ELSEVIER Linear Algebra and its Applications 337 (2001) 87–108

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Transformation to versal deformations of matrices

Alexei A. Mailybaev

Institute of Mechanics, Moscow State Lomonosov University, Michurinsky pr. 1, 117192 Moscow, Russia

Received 31 July 2000; accepted 25 March 2001

Submitted by R.A. Brualdi

Abstract

In the paper versal deformations of matrices are considered. The versal deformation is a matrix family inducing an arbitrary multi-parameter deformation of a given matrix by an appropriate smooth change of parameters and basis. Given a deformation of a matrix, it is suggested to find transformation functions (the change of parameters and the change of basis dependent on parameters) in the form of Taylor series. The general method of construction of recurrent procedures for calculation of coefficients in the Taylor expansions is developed and used for the cases of real and complex matrices, elements of classical Lie and Jordan algebras, and infinitesimally reversible matrices. Several examples are given and studied in detail. Applications of the suggested approach to problems of stability, singularity, and perturbation theories are discussed. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Versal deformation; Normal form; Transformation; Lie algebra; Jordan algebra; Reversible matrix

1. Introduction

Analysis of normal forms and spectra of matrices is a very important problem both from theoretical and practical points of view. This analysis becomes very complicated, when we study multi-parameter families of matrices. Introducing parameters, we obtain many new phenomena like singularities and bifurcations leading to qualitative changes in the behavior of systems described by these matrices. In this paper we study properties of square matrices $\mathbf{A}(\mathbf{p})$ smoothly depending on a vector of parameters $\mathbf{p}$ and determined in the vicinity of the origin $\mathbf{p} = 0$. Such a matrix family is called a deformation of a matrix $\mathbf{A}_0 = \mathbf{A}(0)$. Arnold [1,3] defined and studied normal forms of deformations of complex matrices (called versal deformations). These

E-mail address: mailybaev@imec.msu.ru (A.A. Mailybaev).

PII: S 0 0 2 4 - 3 7 9 5 (0 1) 0 0 3 4 6 - 9

normal forms are special matrix families $\mathbf{B}(\mathbf{q})$ possessing properties of all other families, i.e., we can get an arbitrary deformation of the matrix $\mathbf{A}_0$ from the corresponding versal deformation by a smooth change of parameters $\mathbf{q} = \mathbf{q}(\mathbf{p})$ and a change of basis $\mathbf{x}' = \mathbf{C}(\mathbf{p})\mathbf{x}$ smoothly dependent on parameters

$$\mathbf{A}(\mathbf{p}) = \mathbf{C}(\mathbf{p})\mathbf{B}(\mathbf{q}(\mathbf{p}))\mathbf{C}^{-1}(\mathbf{p}).$$

It was shown that the versal deformation $\mathbf{B}(\mathbf{q})$ is determined only by the matrix $\mathbf{A}_0$. The study of versal deformations for different types of matrices (real, Hamiltonian, reversible, etc.) was continued by many authors [12,13,16,21–23,27]; for more references see a short survey in [4, pp. 172–177]. In papers [6,11,15,16] versal deformations were found in the cases of pairs, triples, and quadruples of matrices, where the change of basis is substituted by another equivalence transformation.

Applications of versal deformations are based on the fact that the spectrum of the matrix family coincides with the spectrum of its versal deformation, while the versal deformation has a very simple form. This property allows using the versal deformation theory for multi-parameter analysis of the spectrum in problems of stability and dynamics. Versal deformations without knowledge about the change of parameters and basis were used for the classification of singularities of bifurcation diagrams, decrement diagrams, and stability boundaries [2,3,12,13,19,21,23].

The problem of finding the change of parameters and basis (the transformation functions $\mathbf{q}(\mathbf{p})$ and $\mathbf{C}(\mathbf{p})$) for a given matrix family $\mathbf{A}(\mathbf{p})$ was considered by several authors. Cushman et al. [9] studied this problem for a specific family of 4×4 Hamiltonian matrices. Schmidt [25,26] used the computer algebra for finding the transformation functions in the case, when the change of parameters can be explicitly found by comparing corresponding characteristic equations. Stolovitch [28] used the Newton method for numerical calculations of the functions $\mathbf{q}(\mathbf{p})$ and $\mathbf{C}(\mathbf{p})$ at given values of the parameters; in the case of simplified versal deformations corresponding to one-parameter matrix families of special type he constructed an algorithm for finding the transformation functions in the form of Taylor series. For real and complex matrix families Mailybaev [18] proposed to find the transformation functions in the form of Taylor series, whose coefficients are calculated by an explicit recurrent procedure. It turns out that Taylor expansions of the functions $\mathbf{q}(\mathbf{p})$ and $\mathbf{C}(\mathbf{p})$ provide the most useful information for applications, where the most important are the first order terms [7,17–20]. This is a consequence of the local nature of a versal deformation.

This paper represents the further development of ideas of paper [18] for versal deformations of matrices of different types (Hamiltonian, reversible, symmetric, etc.). Following [18] it is suggested to find the transformation functions in the form of Taylor series. A general method of construction of recurrent procedures for calculation of coefficients in the Taylor expansions is developed. This method provides “almost ready” recurrent formulae, which can be easily completed for any type of matrices. For several important cases (real and complex matrices, elements of classical Lie and Jordan algebras, and infinitesimally reversible matrices) explicit recurrent procedures are given. Three examples are studied showing efficiency of the method and possibilities for its application to problems of stability, singularity, and perturbation theories.

The paper is organized as follows. General concepts of the versal deformation theory are given in Section 2. The method of construction of recurrent procedures is described in Section 3. In Sections 4 and 5 the developed method is applied to several specific types of matrices. Section 6 shows how the recurrent procedure can be optimized, when we need only a partial information on the transformation functions and the versal deformation. Conclusion gives a short outline of the obtained results.

In the paper the matrices are denoted by bold capital letters, vectors take the form of bold lower-case letters, and scalars are represented by lower-case italic characters.

2. Normal forms and versal deformations

Let M be a manifold and G be a Lie group acting on M by conjugation; for the definition and properties of a Lie group see [8,24]. In this paper we consider the case, when M is a submanifold of the space $\mathrm{gl}(m, \mathbb{D})$ of $m \times m$ matrices with elements from $\mathbb{D}$, and G is a subgroup of the group $\mathrm{GL}(m, \mathbb{D})$ of nonsingular $m \times m$ matrices; $\mathbb{D}$ stands for the space of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or quaternions $\mathbb{H}$. The adjoint action of $\mathbf{C} \in G$ determines an equivalence transformation in M as follows:

$$\mathbf{A} \sim \mathbf{A}' = \mathbf{C}\mathbf{A}\mathbf{C}^{-1} \in M, \quad \mathbf{A} \in M. \tag{1}$$

If the matrix $\mathbf{A} \in M$ is considered as an operator in the space $\mathbf{x} \in \mathbb{D}^m$, then the adjoint action of $\mathbf{C} \in G$ represents the change of basis $\mathbf{x}' = \mathbf{C}\mathbf{x}$.

To simplify the analysis we will consider the case when M is a linear space. This corresponds to many important cases including symmetric, Hamiltonian, infinitesimally reversible, and other types of matrices. The general case, when M is an arbitrary submanifold of $\mathrm{gl}(m, \mathbb{D})$ (this corresponds, for example, to the space of symplectic matrices), can be considered similarly.

The adjoint action of G determines *an orbit* (a G-conjugacy class) of an element $\mathbf{A} \in M$ as follows:

$$\mathrm{Orb}(\mathbf{A}) = \left\{\mathbf{C}\mathbf{A}\mathbf{C}^{-1}:\ \mathbf{C} \in G\right\}. \tag{2}$$

Any element $\mathbf{A}' \in \mathrm{Orb}(\mathbf{A})$ is a representative of $\mathrm{Orb}(\mathbf{A})$ since $\mathrm{Orb}(\mathbf{A}') = \mathrm{Orb}(\mathbf{A})$. The problem of the normal form theory is to find representatives having simple form and to classify them. The choice of the normal form varies in different studies depending on the problem under consideration. A famous example is the Jordan normal form in the case $M = \mathrm{gl}(m, \mathbb{C})$, $G = \mathrm{GL}(m, \mathbb{C})$. Normal forms are well studied and their classification for many important cases of M and G is performed; see [10,27] and references therein.

A deformation $\mathbf{A}(\mathbf{p})$ of a matrix $\mathbf{A}_0 \in M$ is a smooth mapping $\mathbf{A} : (\mathbb{F}^n, 0) \to (M, \mathbf{A}_0)$ determined in the vicinity of the origin $\mathbf{p} = 0$; $\mathbb{F}^n$ is a space of real or complex parameters $\mathbf{p} = (p_1, \ldots, p_n)$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ for $\mathbb{D} = \mathbb{C}$, and $\mathbb{F} = \mathbb{R}$ for $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$). A deformation $\mathbf{A}(\mathbf{p})$ is also called *a matrix family*. A deformation $\mathbf{B}(\mathbf{q})$, $\mathbf{q} \in \mathbb{F}^d$, of a matrix $\mathbf{A}_0$ is called *a versal deformation* if any deformation $\mathbf{A}(\mathbf{p})$ of $\mathbf{A}_0$ can be represented in the vicinity of $\mathbf{p}=0$ in the form

$$\mathbf{A}(\mathbf{p}) = \mathbf{C}(\mathbf{p})\mathbf{B}(\mathbf{q}(\mathbf{p}))\mathbf{C}^{-1}(\mathbf{p}), \tag{3}$$

where $\mathbf{C} : (\mathbb{F}^n, 0) \to (G, \mathbf{I})$ is a smooth mapping (deformation of the identity matrix $\mathbf{I}$ in G); $\mathbf{q}(\mathbf{p})$ is a smooth function from $\mathbb{F}^n$ to the parameter space of the versal deformation $\mathbb{F}^d$ such that $\mathbf{q}(0)=0$ [1,3]. Expression (3) means that any deformation $\mathbf{A}(\mathbf{p})$ can be obtained from the versal deformation $\mathbf{B}(\mathbf{q})$ by a change of basis and a change of parameters. Thus, the versal deformation represents the most general matrix family possessing (in some sense) properties of all deformations of the matrix $\mathbf{A}_0$.

