0.$$

For $j = 0$, we set $Q(\lambda, \mathbf{p}) = c_0\lambda^q + \ldots + c_q$, and $R(\lambda, \mathbf{p}) = d_0\lambda^r + \ldots + d_r$. Let $d_0(\mathbf{p}_0) \ne 0$ and $Q(\lambda, \mathbf{p}_0)$ has a root λ_j

of multiplicity m_j ($c_0(\mathbf{p}_0) = \ldots = c_{m_j-1}(\mathbf{p}_0) = 0$, $c_{m_j}(\mathbf{p}_0) \neq 0$). Then, the system of vectors V_j (2.8) calculated for $P(\lambda, \mathbf{p})$ is a nonsingular linear combination of analogous vectors calculated for the polynomial $Q(\lambda, \mathbf{p})$ (after substitution of c_l for $b_l(\lambda_j)$) in (2.5)). In this case, conditions (2.6) written for $P(\lambda, \mathbf{p})$ and $Q(\lambda, \mathbf{p})$ are equivalent.

Proof. Multiplying $Q(\lambda, \mathbf{p})$ and $R(\lambda, \mathbf{p})$ term by term and taking into account (2.1), we obtain the following relations from the equation $P(\lambda, \mathbf{p}) = Q(\lambda, \mathbf{p})\, R(\lambda, \mathbf{p})$:

$$b_l(\lambda_j) = \sum_{t=0}^{l} d_{l-t} c_t,$$
$$\nabla b_l(\lambda_j) = \sum_{t=0}^{l} (d^0_{l-t} \nabla c_t + c^0_t \nabla d_{l-t}), \tag{3.8}$$

where $d^0_t = d_t(\mathbf{p}_0)$ and $c^0_t = c_t(\mathbf{p}_0)$. By the hypothesis, $c^0_0 = \ldots = c^0_{m_j-1} = 0$. Hence,

$$b^0_{m_j}(\lambda_j) = c^0_{m_j} d^0_0,$$
$$b^0_{m_j+1}(\lambda_j) = c^0_{m_j+1} d^0_0 + c^0_{m_j} d^0_1, \tag{3.9}$$
$$\nabla b_l(\lambda_j) = \sum_{t=0}^{l} d^0_{l-t} \nabla c_t, \quad l = 0, \ldots, m_j - 1.$$

Substituting (3.9) into the expressions for the vectors $\mathbf{f}_l(\lambda_j)$ and $\mathbf{g}_l(\lambda_j)$ (2.5), we obtain

$$\mathbf{f}_l(\lambda_j) + i\mathbf{g}_l(\lambda_j) = \nabla b_l(\lambda_j) = \sum_{t=0}^{l} d^0_{l-t} \nabla c_t$$
$$= \sum_{t=0}^{l} d^0_{l-t} (\mathbf{f}'_t(\lambda_j) + i\,\mathbf{g}'_t(\lambda_j)), \quad l = 0, \ldots, m_j - 3,$$
$$\mathbf{f}_{m_j-2}(\lambda_j) + i\,\mathbf{g}_{m_j-2}(\lambda_j) = \nabla b_{m_j-2}(\lambda_j)/b^0_{m_j}(\lambda_j)$$
$$= \left[\nabla c_{m_j-2} + \sum_{t=0}^{m_j-3} (d^0_{m_j-2-t}/d^0_0) \nabla c_t\right] / c^0_{m_j}$$
$$= \mathbf{f}'_{m_j-2}(\lambda_j) + i\,\mathbf{g}'_{m_j-2}(\lambda_j)$$
$$+ \sum_{t=0}^{m_j-3} \frac{d^0_{m_j-2-t}}{d^0_0 c^0_{m_j}} (\mathbf{f}'_t(\lambda_j) + i\,\mathbf{g}'_t(\lambda_j)), \tag{3.10}$$
$$\mathbf{f}_{m_j-1}(\lambda_j) = \ldots = \mathbf{f}'_{m_j-1}(\lambda_j)$$
$$+ \mathrm{Re}\left[\sum_{t=0}^{m_j-3} \alpha_t (\mathbf{f}'_t(\lambda_j) + i\,\mathbf{g}'_t(\lambda_j))\right],$$
$$\alpha_t = [c^0_{m_j} d^0_0 d^0_{m_j-1-t}$$
$$- (c^0_{m_j+1} d^0_0 + c^0_{m_j} d^0_1) d^0_{m_j-2-t}] / (c^0_{m_j} d^0_0)^2,$$

where $\mathbf{f}'_l(\lambda_j)$ and $\mathbf{g}'_l(\lambda_j)$ denote the vectors (2.5) calculated for the polynomial $Q(\lambda, \mathbf{p})$.

