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MECHANICS

Parametric Resonance in Systems with Weak Dissipation

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We consider a linear oscillatory system with many degrees of freedom, whose periodic coefficients are functions of three independent parameters: the frequency and amplitude of a periodic excitation and the parameter of dissipative forces. The last two quantities are assumed to be small. We analyze the instability of the trivial solution (parametric resonance). For an arbitrary periodic-excitation matrix and a positive-definite matrix of the dissipative forces, we derive general expressions for the regions of the main and combination resonances. We study two particular cases for the periodic-excitation matrix that are often encountered in applications: a symmetric matrix and a stationary matrix multiplied by a scalar periodic function. It is shown that in both cases, the resonance regions represent, to a first approximation, cones in the three-dimensional space of the parameters. The relationships obtained allow us to analyze the influence of both the natural frequencies and the resonance number on the instability region. We employ the method of examining parametric-resonance regions which is based on analyzing the behavior of multipliers and uses formulas for the derivatives of the monodromy matrix with respect to the parameters [1, 2]. As an example, the problem on the dynamical stability of the plane bending of a beam loaded by periodic moments is considered.

In referring to previous studies, we should mention book [3], wherein systems close to Hamiltonian systems were studied, as well as papers [4–7], in which systems were transformed to normal coordinates of a conservative system (i.e., the systems for which a transition matrix is required). This paper differs from the previous studies in the statement of the problem and in both the method of analyzing the problem and the results obtained.

1. We consider a linear oscillatory system with periodic coefficients,

$$\mathbf{M}\ddot{\mathbf{y}} + \gamma\mathbf{D}\dot{\mathbf{y}} + (\mathbf{C} + \delta\mathbf{B}(\Omega t))\mathbf{y} = 0. \tag{1}$$

Here, $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{C}$ are the symmetric positive-definite $(m \times m)$ matrices of mass, damping, and potential forces, respectively; $\mathbf{B}(\tau)$ is a piecewise continuous 2π-periodic matrix of a parametric excitation; $\mathbf{y} = (y_1, \ldots, y_m)^{\mathrm{T}}$ is the vector of the generalized coordinates; and the point stands for the derivative with respect to time.

We now analyze the stability of the trivial solution $\mathbf{y} \equiv 0$ to system (1) as a function of the vector of three parameters $\mathbf{p} = (\gamma, \delta, \Omega)$ whose components describe the amplitude of dissipative forces and the amplitude and frequency of a periodic excitation, respectively. We assume that the quantities γ and $|\delta|$ are small, i.e., that system (1) is close to an autonomous conservative system. The evident restrictions $\gamma > 0$ and $\Omega > 0$ are imposed on the parameters γ and Ω.

We rewrite (1) as a set of equations of the first order:

$$\dot{\mathbf{x}} = \mathbf{A}(\Omega t)\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{pmatrix}. \tag{2}$$

The $(2m \times 2m)$ matrix $\mathbf{A}(\Omega t)$ is a real-valued periodic function with period $T = \dfrac{2\pi}{\Omega}$. The $(2m \times 2m)$ matrix $\mathbf{X}(t)$ satisfying the relationships

$$\dot{\mathbf{X}} = \mathbf{A}(\Omega t)\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I} \tag{3}$$

is referred to as a matriciant of system (2), with $\mathbf{I}$ being the unit matrix. The value of the matriciant at $t = T$ is referred to as a monodromy matrix $\mathbf{F} = \mathbf{X}(T)$ [3]. According to the known theorem on the parameter dependence of solutions to differential equations, the monodromy matrix is a smooth function of the parameter vector $\mathbf{p}$. The eigenvalues ρ of the matrix $\mathbf{F}$ are called multipliers. System (2) is asymptotically stable if all its multipliers are located inside the unit circle on the complex plane, i.e., $|\rho| < 1$. If at least one multiplier is outside of the unit circle, i.e., $|\rho| > 1$, then the system becomes unstable [3].