Versal deformation with the minimal number of parameters d is called *miniversal*. It was proved in [1,3] that the deformation $\mathbf{B}(\mathbf{q})$ of $\mathbf{A}_0$ is versal iff $\mathbf{B}(\mathbf{q})$ is transversal in the space M to the orbit $\mathrm{Orb}(\mathbf{A}_0)$ at $\mathbf{A}_0$. This condition gives a constructive method of finding versal deformations, which was used in all papers devoted to this problem. It is clear that the number of parameters of a miniversal deformation is equal to the codimension of $\mathrm{Orb}(\mathbf{A}_0)$ in M [1,3].

Note that it is sufficient to consider versal deformations of normal forms of matrices. Indeed, let us consider a matrix $\widetilde{\mathbf{A}}_0$ whose normal form is $\mathbf{A}_0$, i.e.,

$$\widetilde{\mathbf{A}}_0 = \mathbf{C}_0\mathbf{A}_0\mathbf{C}_0^{-1}, \quad \mathbf{C}_0 \in G. \tag{4}$$

Then the versal (miniversal) deformation of $\widetilde{\mathbf{A}}_0$ is $\widetilde{\mathbf{B}}(\mathbf{q}) = \mathbf{C}_0\mathbf{B}(\mathbf{q})\mathbf{C}_0^{-1}$, where $\mathbf{B}(\mathbf{q})$ is the versal (miniversal) deformation of the normal form $\mathbf{A}_0$. For any deformation $\widetilde{\mathbf{A}}(\mathbf{p})$ of $\widetilde{\mathbf{A}}_0$ we have

$$\begin{aligned}\widetilde{\mathbf{A}}(\mathbf{p}) &= \mathbf{C}_0\mathbf{C}(\mathbf{p})\mathbf{B}(\mathbf{q}(\mathbf{p}))(\mathbf{C}_0\mathbf{C}(\mathbf{p}))^{-1}\\ &= \widetilde{\mathbf{C}}(\mathbf{p})\widetilde{\mathbf{B}}(\mathbf{q}(\mathbf{p}))\widetilde{\mathbf{C}}^{-1}(\mathbf{p}), \quad \widetilde{\mathbf{C}}(\mathbf{p}) = \mathbf{C}_0\mathbf{C}(\mathbf{p})\mathbf{C}_0^{-1},\end{aligned} \tag{5}$$

where the functions $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$ transform the deformation

$$\mathbf{A}(\mathbf{p}) = \mathbf{C}_0^{-1}\widetilde{\mathbf{A}}(\mathbf{p})\mathbf{C}_0 \tag{6}$$

of the normal form $\mathbf{A}_0$ to the versal deformation $\mathbf{B}(\mathbf{q})$ (3).

3. Transformation to versal deformations

Let a deformation $\mathbf{A}(\mathbf{p})$ and a versal deformation $\mathbf{B}(\mathbf{q})$ of a matrix $\mathbf{A}_0$ be given. In this section we find the transformation functions (the change of basis and the change of parameters) $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$ satisfying relation (3). We will consider miniversal deformations $\mathbf{B}(\mathbf{q})$ (a versal deformation $\mathbf{B}(\mathbf{q})$ can be made miniversal by taking some of the parameters equal to zero).

The smooth functions $\mathbf{A}(\mathbf{p})$, $\mathbf{C}(\mathbf{p})$, and $\mathbf{q}(\mathbf{p})$ are determined in the vicinity of the origin $\mathbf{p}=0$. Thus, they can be represented in the form of Taylor series with the accuracy $o(\|\mathbf{p}\|^k)$, where k is the maximal order of terms kept in the expansion; $\|\mathbf{p}\|$

is the norm in the parameter space $\mathbb{F}^n$. In applications we usually need only a finite number of terms in the Taylor expansion, where the most important are the first order terms.

Let us introduce some notations. Let $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}_+^n$ be a vector with nonnegative integer components h_i. Then we denote $\mathbf{p}^{\mathbf{h}} = p_1^{h_1} \cdots p_n^{h_n}$, $\mathbf{h}! = h_1! \cdots h_n!$, $|\mathbf{h}| = h_1 + \cdots + h_n$, and

$$\mathbf{A}^{(\mathbf{h})} = \frac{\partial^{|\mathbf{h}|}\mathbf{A}}{\partial p_1^{h_1} \cdots \partial p_n^{h_n}}, \quad C_{\mathbf{h}}^{\mathbf{h}'} = C_{h_1}^{h_1'} \cdots C_{h_n}^{h_n'}, \quad C_i^j = \frac{i!}{j!(i-j)!}, \tag{7}$$

where derivatives are evaluated at $\mathbf{p} = 0$; $h_i = 0$ means that we do not take the derivative with respect to p_i. The Taylor series for the function $\mathbf{A}(\mathbf{p})$ is represented in the form

$$\mathbf{A}(\mathbf{p}) = \sum_{|\mathbf{h}| \leqslant k} \frac{\mathbf{A}^{(\mathbf{h})}\mathbf{p}^{\mathbf{h}}}{\mathbf{h}!} + \mathrm{o}(\|\mathbf{p}\|^k), \quad \mathbf{A}^{(0)} = \mathbf{A}_0, \tag{8}$$

where the sum is taken over all $\mathbf{h} \in \mathbb{Z}_+^n$ of order $|\mathbf{h}| \leqslant k$. Substituting $\mathbf{A}$ by $\mathbf{C}$ or $\mathbf{q}$ in (8), we obtain Taylor expansions of the functions $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$, where $\mathbf{C}^{(0)} = \mathbf{I}$ and $\mathbf{q}^{(0)} = 0$. To determine the Taylor series of $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$, the derivatives $\mathbf{C}^{(\mathbf{h})}$ and $\mathbf{q}^{(\mathbf{h})}$ should be found using derivatives $\mathbf{A}^{(\mathbf{h})}$. In this section we derive recurrent formulae for derivatives $\mathbf{C}^{(\mathbf{h})}$ and $\mathbf{q}^{(\mathbf{h})}$ assuming that all the derivatives $\mathbf{C}^{(\mathbf{h}')}$ and $\mathbf{q}^{(\mathbf{h}')}$ of lower orders $\mathbf{h}' < \mathbf{h}$ are known; $\mathbf{h}' < \mathbf{h}$ means that $h_i' \leqslant h_i$ for $i = 1, \ldots, n$ and $h_j' < h_j$ for some j.

First, let us analyze the structure of $\mathbf{C}(\mathbf{p})$. Let $\mathbf{Q} : (\mathrm{GL}(m, \mathbb{D}),\ \mathbf{I}) \to (Q, 0)$ be a smooth function determined in the vicinity of $\mathbf{I}$, where the range Q is a space of matrices or vectors, such that the equation

$$\mathbf{Q}(\mathbf{C}) = 0 \tag{9}$$

determines the manifold G in the vicinity of $\mathbf{I}$. The tangent space to G at $\mathbf{I}$ is denoted by TG and determined by the expression

$$TG = \left\{\mathbf{X} \in \mathrm{gl}(m, \mathbb{D}) \colon\ \mathrm{d}\mathbf{Q}(\mathbf{X}) = 0\right\}, \tag{10}$$

where $\mathrm{d}\mathbf{Q}$ is the differential of $\mathbf{Q}$ at $\mathbf{I}$. The space TG determines the tangent space $T\mathrm{Orb}(\mathbf{A}_0)$ to the orbit $\mathrm{Orb}(\mathbf{A}_0)$ at $\mathbf{A}_0$ in the form [1,3]

$$T\mathrm{Orb}(\mathbf{A}_0) = \left\{[\mathbf{A}_0,\ \mathbf{X}] \colon\ \mathbf{X} \in TG\right\}. \tag{11}$$

The matrix $[\mathbf{A}_0, \mathbf{X}] = \mathbf{A}_0\mathbf{X} - \mathbf{X}\mathbf{A}_0$ is the derivative of the function $\mathbf{A}'(\mathbf{C}) = \mathbf{C}^{-1}\mathbf{A}_0\mathbf{C}$, $\mathbf{C} \in G$, at $\mathbf{I}$ along the direction $\mathbf{X} \in TG$. Since M is a linear space, we have $T\mathrm{Orb}(\mathbf{A}_0) \subset M$.

The miniversal deformation $\mathbf{B}(\mathbf{p})$ is usually chosen to be a linear function of the parameters

$$\mathbf{B}(\mathbf{q}) = \mathbf{A}_0 + \sum_{i=1}^{d} \mathbf{B}_i q_i, \quad \mathbf{q} = (q_1, \ldots, q_d). \tag{12}$$

Let $B \subset M$ be a linear space determined by the matrices $\mathbf{B}_i$, $i = 1, \ldots, d$. According to the versal deformation theory [1,3] $\mathbf{B}(\mathbf{q})$ is a miniversal deformation iff $M = T\mathrm{Orb}(\mathbf{A}_0) + B$ and $d = \dim(M) - \dim(T\mathrm{Orb}(\mathbf{A}_0))$.