The assertions of the lemma immediately follow from (3.10) in view of the relations $c^0_{m_j} \neq 0$ and $d^0_0 \neq 0$. The lemma is proved.

Now, we pass to the proof of Theorems 2–4. By Theorem 1, we represent the polynomial $P(\lambda, \mathbf{p})$ as a product of the nonzero coefficient $\alpha(\mathbf{p})$ and the polynomials $P^j(\lambda, \mathbf{p})$ corresponding to the roots λ_j, $j = 0, \ldots, k$. The stability of $P(\lambda, \mathbf{p})$ is equivalent to the stability of all multipliers $P^j(\lambda, \mathbf{p})$. Since the roots of the polynomials are continuous functions of parameters, the multipliers $P^j(\lambda, \mathbf{p})$ corresponding to the roots with negative real parts λ_j, $j = 2s + 2, \ldots, k$, are stable in a neighborhood of the point $\mathbf{p} = \mathbf{p}_0$. The stability of the polynomial $P^{j+s}(\lambda, \mathbf{p})$ is equivalent to the stability of $P^j(\lambda, \mathbf{p})$, $j = 1, \ldots, s$, since $P^{j+s}(\lambda, \mathbf{p}) = \overline{P^j(\bar{\lambda}, \mathbf{p})}$ by Theorem 1. Hence, the stability of $P(\lambda, \mathbf{p})$ is determined by the simultaneous stability of the polynomials $P^j(\lambda, \mathbf{p})$, $j = 0, 1, \ldots, s, 2s + 1$.

By Lemma 3, conditions (2.6) written for $P(\lambda, \mathbf{p})$ and $Q(\lambda, \mathbf{p}) = P^j(\lambda, \mathbf{p})$, $j \in \{0, 1, \ldots, s, 2s + 1\}$, are equivalent. Taking into account $\beta^0_{m_j}(\lambda_j) = 1$ and $\beta^0_{m_j+1}(\lambda_j) = 0$, we can rewrite relations (2.6) for $P^j(\lambda, \mathbf{p})$ in the form (3.1). By Lemma 1, conditions (3.1) are necessary for the stability of $P^j(\lambda, \mathbf{p})$ along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\varepsilon > 0$. Theorem 2 is proved.

By Lemma 3, the system of vectors V_j (2.8) calculated for $P(\lambda, \mathbf{p})$ is obtained as a result of nonsingular linear combination of analogous vectors calculated for $Q(\lambda, \mathbf{p}) = P^j(\lambda, \mathbf{p})$, which constitute the system V'_j (3.4) with regard to $\beta^0_{m_j}(\lambda_j) = 1$ and $\beta^0_{m_j+1}(\lambda_j) = 0$. Hence, the linear independence of vectors (2.7) implies the linear independence of vectors (3.3). Taking into account the equivalence between relations (2.6) and (3.1) and using Lemma 2, we obtain that, under the hypotheses of Theorem 3, for any direction $\mathbf{e}$ satisfying the necessary conditions of stability (2.6), there exists a curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$, $d\mathbf{p}/d\varepsilon = \mathbf{e}$ that lies in the stability domain for $\varepsilon > 0$. Theorem 3 is proved.

The condition $\mathrm{Re}\lambda < -\delta_0$ for the decrement for $P(\lambda, \mathbf{p})$ is equivalent to the stability condition of the polynomial $P(\lambda - \delta_0, \mathbf{p})$. In this case, the roots of $P(\lambda - \delta_0, \mathbf{p})$ are equal to $\lambda'_j = \lambda_j + \delta_0$ and $\mathrm{Re}\lambda'_j = \mathrm{Re}\lambda_j + \delta_0 = 0$ for $j = 1, \ldots, 2s + 1$. Hence, Theorems 2 and 3 can be applied to the analysis of a domain with bounded decrement, provided that $P(\lambda, \mathbf{p})$ is replaced by $P'(\lambda, \mathbf{p}) =$

$P(\lambda - \delta_0, \mathbf{p})$ and λ_j, by λ'_j. It remains to be noted that, in this case, the quantities $b_l(\lambda_j)$ (2.1)–(2.3) are defined as the coefficients of the polynomial $P'(\lambda + \lambda'_j, \mathbf{p}) = P(\lambda + \lambda'_j - \delta_0, \mathbf{p}) = P(\lambda + \lambda_j, \mathbf{p})$. Theorem 4 is proved.