If $\gamma = \delta = 0$, system (1) is conservative. Seeking its solutions in the form $\mathbf{y} = \mathbf{u}\exp(i\omega t)$, we arrive at the eigenvalue problem

$$\mathbf{C}\mathbf{u} = \omega^2\mathbf{M}\mathbf{u}, \quad \mathbf{u}^{\mathrm{T}}\mathbf{M}\mathbf{u} = 1, \tag{4}$$

where the second equality is a normalization condition. The real-valued natural frequencies ω and vibration

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modes $\mathbf{u}$ are determined from these equations. We assume that all the frequencies $0 < \omega_1 < \omega_2 < \ldots < \omega_m$ are different and denote the corresponding eigenvectors by $\mathbf{u}_j$, with $j = 1, \ldots, m$.

When $\gamma = \delta = 0$, the multipliers of the monodromy matrix $\mathbf{F}_0$ are given by [3]

$$\rho_j, \bar{\rho}_j = \exp(\pm i\omega_j T) = \exp\left(\pm i\frac{2\pi\omega_j}{\Omega}\right), \quad j = 1, \ldots, m. \tag{5}$$

Since all the multipliers ρ_j (5) are at the unit circle $|\rho| = 1$, the stability of system (1) for nonzero γ and δ is determined by perturbations of all the multipliers. In general, all ρ_j are different. Repeated multipliers appear for the following critical values of the frequency Ω:

$$\Omega = \frac{2\omega_j}{k}, \quad j = 1, \ldots, m, \quad k = 1, 2, \ldots; \tag{6}$$

$$\Omega = \frac{\omega_j \pm \omega_l}{k}, \quad j, l = 1, \ldots, m, \quad j > l, \quad k = 1, 2, \ldots \tag{7}$$

Equalities (6) and (7) define, respectively, the main (simple) and combination resonances and correspond to the double multipliers $\rho = (-1)^k$ and $\rho = \exp(i\omega_j T)$. Multipliers of higher order originate only if the quantities $\omega_j \pm \omega_l$ are linked by rational relationships. Here, j, $l = 1, \ldots, m$; $j \geq l$; and $j \neq l$ in the case of $\omega_j - \omega_l$. These cases are nongeneric and will not be considered in this paper.

2. We assume that, for $\gamma = \delta = 0$ and a certain $\Omega = \Omega_0$, all the multipliers of the monodromy matrix $\mathbf{F}_0$ are different. Using the formulas for the derivatives of simple multipliers with respect to parameters [1, 2] and taking (4) into account, we obtain, to a first approximation, the following expression for the modulus of the multiplier of system (2):

$$|\rho_j(\mathbf{p})| = 1 - \frac{\pi\mathbf{u}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_j}{\Omega_0}\gamma + o(\|\mathbf{p} - \mathbf{p}_0\|), \tag{8}$$

where $\mathbf{p}_0 = (0, 0, \Omega_0)$. It follows from the assumption on the positive definiteness of the dissipative matrix $\mathbf{D}$ that the coefficient of γ in (8) is negative. Therefore, the introduction of small dissipative forces results in the displacement of all the simple multipliers into the interior of the unit circle for small $|\delta|$ and $|\Omega - \Omega_0|$. This implies that small dissipative forces stabilize system (1), which is subjected to weak parametric excitations for noncritical values of the frequency Ω.

3. The instability (parametric resonance) can originate at frequencies Ω close to critical values (6) and (7). Under these conditions, the double multipliers appear at the unit circle. Let the frequency $\Omega = \Omega_0$ of a parametric excitation satisfy the relationship

$$\omega_j + \omega_l = k\Omega_0 \tag{9}$$

for certain frequencies ω_j and ω_l of the conservative system and for a certain natural number k. It is worth noting that condition (9) involves both the case of main resonance (6) for $j = l$ and the case of the summed combination resonance (7) for $j > l$.