Let $(\mathbf{A}, \mathbf{B}) = \mathrm{trace}(\mathbf{A}^{\mathrm{T}}\overline{\mathbf{B}})$ be a scalar product in $\mathrm{gl}(m, \mathbb{D})$, where $\mathbf{A}^{\mathrm{T}}$ is the transposed matrix and $\overline{\mathbf{B}}$ is the complex (or quaternionic) conjugate matrix in the case $\mathbb{D} = \mathbb{C}$ (or $\mathbb{D} = \mathbb{H}$). We denote by $N\mathrm{Orb}(\mathbf{A}_0)$ the normal complement of $T\mathrm{Orb}(\mathbf{A}_0)$ in M, i.e., $M = T\mathrm{Orb}(\mathbf{A}_0) + N\mathrm{Orb}(\mathbf{A}_0)$ and $(\mathbf{T}, \mathbf{N}) = 0$ for any $\mathbf{T} \in T\mathrm{Orb}(\mathbf{A}_0)$, $\mathbf{N} \in N\mathrm{Orb}(\mathbf{A}_0)$. Let $\mathbf{N}_i$, $i = 1, \ldots, d$, be a basis of $N\mathrm{Orb}(\mathbf{A}_0)$ normalized with respect to $\mathbf{B}_j$ such that

$$(\mathbf{N}_i, \mathbf{B}_j) = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, \ldots, d. \tag{13}$$

A centralizer of $\mathbf{A}_0$ is a set of matrices $\mathbf{X} \in TG$ commuting with $\mathbf{A}_0$ [1,3]:

$$\mathrm{Cent}(\mathbf{A}_0) = \left\{\mathbf{X} \in TG\colon\ [\mathbf{A}_0, \mathbf{X}] = 0\right\}. \tag{14}$$

Note that $\mathrm{Cent}(\mathbf{A}_0)$ is the tangent space to the *stabilizer* of $\mathbf{A}_0$ at $\mathbf{I}$, where $\mathrm{Stab}(\mathbf{A}_0) = \{\mathbf{C} \in G:\ \mathbf{C}\mathbf{A}_0\mathbf{C}^{-1} = \mathbf{A}_0\}$. Let $f = \dim(\mathrm{Cent}(\mathbf{A}_0))$, where in many (but not all) cases $f = d$. Since $\mathrm{Cent}(\mathbf{A}_0)$ is a linear space, we can choose the orthogonal basis $\mathbf{R}_i$, $i = 1, \ldots, f$, of $\mathrm{Cent}(\mathbf{A}_0)$ such that

$$(\mathbf{R}_i, \mathbf{R}_j) = \delta_{ij}, \quad i, j = 1, \ldots, f. \tag{15}$$

It is known [1,3] that the mapping $\mathbf{C}(\mathbf{p})$ is uniquely determined on any smooth surface $P \subset G$, which is transversal to $\mathrm{Cent}(\mathbf{A}_0)$ at $\mathbf{I}$ and $\dim(P) = \dim(G) - f$. This surface can be taken in the form

$$P = \left\{\mathbf{C} \in G:\ (\mathbf{C} - \mathbf{I}, \mathbf{R}_i) = 0,\ i = 1, \ldots, f\right\}. \tag{16}$$

The surface P is the intersection of G with the plane $\mathbf{I} + N\mathrm{Cent}(\mathbf{A}_0)$, where $N\mathrm{Cent}(\mathbf{A}_0)$ is the normal complement of $\mathrm{Cent}(\mathbf{A}_0)$ in TG. Condition $\mathbf{C}(\mathbf{p}) \in P$ leads to the following equalities for derivatives $\mathbf{C}^{(\mathbf{h})}$:

$$(\mathbf{C}^{(\mathbf{h})}, \mathbf{R}_i) = 0, \quad i = 1, \ldots, f. \tag{17}$$

Taking the derivative of order $|\mathbf{h}| > 0$ of Eq. (9), we get

$$(\mathbf{Q}(\mathbf{C}(\mathbf{p})))^{(\mathbf{h})} = \mathrm{d}\mathbf{Q}(\mathbf{C}^{(\mathbf{h})}) + \mathbf{Q}_{\mathbf{h}} = 0, \tag{18}$$

where $\mathbf{Q}_{\mathbf{h}}$ contains all the other terms obtained after differentiating $\mathbf{Q}(\mathbf{C}(\mathbf{p}))$ as a composite function. The value of $\mathbf{Q}_{\mathbf{h}}$ depends only on derivatives $\mathbf{C}^{(\mathbf{h}')}$ of lower orders $\mathbf{h}' < \mathbf{h}$. Since we assumed that the derivatives $\mathbf{C}^{(\mathbf{h}')}$, $\mathbf{h}' < \mathbf{h}$, are known, the value of $\mathbf{Q}_{\mathbf{h}}$ is determined. Eq. (18) describes all possible values of $\mathbf{C}^{(\mathbf{h})}$. Let $\mathbf{C}_{\mathbf{h}}$ be an arbitrary matrix satisfying (18). Then the matrix $\mathbf{C}^{(\mathbf{h})}$ belongs to the set

$$\mathbf{C}^{(\mathbf{h})} \in \left\{\mathbf{C}_{\mathbf{h}} + \mathbf{X}:\ \mathbf{X} \in TG\right\}. \tag{19}$$

Let us rewrite Eq. (3) in the equivalent form with the use of (12)

$$\mathbf{A}(\mathbf{p})\mathbf{C}(\mathbf{p}) - \mathbf{C}(\mathbf{p})\left(\mathbf{A}_0 + \sum_{i=1}^{d} \mathbf{B}_i q_i(\mathbf{p})\right) = 0. \tag{20}$$

Taking derivative of order $|\mathbf{h}| > 0$ of (20), we obtain

$$\sum_{\mathbf{h}'+\mathbf{h}''=\mathbf{h}} C_{\mathbf{h}}^{\mathbf{h}'} \left(\mathbf{A}^{(\mathbf{h}')}\mathbf{C}^{(\mathbf{h}'')} - \mathbf{C}^{(\mathbf{h}'')} \left(\mathbf{A}_0 + \sum_{i=1}^{d} \mathbf{B}_i q_i(\mathbf{p}) \right)^{(\mathbf{h}')} \right) = 0. \tag{21}$$

Rearranging terms in (21) and substituting

$$\mathbf{C}^{(\mathbf{h})} = \mathbf{C}_{\mathbf{h}} + \mathbf{X}, \quad \mathbf{X} \in TG, \tag{22}$$

we get equation

$$[\mathbf{A}_0, \mathbf{X}] = \mathbf{Y}_{\mathbf{h}}, \quad \mathbf{X} \in TG, \tag{23}$$

where

$$\begin{aligned} \mathbf{Y}_{\mathbf{h}} &= \sum_{i=1}^{d} \mathbf{B}_i q_i^{(\mathbf{h})} + \mathbf{D}_{\mathbf{h}}, \\ \mathbf{D}_{\mathbf{h}} &= -[\mathbf{A}_0, \mathbf{C}_{\mathbf{h}}] - \mathbf{A}^{(\mathbf{h})} \\ &\quad - \sum_{\substack{\mathbf{h}'+\mathbf{h}''=\mathbf{h}, \\ \mathbf{h}', \mathbf{h}'' \neq \mathbf{h}}} C_{\mathbf{h}}^{\mathbf{h}'} \left(\mathbf{A}^{(\mathbf{h}')}\mathbf{C}^{(\mathbf{h}'')} - \mathbf{C}^{(\mathbf{h}'')} \sum_{i=1}^{d} \mathbf{B}_i q_i^{(\mathbf{h}')} \right). \end{aligned} \tag{24}$$

The matrix $\mathbf{D}_{\mathbf{h}} \in M$, since $\mathbf{B}_i \in M$ and $[\mathbf{A}_0, \mathbf{X}] \in M$ for $\mathbf{X} \in TG$. Eq. (23) has a solution $\mathbf{X} \in TG$ iff the matrix $\mathbf{Y}_{\mathbf{h}}$ belongs to $T\mathrm{Orb}(\mathbf{A}_0)$. This condition can be written with the use of the basis $\mathbf{N}_i$, $i = 1, \ldots, d$, of $N\mathrm{Orb}(\mathbf{A}_0)$ as follows:

$$(\mathbf{Y}_{\mathbf{h}}, \mathbf{N}_i) = \left(\sum_{j=1}^{d} \mathbf{B}_j q_j^{(\mathbf{h})} + \mathbf{D}_{\mathbf{h}}, \mathbf{N}_i \right) = 0, \quad i = 1, \ldots, d. \tag{25}$$

Eq. (25) and condition (13) yield

$$q_i^{(\mathbf{h})} = -(\mathbf{D}_{\mathbf{h}}, \mathbf{N}_i), \quad i = 1, \ldots, d. \tag{26}$$

Substituting (26) into (24), we determine the matrix $\mathbf{Y}_{\mathbf{h}}$ such that the solution $\mathbf{X} \in TG$ of Eq. (23) exists. Let $\mathbf{X}_{\mathbf{h}} \in TG$ be an arbitrary particular solution of (23). Then all solutions of (23) form the set $\mathbf{X}_{\mathbf{h}} + \mathrm{Cent}(\mathbf{A}_0)$ and can be expressed as

$$\mathbf{X} = \mathbf{X}_{\mathbf{h}} + \sum_{i=1}^{f} \gamma_i \mathbf{R}_i. \tag{27}$$

The coefficients γ_i can be found from (17) and (22) in the form

$$\gamma_i = -(\mathbf{C}_{\mathbf{h}} + \mathbf{X}_{\mathbf{h}}, \mathbf{R}_i), \quad i = 1, \ldots, f. \tag{28}$$

Therefore, we obtained the recurrent formulae for calculation of the derivatives $\mathbf{C}^{(\mathbf{h})}$ and $\mathbf{q}^{(\mathbf{h})} = (q_1^{(\mathbf{h})}, \ldots, q_d^{(\mathbf{h})})$.