4. EXAMPLES

Let us illustrate the efficiency of the obtained results by several examples from automatic control theory. Consider a closed-loop system consisting of an integrating, oscillatory, and two aperiodic elements connected as shown in Fig. 3. The characteristic equation of such a system is given by [3]

$$\lambda(T_0\lambda^2 + T_1\lambda + 1)(T_2\lambda + 1)(T\lambda + 1) + kk_1k_2 = 0.$$

Under the assumption that the parameters of the aperiodic elements are prescribed and equal to $T = T_2 = 1$, $k = 2$, $k_2 = 1$, let us analyze the stability of the system as a function of the three parameters T_0, T_1, and k_1 of the periodic element:

$$P(\lambda, \mathbf{p}) = T_0\lambda^5 + (2T_0 + T_1)\lambda^4$$
$$+ (T_0 + 2T_1 + 1)\lambda^3 + (T_1 + 2)\lambda^2 + \lambda + 2k_1.$$

In the space of parameters $\mathbf{p} = (T_0, T_1, k_1)$, consider the point $\mathbf{p}_0 = (0, 0, 1)$, which corresponds to the system in the absence of the oscillatory element (the corresponding transfer function is reduced to unity). For $\mathbf{p} = \mathbf{p}_0$, we obtain

$$P_0(\lambda) = \lambda^3 + 2\lambda^2 + \lambda + 2.$$

The polynomial $P_0(\lambda)$ has the simple roots $\lambda_1 = i$, $\lambda_2 = -i$, and $\lambda_3 = -2$, $(m_1 = m_2 = m_3 = 1)$. In addition, there is the infinite root $\lambda_0 = \infty$ of multiplicity $m_0 = 2$ (due to a decrease in the degree of the polynomial). The vectors $\mathbf{f}_0(\lambda_0)$, $\mathbf{f}_1(\lambda_0)$, and $\mathbf{f}_0(\lambda_1)$ calculated according to (2.2)–(2.5) are as follows:

$$\mathbf{f}_0(\lambda_0) = (1, 0, 0), \quad \mathbf{f}_1(\lambda_0) = (0, 1, 0),$$
$$\mathbf{f}_0(\lambda_1) = (-0.2, -0.4, -0.2).$$

The system of vectors $\{\mathbf{f}_0(\lambda_0), \mathbf{f}_1(\lambda_0), \mathbf{f}_0(\lambda_1)\}$ (2.7) is linearly independent. Hence, by Theorem 3, the tangent cone to the stability domain is determined by conditions (2.6) written for the roots λ_0 and λ_1:

$$K = \{\mathbf{e}: (\mathbf{f}_0(\lambda_0), \mathbf{e}) \geq 0, \quad (\mathbf{f}_1(\lambda_0), \mathbf{e}) \geq 0, (\mathbf{f}_0(\lambda_1), \mathbf{e}) \geq 0\}. \tag{4.1}$$

Inequalities (4.1) determine, in the space of parameters $\mathbf{p} = (T_0, T_1, k_1)$, a trihedral angle (the intersection of three half-spaces) (Fig. 4a), which represents an approximation of the stability domain in a neighborhood of the point $\mathbf{p}_0 = (0, 0, 1)$. For comparison, Fig. 4b shows the boundary of the stability domain calculated numerically. The numerical results confirm that there is a trihedral angle (4.1) at the point $\mathbf{p}_0 = (0, 0, 1)$.

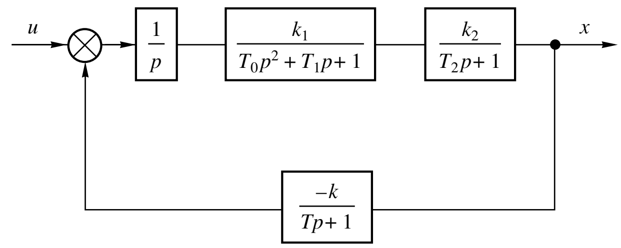

Fig. 3. Block diagram of the automatic control system.

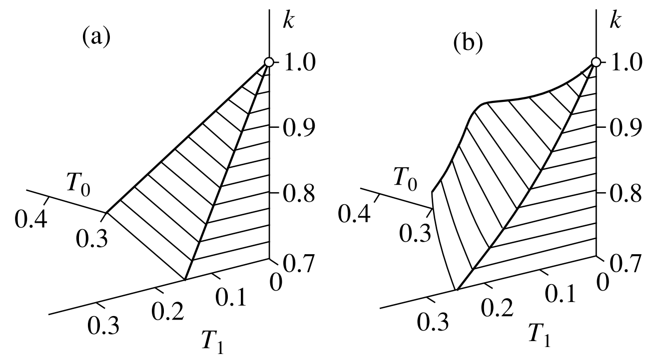

Fig. 4. (a) Approximation of the stability domain and (b) the stability domain determined numerically.