Condition (9) implies that two multipliers coincide:

$$\rho_j = \bar{\rho}_l = \exp(i\omega_j T_0), \quad T_0 = \frac{2\pi}{\Omega_0}.$$

Denoting $\rho_0 = \rho_j = \bar{\rho}_l$, we obtain that $\rho_0 = (-1)^k$ for $j = l$ (main resonance), while for $j > l$, ρ_0 is a complex-valued multiplier (combination resonance). The double multiplier ρ_0 is semisimple because two linearly independent eigenvectors correspond to it. Employing the theory of perturbations for multiple eigenvalues [8] and the formulas for the derivatives of the monodromy matrix with respect to the parameters [1, 2], we find the equation for the stability region in the first approximation:

$$\gamma^2(\eta_j + \eta_l)^2\left[\eta_j\eta_l\gamma^2 - \xi_1\delta^2 + k^2\left(\Delta\Omega + \frac{\sigma_+\delta}{k}\right)^2\right] - \left[\xi_2\delta^2 + k(\eta_j - \eta_l)\left(\Delta\Omega + \frac{\sigma_+\delta}{k}\right)\gamma\right]^2 > 0. \tag{10}$$

Here, $\Delta\Omega = \Omega - \Omega_0$, and the coefficients η_j, η_l, σ_+, ξ_1, and ξ_2 are real quantities determined by the relationships

$$\eta_j = \mathbf{u}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_j, \quad \eta_l = \mathbf{u}_l^{\mathrm{T}}\mathbf{D}\mathbf{u}_l, \quad \sigma_+ = -\frac{\omega_j c_0^{(ll)} + \omega_l c_0^{(jj)}}{2\omega_j\omega_l},$$
$$\xi_1 + i\xi_2 = \frac{c_{-k}^{(jl)} c_k^{(lj)}}{\omega_j\omega_l}, \quad c_k^{(lj)} = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{u}_l^{\mathrm{T}}\mathbf{B}(\tau)\mathbf{u}_j e^{ik\tau} d\tau. \tag{11}$$

It is worth noting that the constants η_j and η_l are positive due to the assumption on the positive definiteness of the dissipative matrix $\mathbf{D}$. Inequality (10) defines the form of the stability region in the three-dimensional space of the parameters $\mathbf{p} = (\gamma, \delta, \Omega)$.

We now consider the critical frequency Ω_0 satisfying the condition

$$\omega_j - \omega_l = k\Omega_0, \quad j > l \tag{12}$$

for a certain natural number k (difference combination resonance). In this case, there is a semisimple double multiplier $\rho_0 = \rho_j - \rho_l$ for $\gamma = \delta = 0$. Then, the first

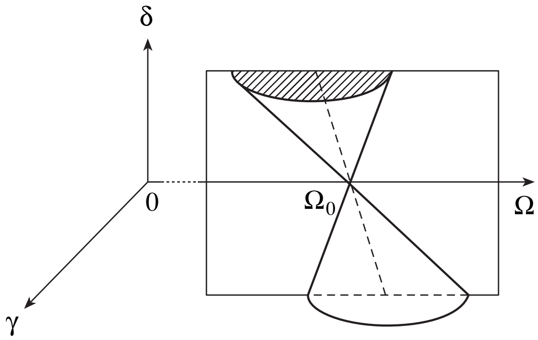

Fig. 1.

approximation for the stability region in the neighborhood of the point $\mathbf{p}_0 = (0, 0, \Omega_0)$ takes the form

$$\gamma^2(\eta_j+\eta_l)^2\left[\eta_j\eta_l\gamma^2+\xi_1\delta^2+k^2\left(\Delta\Omega+\frac{\sigma_-\delta}{k}\right)^2\right] - \left[\xi_2\delta^2-k(\eta_j-\eta_l)\left(\Delta\Omega+\frac{\sigma_-\delta}{k}\right)\gamma\right]^2>0. \quad (13)$$

Here, the real coefficients η_j, η_l, ξ_1, and ξ_2 are determined by relationships (11) and the real constant σ_- is given by the equality $\sigma_- = \dfrac{\omega_j c_0^{(ll)} - \omega_l c_0^{(jj)}}{2\omega_j\omega_l}$. Changing the sign of the inequalities in (10) and (13) yields the equation for the boundary of the stability region in the first approximation.