Theorem 1. *Derivatives* $\mathbf{C}^{(\mathbf{h})}$ *and* $\mathbf{q}^{(\mathbf{h})}$ *of the functions transforming the matrix family* $\mathbf{A}(\mathbf{p})$ *to the miniversal deformation* $\mathbf{B}(\mathbf{q})$ (3) *can be found using recurrent expressions* (22), (24) *and* (26)–(28), *where* $\mathbf{C}^{(0)} = \mathbf{I}$, $\mathbf{q}^{(0)} = 0$, *and* $\mathbf{C}_\mathbf{h}$, $\mathbf{X}_\mathbf{h}$ *are any particular solutions of* (18) *and* (23).

For practical realization of the method in each particular case of M and G we should specify the matrices $\mathbf{B}_i, \mathbf{N}_i, \mathbf{R}_j$ $(i = 1, \ldots, d,\ j = 1, \ldots, f)$ and obtain formulae for the matrices $\mathbf{C}_\mathbf{h}$, $\mathbf{X}_\mathbf{h}$. This is usually a straightforward technical work. To find a particular solution $\mathbf{X}_\mathbf{h}$ it is convenient to use a matrix $\mathbf{M}(\mathbf{A}_0, \mathbf{Y}_\mathbf{h})$ given in Appendix A, which provides a particular solution of Eq. (23) in $\mathrm{gl}(m, \mathbb{D})$. Then the matrix $\mathbf{X}_\mathbf{h}$ can be expressed in the form $\mathbf{X}_\mathbf{h} = \mathbf{M}(\mathbf{A}_0, \mathbf{Y}_\mathbf{h}) + \mathbf{Z}_\mathbf{h}$, where $\mathbf{Z}_\mathbf{h} \in \mathrm{gl}(m, \mathbb{D})$ is any matrix commuting with $\mathbf{A}_0$ such that $\mathbf{X}_\mathbf{h} \in TG$. Explicit form of matrices $\mathbf{Z}_\mathbf{h}$ is given in Appendix A.

The case, when the manifold M is not a linear space, is studied similarly. Nonlinear manifolds M are considered, when we study symplectic, reversible, or other types of matrices. In this case the miniversal deformation $\mathbf{B}(\mathbf{q})$ is generally not a linear function of $\mathbf{q}$. The procedure of Theorem 1 can be used in the case of the nonlinear M if we take the matrix $\mathbf{D}_\mathbf{h}$ in the form

$$\begin{aligned}\mathbf{D}_\mathbf{h} = &- (\mathbf{A}(\mathbf{p})\mathbf{C}(\mathbf{p}) - \mathbf{C}(\mathbf{p})\mathbf{B}(\mathbf{q}(\mathbf{p})))^{(\mathbf{h})} \\ &+ [\mathbf{A}_0, \mathbf{C}^{(\mathbf{h})} - \mathbf{C}_\mathbf{h}] - \sum_{i=1}^{d} \mathbf{B}_i q_i^{(\mathbf{h})},\end{aligned} \tag{29}$$

where $\mathbf{B}_i = \partial \mathbf{B}/\partial q_i$ (the derivative is calculated at $\mathbf{q} = 0$). Expression (29) after taking the derivative of the expression in parenthesis contains only derivatives $\mathbf{C}^{(\mathbf{h}')}$ and $q_i^{(\mathbf{h}')}$ of lower orders $\mathbf{h}' < \mathbf{h}$. Thus, the matrix $\mathbf{D}_\mathbf{h}$ is determined at the corresponding step of the recurrent procedure.

In the following two sections it will be shown how the recurrent procedure for calculation of the derivatives $\mathbf{C}^{(\mathbf{h})}$ and $\mathbf{q}^{(\mathbf{h})}$ can be completed in several important cases of M and G.

4. Versal deformations of real and complex matrices

Let us consider the case $M = \mathrm{gl}(m, \mathbb{D})$ and $G = \mathrm{GL}(m, \mathbb{D})$. The normal form of a matrix $\mathbf{A}_0 \in \mathrm{gl}(m, \mathbb{D})$ is the Jordan normal form in the case $\mathbb{D} = \mathbb{C}$ or its modification in the cases $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$. Miniversal deformations of complex and real matrices transformed to the normal form are studied in [1,3,12]. Although miniversal deformations of quaternionic matrices $\mathbf{A}_0 \in \mathrm{gl}(m, \mathbb{H})$ are not given in the literature, their construction is an easy straightforward matter after the works of Arnold [1,3] and Patera et al. [23]. Using results of these papers, it is straightforward to construct the matrices $\mathbf{B}_i$, $\mathbf{N}_i$, and $\mathbf{R}_j$. In the case under consideration $f = d$ and we can take

$\mathbf{C_h} = 0$ and $\mathbf{X_h} = \mathbf{M}(\mathbf{A}_0, \mathbf{Y_h})$, where the matrix $\mathbf{M}(\mathbf{A}_0, \mathbf{Y_h})$ is defined in Appendix A. Therefore, we obtain explicit recurrent formulae for calculation of derivatives $\mathbf{C}^{(\mathbf{h})}$ and $\mathbf{q}^{(\mathbf{h})}$ in the form

$$\begin{aligned}
q_i^{(\mathbf{h})} &= -(\mathbf{D_h}, \mathbf{N}_i), \quad i = 1, \ldots, d, \\
\mathbf{D_h} &= -\mathbf{A}^{(\mathbf{h})} - \sum_{\substack{\mathbf{h}'+\mathbf{h}''=\mathbf{h},\\ \mathbf{h}',\mathbf{h}''\neq\mathbf{h}}} C_{\mathbf{h}}^{\mathbf{h}'} \left(\mathbf{A}^{(\mathbf{h}')}\mathbf{C}^{(\mathbf{h}'')} - \mathbf{C}^{(\mathbf{h}'')} \sum_{i=1}^{d} \mathbf{B}_i q_i^{(\mathbf{h}')} \right), \\
\mathbf{C}^{(\mathbf{h})} &= \mathbf{M}(\mathbf{A}_0, \mathbf{Y_h}) + \sum_{i=1}^{d} \gamma_i \mathbf{R}_i, \qquad (30) \\
\mathbf{Y_h} &= \sum_{i=1}^{d} \mathbf{B}_i q_i^{(\mathbf{h})} + \mathbf{D_h}, \quad \gamma_i = -\left(\mathbf{M}(\mathbf{A}_0, \mathbf{Y_h}),\ \mathbf{R}_i\right).
\end{aligned}$$

Formulae (30) represent the algorithm similar to the that given in [18].

4.1. *Example*

For example, let us consider a two-parameter real matrix family

$$\widetilde{\mathbf{A}}(\mathbf{p}) = \begin{pmatrix} 1-p_1^2 & p_1 & 1+p_1^2 \\ 1-p_2 & p_1p_2 & -1+p_2 \\ 1+p_2^2 & -p_1p_2 & 1+p_2^2 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \qquad (31)$$

The matrix $\widetilde{\mathbf{A}}_0 = \widetilde{\mathbf{A}}(0)$ has the Jordan normal form $\mathbf{A}_0$ consisting of two Jordan blocks of dimensions 2 and 1 corresponding to a double eigenvalue $\lambda = 0$ and a simple eigenvalue $\lambda = 2$. The matrices $\mathbf{A}_0$ and $\mathbf{C}_0$ in the normal form transformation (4) have the form

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{C}_0 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 1 & 0 \\ 0 & -1/2 & 1/2 \end{pmatrix}. \qquad (32)$$

Then the matrix family $\mathbf{A}(\mathbf{p}) = \mathbf{C}_0^{-1}\widetilde{\mathbf{A}}(\mathbf{p})\mathbf{C}_0$, which is a deformation of the normal form $\mathbf{A}_0$, is equal to

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} -p_1 & 1-p_1-p_2+p_1^2 & p_2^2 \\ p_1+p_1p_2 & p_1-p_1^2+p_1p_2 & -p_2^2 \\ p_1-p_1p_2 & p_1-p_1^2-p_1p_2 & 2+p_2^2 \end{pmatrix}. \qquad (33)$$

In this case $d = 3$ and the matrices $\mathbf{B}_i, \mathbf{N}_i$, and $\mathbf{R}_i$, $i = 1, 2, 3$, can be taken as follows [3,12]:

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (34)$$

$$\mathbf{N}_1 = \mathbf{B}_1\big/2, \quad \mathbf{R}_1 = \mathbf{B}_1\big/\sqrt{2}, \quad \mathbf{N}_2 = \mathbf{R}_2^{\mathrm{T}} = \mathbf{B}_2, \quad \mathbf{N}_3 = \mathbf{R}_3 = \mathbf{B}_3.$$

Using (32)–(34) in (30), after three iterations we find

$$\mathbf{B}(\mathbf{q}(\mathbf{p})) = \begin{pmatrix} q_1(\mathbf{p}) & 1 & 0 \\ q_2(\mathbf{p}) & q_1(\mathbf{p}) & 0 \\ 0 & 0 & 2+q_3(\mathbf{p}) \end{pmatrix}, \quad \mathbf{q} = (q_1, q_2, q_3),$$

$$\begin{aligned} q_1(\mathbf{p}) &= -p_1^2/2 + p_1p_2/2 + p_1p_2^2/8 + \mathrm{o}(\|\mathbf{p}\|^3), \\ q_2(\mathbf{p}) &= p_1 - p_1p_2^2/2 + \mathrm{o}(\|\mathbf{p}\|^3), \\ q_3(\mathbf{p}) &= p_2^2 - p_1p_2^2/4 + \mathrm{o}(\|\mathbf{p}\|^3), \end{aligned} \tag{35}$$

$$\mathbf{C}(\mathbf{p}) = \begin{pmatrix} 1 - p_1/2 - p_2/2 & 0 & 0 \\ p_1 & 1 + p_1/2 + p_2/2 & 0 \\ -p_1/2 & -3p_1/4 & 1 \end{pmatrix} + \mathrm{o}(\|\mathbf{p}\|).$$

Since the expression for $\mathbf{C}(\mathbf{p})$ is too large, only the first order terms of the expansion of $\mathbf{C}(\mathbf{p})$ are given in (35). The original matrix family $\widetilde{\mathbf{A}}(\mathbf{p})$ is expressed in the form (5) with the matrices $\mathbf{C}_0$, $\mathbf{C}(\mathbf{p})$, and $\mathbf{B}(\mathbf{q}(\mathbf{p}))$ from (32) and (35).