In many problems of automatic control, it is required to determine the values of the parameters for which the damping in the system exceeds a certain prescribed value, i.e., $\operatorname{Re}\lambda < -\delta_0$ for all roots λ. Denote by $S(\delta_0)$ the set of parameters satisfying this condition. Consider a system consisting of a single one-capacitance object connected with a direct digital controller. This system is described by the following characteristic polynomial [2]:

$$P(\lambda, \mathbf{p}) = (T_1\lambda + 1)(T'^2\lambda^2 + T_k\lambda + 1) + k_1k_2.$$

Let us fix the parameters $T_1 = 1$, $T'^2 = 0.5$, and $k_1 = 1$ and analyze the domain $S(1)$ in the space of parameters $\mathbf{p} = (T_k, k_2)$. Consider the point $\mathbf{p}_0 = (3/2, 0)$ at which the polynomial $P_0(\lambda) = P(\lambda, \mathbf{p}_0)$ has the roots $\lambda_1 = -1$ and $\lambda_2 = -2$ of multiplicities $m_1 = 2$ and $m_2 = 1$, respectively. Let us calculate the vectors $\mathbf{f}_0(\lambda_1)$ and $\mathbf{f}_1(\lambda_1)$ by formulas (2.2)–(2.5):

$$\mathbf{f}_0(\lambda_1) = (0, 2), \quad \mathbf{f}_1(\lambda_1) = (-2, -2).$$

The vectors $\mathbf{f}_0(\lambda_1)$ and $\mathbf{f}_1(\lambda_1)$ are linearly independent. Hence, by Theorem 4, the tangent cone to the domain $S(1)$ at the point $\mathbf{p}_0 = (3/2, 0)$ is given by

$$K = \{\mathbf{e}: (\mathbf{f}_0(\lambda_1), \mathbf{e}) \geq 0, (\mathbf{f}_1(\lambda_1), \mathbf{e}) \geq 0\}. \tag{4.2}$$

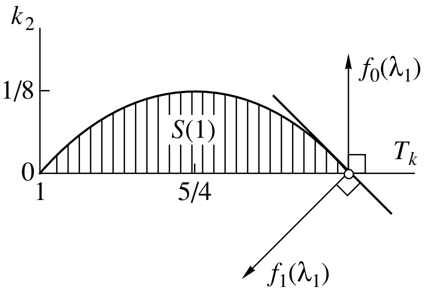

Fig. 5. Domain $S(1)$ with bounded decrement ($\mathrm{Re}\,\lambda < -1$) for the automatic control system.

Taking into account that $\mathbf{p} = \mathbf{p}_0 + \mathbf{e}\varepsilon + o(\varepsilon)$, we can rewrite expression (4.2) as

$$\begin{cases} 2k_2 + o(\|\mathbf{p} - \mathbf{p}_0\|) \geq 0, \\ -2(T_k - 3/2) - 2k_2 + o(\|\mathbf{p} - \mathbf{p}_0\|) \geq 0. \end{cases} \quad (4.3)$$

In the case considered, the domain $S(1)$ can be determined analytically (by applying the Routh–Hurwitz condition to the polynomial $P(\lambda - 1, \mathbf{p})$):

$$S(1) = \{\mathbf{p} = (T_k, k_2)\colon 0 < k_2 < -2T_k^2 + 5T_k - 3\},$$

which confirms the obtained approximation (4.3) (Fig. 5).

Note that, for determining the approximations to the stability domain or to the domain $S(\delta_0)$ in the neighborhood of the point $\mathbf{p} = \mathbf{p}_0$, we needed only the information about the system at this point (the values of the roots, the coefficients of the polynomial, and the first-order derivatives of this polynomial with respect to parameters, calculated for $\mathbf{p} = \mathbf{p}_0$).

CONCLUSION

In this paper, an efficient method is developed for a local quantitative analysis of the stability domain of multiparameter families of polynomials of arbitrary degree. We studied the cases when the leading coefficients of the polynomial vanish and when the polynomial has imaginary roots of arbitrary multiplicity. Similar results are obtained for the domains with a bounded degree of stability. The approximations of the stability domain determined by the proposed method can be used in various problems of stabilization of mechanical and control systems, as well as in optimization problems with stability criterion.

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