We now analyze the configuration of the instability regions (parametric resonance) in the following most common cases.

(a) Let the parametric-excitation matrix $\mathbf{B}(\Omega t)$ be symmetric. Then, the quantities $c_{-k}^{(jl)}$ and $c_k^{(lj)}$ are complex conjugate. Therefore, $\xi_2 = 0$ and the quantity ξ_1 in (11) takes the form

$$\xi_1 = \frac{c_{-k}^{(jl)}c_k^{(lj)}}{\omega_j\omega_l} = \frac{\left(a_k^{(jl)}\right)^2+\left(b_k^{(jl)}\right)^2}{4\omega_j\omega_l} \ge 0,$$

$$a_k^{(jl)} = \frac{1}{\pi}\int_0^{2\pi}\mathbf{u}_j^{\mathrm{T}}\mathbf{B}(\tau)\mathbf{u}_l\cos(k\tau)d\tau,$$

$$b_k^{(jl)} = \frac{1}{\pi}\int_0^{2\pi}\mathbf{u}_j^{\mathrm{T}}\mathbf{B}(\tau)\mathbf{u}_l\sin(k\tau)d\tau.$$

In the case of main and combined resonances of the summation type (9), stability condition (10) leads to an inequality defining the parametric resonance region (after the reduction of the positive factor):

$$\eta_j\eta_l\gamma^2-\xi_1\delta^2+4k^2\frac{\eta_j\eta_l}{(\eta_j+\eta_l)^2}\left(\Delta\Omega+\frac{\sigma_+\delta}{k}\right)^2\le 0. \quad (14)$$

Since the quantities η_j and η_l are positive and $\xi_1 \ge 0$, condition (14) for $\xi_1 \ne 0$ defines the interior of a cone in the three-dimensional space of the parameters $\mathbf{p} = (\gamma, \delta, \Omega)$ (Fig. 1). The cone axis defined by the centers of the conic plane sections at δ = const is determined by the equations $\gamma = 0$ and $\Omega - \Omega_0 + \dfrac{\sigma_+\delta}{k} = 0$. In the case of parametric excitation, with a zero mean $c_0^{(jj)} = c_0^{(ll)} = 0$, we have $\sigma_+ = 0$ and, therefore, the cone axis is parallel to the $O\delta$-axis. The stability region corresponds to the exterior of the cone.

When the number k specifying the number of resonances (9) increases at fixed values of ω_j and ω_l, the coefficient ξ_1 decreases as the modulus squared of the Fourier series expansion term. As a result, the instability cone rapidly shrinks with increasing k and the cone axis is straightened. The section of the cone by the plane δ = const (with the parametric excitation amplitude fixed) is an ellipse. Because of the factor k^2 in (14), the ellipse shrinks in the direction of the $O\Omega$-axis with increasing k. Since the denominators in (11) contain the products $\omega_j\omega_l$, the quantities ξ_1 and $|\sigma_+|$ decrease with increasing j and l. As a result, the instability cone shrinks and its axis is straightened with increasing j and l, i.e., for resonances at higher frequencies.

In the case of the difference-type combination resonance (12), it follows from stability condition (13) that the parametric-resonance region is defined by the inequality

$$\eta_j\eta_l\gamma^2+\xi_1\delta^2+4k^2\frac{\eta_j\eta_l}{(\eta_j+\eta_l)^2}\left(\Delta\Omega+\frac{\sigma_-\delta}{k}\right)^2\le 0. \quad (15)$$

It is noteworthy that inequality (15) differs from (14) only in the sign of the second term and in the coefficient σ_-, which is used in place of σ_+. Consequently, for $\xi_1 \ne 0$ (nondegenerate case), only one of the inequalities (14) and (15) defines a cone, while the other gives the point $\gamma = \delta = \Delta\Omega = 0$ (absence of resonance). Therefore, for positive ξ_1, the region of the difference-type combination resonance is lacking. It is worth noting that, in the case of Hamiltonian systems (without dissipation), the absence of the difference-type combination resonances has already been noted [3].