Using transformation to the miniversal deformation, we can obtain important local information about the behavior of the spectrum of $\widetilde{\mathbf{A}}(\mathbf{p})$. Since $\widetilde{\mathbf{A}}(\mathbf{p}) \sim \mathbf{B}(\mathbf{q}(\mathbf{p}))$, the spectra of the matrices $\widetilde{\mathbf{A}}(\mathbf{p})$ and $\mathbf{B}(\mathbf{q}(\mathbf{p}))$ coincide. From (35) it follows that $2 + q_3(\mathbf{p})$ represents the value of the perturbed simple eigenvalue $\lambda = 2$. The bifurcation diagram (the set of values of $\mathbf{p}$, where the matrix $\widetilde{\mathbf{A}}(\mathbf{p})$ has a multiple eigenvalue) is given by the equation $q_2(\mathbf{p}) = 0$, and the multiple (double) eigenvalue on the bifurcation diagram is equal to $q_1(\mathbf{p})$.

5. Versal deformations of elements of classical Lie and Jordan algebras and infinitesimally reversible matrices

Let us consider a classical Lie algebra M under the adjoint action of a corresponding classical Lie group G [8,24]. Following notations of [10], we define

$$\begin{aligned} M &= \left\{\mathbf{A} \in \mathrm{gl}(m, \mathbb{D}) \colon\ \mathbf{K}\mathbf{A}^\dagger + \mathbf{A}\mathbf{K} = 0\right\}, \\ G &= \left\{\mathbf{C} \in \mathrm{GL}(m, \mathbb{D}) \colon\ \mathbf{C}\mathbf{K}\mathbf{C}^\dagger = \mathbf{K}\right\}, \end{aligned} \tag{36}$$

where $\mathbf{K} \in \mathrm{gl}(m, \mathbb{D})$ is a nonsingular matrix such that $\mathbf{K}^\dagger = \varepsilon\mathbf{K}$, $\varepsilon = \pm 1$, and the matrix $\mathbf{A}^\dagger$ is defined as $\mathbf{A}^\dagger = \sigma(\mathbf{A}^{\mathrm{T}})$ with σ being the operator of complex (quaternionic) conjugation or the identity operator. Choosing different $\mathbf{K}, \varepsilon = \pm 1, \sigma$, and $\mathbb{D}, \mathbb{F}$, we get different types of classical Lie algebras M with the involution $\alpha(\mathbf{A}) = \mathbf{K}\mathbf{A}^\dagger\mathbf{K}^{-1}$ ($\alpha(\mathbf{A}) = -\mathbf{A}$ for $\mathbf{A} \in M$) and the corresponding classical Lie groups G; see Table 1 [10].

The normal form theory for classical Lie algebras suggests using transformation (4) with a matrix $\mathbf{C}_0 \in \mathrm{GL}(m, \mathbb{D})$. Then an element $\widetilde{\mathbf{A}}$ of a Lie algebra $\widetilde{M}$ determined

Table 1
Classical Lie algebras with involutions

Algebra M	Group G	m	$\mathbb{D}$	$\mathbb{F}$	$\sigma(\eta)$	ε	Signature of $\mathbf{K}$
$\mathrm{o}(p,q)$	$\mathrm{O}(p,q)$	$p+q$	$\mathbb{R}$	$\mathbb{R}$	η	1	(p,q)
$\mathrm{sp}(2k,\mathbb{R})$	$\mathrm{SP}(2k,\mathbb{R})$	$2k$	$\mathbb{R}$	$\mathbb{R}$	η	-1	
$\mathrm{o}(k,\mathbb{C})$	$\mathrm{O}(k,\mathbb{C})$	k	$\mathbb{C}$	$\mathbb{C}$	η	1	
$\mathrm{sp}(2k,\mathbb{C})$	$\mathrm{SP}(2k,\mathbb{C})$	$2k$	$\mathbb{C}$	$\mathbb{C}$	η	-1	
$\mathrm{u}(p,q)$	$\mathrm{U}(p,q)$	$p+q$	$\mathbb{C}$	$\mathbb{R}$	$\overline{\eta}$	1	(p,q)
$\mathrm{sp}(p,q)$	$\mathrm{SP}(p,q)$	$p+q$	$\mathbb{H}$	$\mathbb{R}$	$\overline{\eta}$	1	(p,q)
$\mathrm{o}^*(2k)$	$\mathrm{O}^*(2k)$	k	$\mathbb{H}$	$\mathbb{R}$	$\overline{\eta}$	-1	

by a matrix $\widetilde{\mathbf{K}}$ is transformed to an element $\mathbf{A}_0$ of another Lie algebra M determined by a matrix $\mathbf{K}$ such that

$$\widetilde{\mathbf{A}} = \mathbf{C}_0\mathbf{A}_0\mathbf{C}_0^{-1}, \quad \widetilde{\mathbf{K}} = \mathbf{C}_0\mathbf{K}\mathbf{C}_0^{\dagger}. \tag{37}$$

Therefore, normal forms of elements of classical Lie algebras are represented by pairs $(\mathbf{A},\ \mathbf{K})$, where both $\mathbf{A}$ and $\mathbf{K}$ have simple form and $\mathbf{K}$ defines the Lie algebra. Such normal forms are studied and listed in [10]. Note that the same matrix $\mathbf{C}_0$ transforms a deformation of the matrix $\widetilde{\mathbf{A}}_0$ in the space $\widetilde{M}$ to a deformation of the normal form $\mathbf{A}_0$ in the space M (6).

Miniversal deformations of elements of classical Lie algebras (transformed to the normal form) are given in [21,23] together with explicit forms of the centralizers $\mathrm{Cent}(\mathbf{A}_0)$. In this case $N\mathrm{Orb}(\mathbf{A}_0) = (\overline{\mathrm{Cent}(\mathbf{A}_0)})^{\mathrm{T}}$ under the scalar product considered, i.e., $N\mathrm{Orb}(\mathbf{A}_0) = \{\overline{\mathbf{X}}^{\mathrm{T}} :\ \mathbf{X} \in \mathrm{Cent}(\mathbf{A}_0)\}$. Hence, it is straightforward to construct the matrices $\mathbf{B}_i$, $\mathbf{N}_i$, and $\mathbf{R}_i$.

The manifold $G \subset \mathrm{GL}(m,\mathbb{D})$ is defined by the equation

$$\mathbf{Q}(\mathbf{C}) = \mathbf{C}\mathbf{K}\mathbf{C}^{\dagger} - \mathbf{K} = 0. \tag{38}$$

The tangent space to G at $\mathbf{I}$ is determined by

$$\mathrm{d}\mathbf{Q}(\mathbf{X}) = \mathbf{K}\mathbf{X}^{\dagger} + \mathbf{X}\mathbf{K} = 0, \tag{39}$$

i.e., $TG = M$. Let $\mathbf{C}(\mathbf{p})$ be a deformation of $\mathbf{I}$ in G. Taking the derivative of order $|\mathbf{h}| > 0$ of (38), we obtain

$$(\mathbf{Q}(\mathbf{C}(\mathbf{p})))^{(\mathbf{h})} = \mathbf{K}\left(\mathbf{C}^{(\mathbf{h})}\right)^{\dagger} + \mathbf{C}^{(\mathbf{h})}\mathbf{K} + \sum_{\substack{\mathbf{h}'+\mathbf{h}''=\mathbf{h},\\ \mathbf{h}',\,\mathbf{h}''\neq\mathbf{h}}} C_{\mathbf{h}}^{\mathbf{h}'}\mathbf{C}^{(\mathbf{h}')}\mathbf{K}\left(\mathbf{C}^{(\mathbf{h}'')}\right)^{\dagger} = 0. \tag{40}$$

A particular solution $\mathbf{C}^{(\mathbf{h})} = \mathbf{C}_{\mathbf{h}}$ of (40) can be taken in the form

$$\mathbf{C}_{\mathbf{h}} = -\frac{1}{2}\sum_{\substack{\mathbf{h}'+\mathbf{h}''=\mathbf{h},\\ \mathbf{h}',\,\mathbf{h}''\neq\mathbf{h}}} C_{\mathbf{h}}^{\mathbf{h}'}\mathbf{C}^{(\mathbf{h}')}\mathbf{K}\left(\mathbf{C}^{(\mathbf{h}'')}\right)^{\dagger}\mathbf{K}^{-1}, \tag{41}$$

which can be checked by substitution of (41) into (40) and using equality $\mathbf{K}^\dagger = \varepsilon\mathbf{K}$, $\varepsilon = \pm 1$.

To complete the procedure of Theorem 1, we need to find a particular solution $\mathbf{X_h}$ of the equation

$$[\mathbf{A}_0, \mathbf{X_h}] = \mathbf{Y_h}, \quad \mathbf{X_h} \in TG = M, \tag{42}$$

where $\mathbf{Y_h} \in M$ is defined in (24) and the existence of the solution is assumed. Using the matrix $\mathbf{M}(\mathbf{A}_0, \mathbf{Y_h})$ from Appendix A, we can take $\mathbf{X_h}$ in the form

$$\mathbf{X_h} = \tfrac{1}{2}(\mathbf{M}(\mathbf{A}_0, \mathbf{Y_h}) - \mathbf{K}\,(\mathbf{M}(\mathbf{A}_0, \mathbf{Y_h}))^\dagger\, \mathbf{K}^{-1}). \tag{43}$$

Substituting (43) into equations $[\mathbf{A}_0, \mathbf{X_h}] = \mathbf{Y_h}$, $\mathbf{K}\mathbf{X}_\mathbf{h}^\dagger + \mathbf{X_h}\mathbf{K} = 0$ and using equalities $\mathbf{K}\mathbf{A}_0^\dagger + \mathbf{A}_0\mathbf{K} = 0$, $\mathbf{K}\mathbf{Y}_\mathbf{h}^\dagger + \mathbf{Y_h}\mathbf{K} = 0$, and $[\mathbf{A}_0, \mathbf{M}(\mathbf{A}_0, \mathbf{Y_h})] = \mathbf{Y_h}$, it can be easily shown that $\mathbf{X_h}$ is a particular solution of (42).

Therefore, we completed the recurrent procedure for finding derivatives $\mathbf{C}^{(\mathbf{h})}$ and $\mathbf{q}^{(\mathbf{h})}$ in the case of classical Lie algebras. This procedure consists of expressions of Theorem 1 and formulae (41) and (43).

The same formulae (41) and (43) can be used in the case of classical Jordan algebras determined by the expressions

$$\begin{aligned} M &= \left\{\mathbf{A} \in \mathrm{gl}(m, \mathbb{D}):\ \mathbf{K}\mathbf{A}^\dagger - \mathbf{A}\mathbf{K} = 0\right\}, \\ G &= \left\{\mathbf{C} \in \mathrm{GL}(m, \mathbb{D}):\ \mathbf{C}\mathbf{K}\mathbf{C}^\dagger = \mathbf{K}\right\}. \end{aligned} \tag{44}$$

Miniversal deformations and centralizers of elements of classical Jordan algebras are given in [22]. Note that in this case the tangent space $TG = \{\mathbf{X} \in \mathrm{gl}(m, \mathbb{D}):\ \mathbf{K}\mathbf{X}^\dagger + \mathbf{X}\mathbf{K} = 0\}$ is the corresponding Lie algebra determined by the matrix $\mathbf{K}$.

Analogously, we can study miniversal deformations of infinitesimally reversible matrices

$$\begin{aligned} M &= \left\{\mathbf{A} \in \mathrm{gl}(m, \mathbb{R}):\ \mathbf{K}\mathbf{A} + \mathbf{A}\mathbf{K} = 0\right\}, \\ G &= \left\{\mathbf{C} \in \mathrm{GL}(m, \mathbb{R}):\ \mathbf{K}\mathbf{C} - \mathbf{C}\mathbf{K} = 0\right\}, \end{aligned} \tag{45}$$

where the matrix $\mathbf{K} \in \mathrm{GL}(m, \mathbb{R})$ satisfies the condition $\mathbf{K}^2 = \mathbf{I}$. Normal forms and miniversal deformations of infinitesimally reversible matrices are listed in [27]. In this case the matrices $\mathbf{C_h}$ and $\mathbf{X_h}$ can be chosen as follows:

$$\begin{aligned} \mathbf{C_h} &= 0, \\ \mathbf{X_h} &= \tfrac{1}{2}\left(\mathbf{M}(\mathbf{A}_0, \mathbf{Y_h}) + \mathbf{K}\,\mathbf{M}(\mathbf{A}_0, \mathbf{Y_h})\,\mathbf{K}\right). \end{aligned} \tag{46}$$

5.1. *Example*

Let us consider the symplectic Lie algebra $\widetilde{M} = \mathrm{sp}(4, \mathbb{R})$ with the corresponding symplectic Lie group $\widetilde{G} = \mathrm{SP}(4, \mathbb{R})$, where the matrix $\widetilde{\mathbf{K}}$ is as follows:

$$\widetilde{\mathbf{K}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{47}$$

The matrices $\mathbf{A} \in \mathrm{sp}(2k, \mathbb{R})$ are also called Hamiltonian matrices. Let $\widetilde{\mathbf{A}}(\mathbf{p})$, $\mathbf{p} = (p_1, p_2)$, be a two-parameter matrix family in $\widetilde{M}$ of the form

$$\widetilde{\mathbf{A}}(\mathbf{p}) = \begin{pmatrix} 0 & \rho & 1 & 0 \\ -\rho & 0 & 0 & 1 \\ 3 + p_1 - \rho^2 & 0 & 0 & \rho \\ 0 & 4p_1 - \rho^2 & -\rho & 0 \end{pmatrix}, \tag{48}$$

where $\rho = \sqrt{(4 + p_1)(3/4 + p_2)}\big/2$. Matrix family (48) corresponds to the Hamiltonian equation $\dot{\mathbf{x}} = \widetilde{\mathbf{A}}(\mathbf{p})\mathbf{x}$ describing oscillations of a simply supported elastic pipe conveying fluid with $p_1 = \Lambda - 4$, $p_2 = \alpha - 3/4$; Λ is the parameter proportional to a squared velocity of the fluid; α depends on the ratio of masses of the tube and the fluid [19,29].

The matrix $\widetilde{\mathbf{A}}_0 = \widetilde{\mathbf{A}}(0)$ has the quadruple zero eigenvalue with the corresponding Jordan block of dimension 4. Hence, the normal form $(\mathbf{A}_0, \mathbf{K})$ for the matrices $\widetilde{\mathbf{A}}_0, \widetilde{\mathbf{K}}$ is [10]

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{49}$$

Calculating the Jordan chain for the zero eigenvalue of $\widetilde{\mathbf{A}}_0$ (an eigenvector and associated vectors), we find the matrix $\mathbf{C}_0$ in transformation (37) as follows:

$$\mathbf{C}_0 = \begin{pmatrix} 0 & 1 & 0 & 1/6 \\ -\sqrt{3} & 0 & \sqrt{3}/6 & 0 \\ 3/2 & 0 & 3/4 & 0 \\ 0 & -\sqrt{3}/2 & 0 & \sqrt{3}/4 \end{pmatrix}. \tag{50}$$

The versal deformation corresponding to $(\mathbf{A}_0, \mathbf{K})$ depends on two parameters $\mathbf{q} = (q_1, q_2)$ and has the form $\mathbf{B}(\mathbf{q}) = \mathbf{A}_0 + q_1\mathbf{B}_1 + q_2\mathbf{B}_2$. The matrices $\mathbf{B}_i, \mathbf{N}_i, \mathbf{R}_i$, $i = 1, 2$, can be taken in the form [23]

$$\mathbf{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{N}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{N}_1 = \mathbf{B}_1, \quad \mathbf{R}_1 = \mathbf{B}_1^{\mathrm{T}}, \quad \mathbf{R}_2 = \mathbf{N}_2^{\mathrm{T}}\big/\sqrt{3}.$$

The original matrix family $\widetilde{\mathbf{A}}(\mathbf{p})$ is represented using the versal deformation as follows:

$$\widetilde{\mathbf{A}}(\mathbf{p}) = \mathbf{C}_0\mathbf{C}(\mathbf{p})\mathbf{B}(\mathbf{q}(\mathbf{p}))\mathbf{C}^{-1}(\mathbf{p})\mathbf{C}_0^{-1}.$$

Using the recurrent procedure of Theorem 1 and formulae (41) and (43), we can find Taylor expansions of the functions $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$ up to the terms of arbitrary order. The results of calculations for $\mathbf{q}(\mathbf{p}) = (q_1(\mathbf{p}), q_2(\mathbf{p}))$ are as follows:

$$\mathbf{B}(\mathbf{q}(\mathbf{p})) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q_2(\mathbf{p}) & 0 & 1 \\ q_1(\mathbf{p}) & 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} q_1(\mathbf{p}) = -12p_1 - 4p_1^2, \\ q_2(\mathbf{p}) = 17p_1/4 - 4p_2 - p_1p_2. \end{array} \tag{51}$$

Calculations show that the higher order terms in expansions of $q_1(\mathbf{p})$ and $q_2(\mathbf{p})$ are zeros.

Eigenvalues of the matrices $\widetilde{\mathbf{A}}(\mathbf{p})$ and $\mathbf{B}(\mathbf{q}(\mathbf{p}))$ coincide. The characteristic equation for the miniversal deformation has the form

$$|\mathbf{B}(\mathbf{q}(\mathbf{p})) - \lambda\mathbf{I}| = \lambda^4 - q_2(\mathbf{p})\lambda^2 - q_1(\mathbf{p}). \tag{52}$$

Hence, the stability domain (the set of values of the parameters $\mathbf{p}$ such that all eigenvalues of $\widetilde{\mathbf{A}}(\mathbf{p})$ are purely imaginary and semisimple) in the vicinity of $\mathbf{p} = 0$ is described by the inequalities

$$D(\mathbf{p}) = (q_2(\mathbf{p}))^2 + 4q_1(\mathbf{p}) > 0, \quad q_1(\mathbf{p}) < 0, \;\; q_2(\mathbf{p}) < 0. \tag{53}$$

Direct calculations show that relations (51) and (53) determine exact form of the stability domain near $\mathbf{p} = 0$ [19].

Though the stability analysis for simple matrix family (48) can be performed without miniversal deformations, the same method can be used for a matrix family of any dimension $m = 2k$ dependent on arbitrary number of parameters n. Due to the block-diagonal structure of the versal deformation, the stability conditions (in the case, when $\widetilde{\mathbf{A}}_0$ has the quadruple zero eigenvalue with the Jordan block of dimension 4) have the same form (53), where Taylor series of the functions $q_1(\mathbf{p})$ and $q_2(\mathbf{p})$ can be found by the explicit recurrent procedure.