(b) Let the parametric-excitation matrix have the form $\mathbf{B}(\Omega t) = \varphi(\Omega t)\mathbf{B}_0$, where $\mathbf{B}_0$ is an arbitrary time-independent matrix and $\varphi(\tau)$ is a 2π-periodic scalar function. In this case, the product $c_{-k}^{(jl)}c_k^{(lj)}$ in (11) is real. Therefore, $\xi_2 = 0$ and the coefficient ξ_1 is given by

$$\xi_1 = \frac{c_{jl}[\alpha_k^2 + \beta_k^2]}{4\omega_j\omega_l}, \quad c_{jl} = \mathbf{u}_j^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_l\mathbf{u}_l^{\mathrm{T}}\mathbf{B}_0\mathbf{u}_j,$$
$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi}\varphi(\tau)\cos(k\tau)d\tau, \quad \beta_k = \frac{1}{\pi}\int_0^{2\pi}\varphi(\tau)\sin(k\tau)d\tau. \tag{16}$$

In the case of the main resonance and the summed combination resonance (9), the stability condition (10) defines the parametric-resonance region (14). For the difference-type combination resonance (12), the instability region is given by (15). In the nondegenerate case, when $\xi_1 \neq 0$, the sign of ξ_1 coincides with that of c_{jl}. For the main resonance, $c_{jj} \geq 0$; therefore, when $c_{jj} \neq 0$, there exists the main-resonance region defined by cone (14). The existence of combination resonance regions depends on the sign of c_{jl}. Namely, for $c_{jl} > 0$ or $c_{jl} < 0$, only the region of the summed or difference-type combination resonance exists, respectively. The shape of the parametric-resonance regions (cones) depends on the resonance number k and the frequencies ω_j and ω_l just as in the case **(a)** described above. When $c_{jl} = 0$, the resonance region is either absent or degenerate (i.e., the first approximation represents a straight line).

We formulate the results obtained in the following statement.

Theorem. *For a symmetric matrix* $\mathbf{B}(\tau) = \mathbf{B}^T(\tau)$, *system* (1) *exhibits only the main resonance* (6) *and the summed combination resonance* (7). *In the case of* $\mathbf{B}(\tau) = \varphi(\tau)\mathbf{B}_0$, *where* $\varphi(\tau)$ *is a periodic scalar function and* $\mathbf{B}_0$ *is a constant matrix, either the main resonances* (6), *the summed combination resonances* (*for* $c_{jl} > 0$), *or the difference-type combination resonances* (*for* $c_{jl} < 0$) *are realized in the system. In the three-dimensional space of the parameters* γ, δ, *and* Ω, *the regions of both main and summed combination resonances are described by cones* (14), *while those of difference-type combined resonances are given by cones* (15).

The cases **(a)** and **(b)** considered above correspond to the most conventional forms of parametric excitations. In other cases, stability conditions (10) and (13) can be used to find the three-dimensional resonance regions. Condition (10) defines the parametric-resonance region (14) (cone) for $\xi_1 > 0$ and $\xi_2 = 0$, while condition (13) describes the resonance region (15) for $\xi_1 < 0$ and $\xi_2 = 0$.

4. We now fix the parameter $\gamma > 0$ and consider the case of $\xi_2 = 0$ [for example, cases **(a)** and **(b)** considered above]. Then, to a first approximation, the parametric-resonance regions are defined by (14) and (15). Depending on the sign of ξ_1, the parametric-resonance region is either absent or occupies the interior of the hyperbolas (conic sections by the plane γ = const) on the two-dimensional plane of the parameters δ and Ω (Fig. 2). The asymptotes of these parabolas are determined by the equation

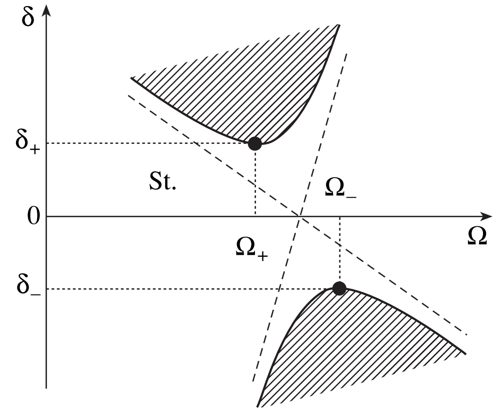

Fig. 2.

$$|\xi_1|^{1/2}\delta \pm 2k\frac{(\eta_j\eta_l)^{1/2}}{\eta_j + \eta_l}\left(\Delta\Omega + \frac{\sigma_s\delta}{k}\right) = 0, \tag{17}$$

where the subscript s stands for "+" and "–" for resonances (9) and (12), respectively. If the average over a period for the matrix $\mathbf{B}(\Omega t)$ is zero, then $\sigma_+ = \sigma_- = 0$. The parametric-resonance region on the plane (δ, Ω) decreases with increasing the dissipative parameter γ. In the first approximation, using (14) or (15), we find the minimum (critical) excitation amplitude $|\delta|$ and the corresponding frequencies at which the parametric resonance is possible:

$$\delta_\pm = \pm\left|\frac{\eta_j\eta_l}{\xi_1}\right|^{1/2}\gamma, \quad \Omega_\pm = \Omega_0 - \frac{\sigma_s\delta_\pm}{k}. \tag{18}$$

Here, Ω_0 is the resonance frequency given by (9) or (12), depending on the resonance type (Fig. 2).

5. As an example, we now consider Bolotin's problem [5, 9] on the dynamic stability of the plane bending of a beam. The elastic beam is assumed to be simply supported at its ends and loaded by the periodic moments $M(\Omega t) = \delta\varphi(\Omega t)$ in the plane of its maximum stiffness, where $\varphi(\tau)$ is a 2π-periodic function. Bending-torsional vibrations off this plane are described by the equations [9]

$$m\frac{\partial^2 w}{\partial t^2} + \gamma m d_1\frac{\partial w}{\partial t} + \mathrm{EJ}\frac{\partial^4 w}{\partial x^4} + \delta\varphi(\Omega t)\frac{\partial^2\theta}{\partial x^2} = 0,$$
$$mr^2\frac{\partial^2\theta}{\partial t^2} + \gamma m r^2 d_2\frac{\partial\theta}{\partial t} + \delta\varphi(\Omega t)\frac{\partial^2 w}{\partial x^2} - \mathrm{GI}\frac{\partial^2\theta}{\partial x^2} = 0. \tag{19}$$

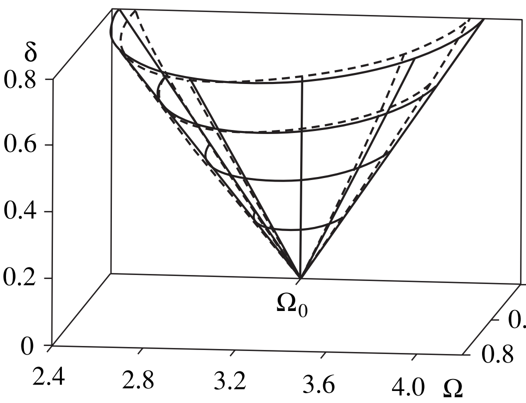

Fig. 3.