6. Partial transformation to a versal deformation

Transformation of a matrix family $\widetilde{\mathbf{A}}(\mathbf{p})$ to a versal deformation (5) consists of two steps: transformation of $\widetilde{\mathbf{A}}_0$ to the normal form (4) and then computation of the functions $\mathbf{C}(\mathbf{p}), \mathbf{q}(\mathbf{p})$ by the recurrent procedure of Theorem 1. It should be noted that though the first step (transformation of the matrix $\widetilde{\mathbf{A}}_0$ to the normal form) is clear from the theoretical point of view, it represents a very complicated problem of numerical computations due to its instability (high sensitivity of multiple eigenvalues to perturbations of parameters) [3,30]. Below it will be shown how in some cases we can avoid a considerable part of calculations associated with the normal form transformation.

Applications of versal deformations are based on the fact that the spectra of the matrices $\mathbf{A}(\mathbf{p})$ and $\mathbf{B}(\mathbf{q}(\mathbf{p}))$ coincide, while $\mathbf{B}(\mathbf{q}(\mathbf{p}))$ has a very simple structure.

In many stability and dynamics problems we are interested only in a part of the spectrum, which lies, for example, on the imaginary axis or the unit circle. Versal deformations have usually a block-diagonal structure. Hence, we need for applications only the blocks of $\mathbf{B}(\mathbf{q}(\mathbf{p}))$ corresponding to the eigenvalues under consideration while the information about other blocks in not important. Let us modify the recurrent procedure for calculations of a specific block of $\mathbf{B}(\mathbf{q}(\mathbf{p}))$ in the case $M = \mathrm{gl}(m, \mathbb{D})$, $G = \mathrm{GL}(m, \mathbb{D})$.

Eq. (5) of the transformation to the versal deformation can be written in the form

$$\widetilde{\mathbf{A}}(\mathbf{p})\mathbf{S}(\mathbf{p}) = \mathbf{S}(\mathbf{p})\mathbf{B}(\mathbf{q}(\mathbf{p})), \quad \mathbf{S}(\mathbf{p}) = \mathbf{C}_0\mathbf{C}(\mathbf{p}). \tag{54}$$

Let the versal deformation be a block-diagonal matrix family

$$\mathbf{B}(\mathbf{q}) = \begin{pmatrix} \mathbf{B}'(\mathbf{q}') & \\ & \mathbf{B}''(\mathbf{q}'') \end{pmatrix}, \quad \mathbf{q} = (\mathbf{q}', \mathbf{q}''). \tag{55}$$

We will be interested in the $s \times s$ block

$$\mathbf{B}'(\mathbf{q}'(\mathbf{p})) = \mathbf{A}'_0 + \sum_{i=1}^{d'} \mathbf{B}'_i q'_i(\mathbf{p}), \tag{56}$$

where the vector $\mathbf{q}' = (q'_1, \ldots, q'_{d'})$ has dimension d'; $\mathbf{A}'_0$ is a part of the Jordan normal form $\mathbf{A}_0 = \mathbf{B}(0) = \mathrm{diag}(\mathbf{A}'_0, \mathbf{A}''_0)$ (in the cases $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$ the matrix $\mathbf{A}_0$ represents the real or quaternionic equivalent of the Jordan normal form [10]).

Eq. (54) splits into two independent parts corresponding to the blocks $\mathbf{B}'$ and $\mathbf{B}''$, where the first part takes the form

$$\widetilde{\mathbf{A}}(\mathbf{p})\mathbf{S}'(\mathbf{p}) = \mathbf{S}'(\mathbf{p})\mathbf{B}'(\mathbf{q}'(\mathbf{p})). \tag{57}$$

Here $\mathbf{S}'$ is an $m \times s$ matrix consisting of the first s columns of $\mathbf{S}$. Let us assume that the matrix $\mathbf{A}'_0$ is known as well as the $m \times s$ matrices $\mathbf{S}'_0 = \mathbf{S}'(0)$ and $\mathbf{V}$ satisfying equations

$$\widetilde{\mathbf{A}}_0\mathbf{S}'_0 = \mathbf{S}'_0\mathbf{A}'_0, \quad \mathbf{V}^{\mathrm{T}}\widetilde{\mathbf{A}}_0 = \mathbf{A}'_0\mathbf{V}^{\mathrm{T}}, \quad \mathbf{V}^{\mathrm{T}}\mathbf{S}'_0 = \mathbf{I}. \tag{58}$$

Columns of the matrices $\mathbf{S}'_0$ and $\mathbf{V}$ form the right and left Jordan chains corresponding to the part $\mathbf{A}'_0$ of the Jordan normal form. The last equality of (58) represents the normalization condition uniquely determining $\mathbf{V}$ for given $\mathbf{S}'_0$. As in Section 3 let us define $s \times s$ matrices $\mathbf{B}'_i$, $\mathbf{N}'_i$, and $\mathbf{R}'_i$, $i = 1, \ldots, d'$, for the matrix $\mathbf{A}'_0$, where $\{\mathbf{B}'_i\}$ is a basis of some complement of $T\mathrm{Orb}(\mathbf{A}'_0)$ in $\mathrm{gl}(s, \mathbb{D})$, $\{\mathbf{N}'_i\}$ is a basis of $N\mathrm{Orb}(\mathbf{A}'_0)$ satisfying conditions $(\mathbf{N}'_i, \mathbf{B}'_j) = \delta_{ij}$, and $\{\mathbf{R}'_i\}$ is a basis of $\mathrm{Cent}(\mathbf{A}'_0) = \overline{(N\mathrm{Orb}(\mathbf{A}'_0))}^{\mathrm{T}}$ satisfying $(\mathbf{R}'_i, \mathbf{R}'_j) = \delta_{ij}$. Taking derivative of Eq. (57) with respect to the parameters, we find

$$\widetilde{\mathbf{A}}_0\mathbf{S}'^{(\mathbf{h})} - \mathbf{S}'^{(\mathbf{h})}\mathbf{A}'_0 = \mathbf{Y}'_{\mathbf{h}}, \tag{59}$$

where

$$\mathbf{Y}'_{\mathbf{h}} = \mathbf{S}'_0 \sum_{i=1}^{d'} \mathbf{B}'_i q_i'^{(\mathbf{h})} + \mathbf{D}'_{\mathbf{h}},$$
$$\mathbf{D}'_{\mathbf{h}} = -\mathbf{A}^{(\mathbf{h})}\mathbf{S}'_0 - \sum_{\substack{\mathbf{h}'+\mathbf{h}''=\mathbf{h},\\ \mathbf{h}',\mathbf{h}''\neq\mathbf{h}}} C_{\mathbf{h}}^{\mathbf{h}'} \left(\widetilde{\mathbf{A}}^{(\mathbf{h}')}\mathbf{S}'^{(\mathbf{h}'')} - \mathbf{S}'^{(\mathbf{h}'')} \sum_{i=1}^{d'} \mathbf{B}'_i q_i'^{(\mathbf{h}')} \right). \tag{60}$$

Multiplication of (59) by $\mathbf{V}^{\mathrm{T}}$ from left with the use of (58) yields

$$\left[\mathbf{A}'_0, \mathbf{V}^{\mathrm{T}}\mathbf{S}'^{(\mathbf{h})}\right] = \mathbf{V}^{\mathrm{T}}\mathbf{Y}'_{\mathbf{h}}. \tag{61}$$

Hence, the right-hand side of (61) belongs to the tangent space $T\mathrm{Orb}(\mathbf{A}'_0)$. Using the basis $\mathbf{N}'_i$, $i = 1, \ldots, d'$, of the normal complement $N\mathrm{Orb}(\mathbf{A}'_0)$ and the first expression of (60), we obtain

$$q_i'^{(\mathbf{h})} = -\left(\mathbf{V}^{\mathrm{T}}\mathbf{D}'_{\mathbf{h}}, \mathbf{N}'_i\right), \quad i = 1, \ldots, d'. \tag{62}$$

Substituting (62) into (60), we find the matrix $\mathbf{Y}_{\mathbf{h}}$ standing in the right-hand side of (59). Then the matrix $\mathbf{S}'^{(\mathbf{h})}$ has the form

$$\mathbf{S}'^{(\mathbf{h})} = \mathbf{M}'(\widetilde{\mathbf{A}}_0, \mathbf{A}'_0, \mathbf{Y}'_{\mathbf{h}}) + \mathbf{S}'_0 \sum_{i=1}^{d'} \gamma_i \mathbf{R}'_i, \tag{63}$$

where the matrix $\mathbf{M}'(\widetilde{\mathbf{A}}_0, \mathbf{A}'_0, \mathbf{Y}'_{\mathbf{h}})$ is a particular solution of (59) defined in Appendix A. Using relations $\mathbf{S}(\mathbf{p}) = \mathbf{C}_0\mathbf{C}(\mathbf{p})$ and (58) in conditions (17), we find

$$\begin{aligned}
(\mathbf{C}^{(\mathbf{h})}, \mathbf{R}_i) &= \mathrm{trace}((\mathbf{C}^{(\mathbf{h})})^{\mathrm{T}}\overline{\mathbf{R}}_i) \\
&= \mathrm{trace}((\mathbf{S}^{(\mathbf{h})})^{\mathrm{T}}(\mathbf{C}_0^{-1})^{\mathrm{T}}\overline{\mathbf{R}}_i) \\
&= \mathrm{trace}((\mathbf{S}'^{(\mathbf{h})})^{\mathrm{T}}\mathbf{V}\overline{\mathbf{R}}'_i) \\
&= (\mathbf{V}^{\mathrm{T}}\mathbf{S}'^{(\mathbf{h})}, \mathbf{R}'_i) = 0,
\end{aligned} \tag{64}$$

where the $s \times s$ upper left block of the $m \times m$ matrix $\mathbf{R}_i$ is equal to $\mathbf{R}'_i$ and other entries of $\mathbf{R}_i$ are zeros. Substituting (63) into (64), we find the coefficients γ_i in the form

$$\gamma_i = -\left(\mathbf{V}^{\mathrm{T}}\mathbf{M}'(\widetilde{\mathbf{A}}_0, \mathbf{A}'_0, \mathbf{Y}'_{\mathbf{h}}), \mathbf{R}'_i\right), \quad i = 1, \ldots, d'. \tag{65}$$

Expressions (60), (62), (63), and (65) represent the explicit recurrent procedure for calculation of Taylor expansions of the function $\mathbf{q}'(\mathbf{p})$ in the block of the versal deformation $\mathbf{B}'(\mathbf{q}'(\mathbf{p}))$ and the corresponding part $\mathbf{S}'(\mathbf{p})$ of the matrix family $\mathbf{S}(\mathbf{p})$. The advantage of the described method is that we do not need the whole Jordan normal form $\mathbf{A}_0$ and the calculations are restricted to a part of the versal deformation under consideration.