Here, $w(x, t)$ is the transverse deflection of the beam; $\theta(x, t)$ and r are the torsion angle and the radius of inertia for the beam's cross section, respectively; EJ and GI are the bending and torsion stiffnesses of the beam, respectively; m is the mass per unit length of the beam; γ is the parameter of dissipative force (viscous friction); and d_1 and d_2 are fixed constants defining the bending and torsional friction forces. The boundary conditions take the form

$$x = 0, l: w = \frac{\partial^2 w}{\partial x^2} = \theta = 0, \tag{20}$$

where l is the beam length. We seek the solution to system (19), (20) in the form of a series [5]:

$$\begin{aligned} w(x, t) &= \sum_{n=1}^{\infty} W_n(t) \sin\frac{n\pi x}{l}, \\ \theta(x, t) &= \sum_{n=1}^{\infty} \Theta_n(t) \sin\frac{n\pi x}{l}, \end{aligned} \tag{21}$$

where $W_n(t)$ and $\Theta_n(t)$ are unknown functions of time. Substituting (21) into Eq. (19), we obtain a set of ordinary differential equations with respect to $W_n(t)$ and $\Theta_n(t)$ of the form of (1) with

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{pmatrix},$$

$$\mathbf{B}(\Omega t) = \varphi(\Omega t)\begin{pmatrix} 0 & -\dfrac{\pi^2 n^2}{l^2 m} \\ -\dfrac{\pi^2 n^2}{r^2 l^2 m} & 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} W_n \\ \Theta_n \end{pmatrix}. \tag{22}$$

Here, ω_{n1} and ω_{n2} are the natural frequencies of bending and torsional vibrations of the beam, respectively:

$$\omega_{n1} = \frac{n^2\pi^2}{l^2}\sqrt{\frac{\text{EJ}}{m}}, \quad \omega_{n2} = \frac{n\pi}{rl}\sqrt{\frac{\text{GI}}{m}}, \quad n = 1, 2, \ldots \tag{23}$$

The eigenvectors corresponding to the frequencies ω_{n1} and ω_{n2} are equal to $\mathbf{u}_{n1} = (1, 0)^T$ and $\mathbf{u}_{n2} = (0, 1)^T$, respectively.

We now analyze the stability of system (1) with matrices (22) for a certain fixed n. Since $\mathbf{B}(\Omega t) = \varphi(\Omega t)\mathbf{B}_0$, where $\mathbf{B}_0$ is a fixed matrix, this system belongs to the type considered in item **(b)**. The quantities determined from Eqs. (16) take the form

$$c_{11} = c_{22} = 0, \quad c_{12} = \frac{\pi^4 n^4}{l^4 r^2 m^2} > 0. \tag{24}$$

Therefore, the regions of the difference-type combination resonance are absent, while the main resonance regions are degenerate (in order to analyze them, higher approximations are required). According to (14), the regions of the summed combination resonance at frequencies close to $\Omega_0 = \dfrac{\omega_{n1} + \omega_{n2}}{k}$, $k = 1, 2, \ldots$ are defined by

$$d_1 d_2 \gamma^2 - \frac{c_{12}(\alpha_k^2 + \beta_k^2)}{4\omega_{n1}\omega_{n2}}\delta^2 + 4k^2\frac{d_1 d_2}{(d_1 + d_2)^2}\Delta\Omega^2 \le 0, \tag{25}$$

where the quantities α_k and β_k are determined from (16).

To carry out numerical calculations, we set $n = 1$, $\varphi(\tau) = \cos\tau$, $d_1 = d_2 = 1$, $\omega_{n1} = 1$ s^{-1}, $\omega_{n2} = \sqrt{5}$ s^{-1}, $l^2 m = \dfrac{\pi^2}{4}$ kg cm, and $r^2 = \dfrac{4}{\sqrt{5}}$ cm^2. The boundary of the combination resonance region (with $k = 1$) given by (25) in the first approximation is shown in Fig. 3 (solid curves). The same quantity obtained by numerical evaluation of the monodromy matrix for various values of the parameters γ, δ, and Ω is also shown (dashed curves). The Runge–Kutta method was employed to integrate Eqs. (3). It is seen from Fig. 3 that the exact (obtained numerically) and approximate regions of the combination resonance are well matched up to the amplitudes $\delta \approx 0.8$.

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