Similar procedures can be obtained in some other cases of M and G, when the equation $\mathbf{f}(\mathbf{S}) = 0$ determining G can be divided into two independent parts $\mathbf{f}'(\mathbf{S}') = 0$ and $\mathbf{f}''(\mathbf{S}'') = 0$ corresponding to $m \times s$ and $m \times (m - s)$ blocks of the matrix $\mathbf{S} = [\mathbf{S}', \mathbf{S}'']$.

6.1. Example

Let the matrix $\widetilde{\mathbf{A}}_0 \in \mathrm{gl}(m, \mathbb{R})$ have a real double eigenvalue λ_0 with the Jordan chain $\mathbf{u}_1, \mathbf{u}_2$ (an eigenvector and an associated vector) satisfying equation $\widetilde{\mathbf{A}}_0\mathbf{U} - \mathbf{U}\mathbf{A}'_0 = 0$, where $\mathbf{A}'_0 = \mathbf{J}_{\lambda_0}(2)$ is the Jordan block of dimension 2 with the eigenvalue λ_0, and $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2]$. Let $\mathbf{v}_1, \mathbf{v}_2$ be the left Jordan chain (an eigenvector and an associated vector of the matrix $\widetilde{\mathbf{A}}_0^{\mathrm{T}}$) satisfying equations $\mathbf{V}^{\mathrm{T}}\widetilde{\mathbf{A}}_0 - \mathbf{A}'_0\mathbf{V}^{\mathrm{T}} = 0$, $\mathbf{V}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$, where $\mathbf{V} = [\mathbf{v}_2, \mathbf{v}_1]$. In this case the versal deformation $\mathbf{B}(\mathbf{q})$ has a block [3,12]

$$\mathbf{B}'(\mathbf{q}') = \begin{pmatrix} \lambda_0 + q'_1 & 1 \\ q'_2 & \lambda_0 + q'_1 \end{pmatrix}, \quad \mathbf{q}' = (q'_1, q'_2), \tag{66}$$

and we can take $\mathbf{S}'_0 = \mathbf{U}$, $\mathbf{N}'_1 = \mathbf{B}'_1/2$, $\mathbf{R}'_1 = \mathbf{B}'_1\big/\sqrt{2}$, $\mathbf{N}'_2 = (\mathbf{R}'_2)^{\mathrm{T}} = \mathbf{B}'_2$. Using the recurrent procedure of this section, we can find Taylor series for the functions $q'_1(\mathbf{p})$, $q'_2(\mathbf{p})$, and the $m \times 2$ matrix $\mathbf{S}'(\mathbf{p})$. In particular, for matrix family (31) we will obtain the expansions of $q_1(\mathbf{p})$ and $q_2(\mathbf{p})$ given in (35).

Perturbation of the double eigenvalue λ_0 in the vicinity of $\mathbf{p} = 0$ is determined by the characteristic equation $|\mathbf{B}'(\mathbf{q}'(\mathbf{p})) - \lambda\mathbf{I}| = 0$, which yields

$$\lambda = \lambda_0 + q'_1(\mathbf{p}) \pm \sqrt{q'_2(\mathbf{p})}. \tag{67}$$

On the surface $q'_2(\mathbf{p}) = 0$, where the eigenvalue $\lambda = \lambda_0 + q'_1(\mathbf{p})$ is double, columns of the matrix $\mathbf{S}'(\mathbf{p})$ represent the corresponding Jordan chain. Note that the advantage of the suggested method is that for local multi-parameter analysis of a double eigenvalue we need only the corresponding right and left Jordan chains of the matrix $\widetilde{\mathbf{A}}_0$ and derivatives of the matrix family $\widetilde{\mathbf{A}}(\mathbf{p})$ at $\mathbf{p} = 0$. As a result, we get information about the bifurcation of the double eigenvalue λ_0 (67) and the corresponding eigenspace determined by the columns of $\mathbf{S}'(\mathbf{p})$ with the accuracy up to the terms of any order.

Similarly, we can study an arbitrary multiple eigenvalue. Note that in the case of a simple eigenvalue λ_0 a block $\mathbf{B}'(\mathbf{q}') = \lambda_0 + q'_1$ and, hence, the recurrent procedure gives the Taylor expansion of the simple eigenvalue $\lambda = \lambda_0 + q'_1(\mathbf{p})$ and the corresponding eigenvector $\mathbf{S}'(\mathbf{p})$.

7. Conclusion

In this paper the problem of finding the functions $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$ transforming a matrix family to a versal deformation is considered. A general method for construction of recurrent procedures, which allow determining the functions $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$ in the form of Taylor series, is developed and used for the cases of real and complex matrices, elements of classical Lie and Jordan algebras, and infinitesimally reversible matrices. Transformation to the versal deformation consists of determining the versal deformation $\mathbf{B}(\mathbf{q})$ and the transformation functions $\mathbf{C}(\mathbf{p})$, $\mathbf{q}(\mathbf{p})$. Therefore, computation of the functions $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$ represents the important part of the versal deformation theory (like calculation of a change of basis in the transformation of a matrix to the Jordan normal form). The suggested approach can be extended to the case of versal deformations studied in [6,11,15,16], where pairs, triples, or quadruples of matrices were considered and different equivalence transformations were used instead of the change of basis.

Results of this paper extend the application area of the versal deformation theory. Information about the transformation functions $\mathbf{C}(\mathbf{p})$ and $\mathbf{q}(\mathbf{p})$ can be used for analytical and numerical analysis of multi-parameter matrix families in stability, dynamics, and perturbation problems.

Appendix A

A.1. *Equation* $[\mathbf{A}, \mathbf{X}] = \mathbf{Y}$

Let us consider the equation

$$[\mathbf{A}, \mathbf{X}] = \mathbf{Y}, \quad \mathbf{X} \in \mathrm{gl}(m, \mathbb{C}), \tag{A.1}$$

where $\mathbf{A}$ and $\mathbf{Y}$ are given matrices from $\mathrm{gl}(m, \mathbb{C})$. It is assumed that the solution $\mathbf{X}$ of (A.1) exists, i.e., $\mathbf{Y}$ belongs to $T\,\mathrm{Orb}(\mathbf{A})$ in $\mathrm{gl}(m, \mathbb{C})$.

First, let us study the case, when $\mathbf{A}$ is the Jordan normal form $\mathbf{A} = \mathrm{diag}(\mathbf{J}_1, \ldots, \mathbf{J}_s)$, $\mathbf{J}_i = \mathrm{diag}(\mathbf{J}_{\lambda_i}(m_i^1), \ldots, \mathbf{J}_{\lambda_i}(m_i^{n_i}))$, and

$$\mathbf{J}_{\lambda_i}(m_i^j) = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda_i \end{pmatrix} \tag{A.2}$$

is the $m_i^j \times m_i^j$ Jordan block corresponding to the eigenvalue λ_i; the eigenvalues $\lambda_1, \ldots, \lambda_s$ are different; $m_i^1 \geqslant \cdots \geqslant m_i^{n_i}$ are sizes of the Jordan blocks corresponding to the eigenvalue λ_i. The Jordan structure of $\mathbf{A}$ induces a partition of an $m \times m$ matrix $\mathbf{X}$ into the blocks $\mathbf{X}_{ij}^{kl}$. The $m_i^k \times m_j^l$ block $\mathbf{X}_{ij}^{kl}$ stands on the intersection of rows corresponding to $\mathbf{J}_{\lambda_i}(m_i^k)$ and columns corresponding to $\mathbf{J}_{\lambda_j}(m_j^l)$. Elements of $\mathbf{X}_{ij}^{kl}$ are denoted by $x_{ij}^{kl}(r, s)$, $r = 1, \ldots, m_i^k$, $s = 1, \ldots, m_j^l$. Analogous notations are used for the matrix $\mathbf{Y}$. Then the particular solution of (A.1) can be found in the form [18]

$$x_{ij}^{kl}(r, s) = \sum_{r_1=0}^{m_i^k - r} \sum_{s_1=0}^{s-1} \frac{(-1)^{r_1} C_{r_1+s_1}^{r_1}}{(\lambda_i - \lambda_j)^{r_1+s_1+1}}\, y_{ij}^{kl}(r + r_1, s - s_1), \quad i \neq j,$$

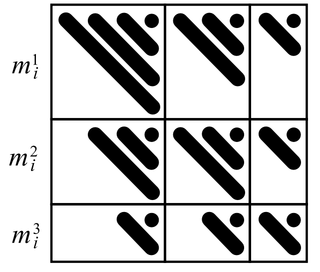

Fig. 1 Structure of matrices commuting with the Jordan normal form.

$$x_{ii}^{kl}(r,s) = \begin{cases} 0, & k \leqslant l, \quad r = 1, \\ \displaystyle\sum_{s_1=s'(r,s)}^{s} y_{ii}^{kl}(r-s+s_1-1, s_1), & k \leqslant l, \quad r \neq 1, \\ \displaystyle -\sum_{r_1=r}^{r'(r,s)} y_{ii}^{kl}(r_1, s-r+r_1+1), & k > l, \quad s \neq m_i^l, \\ 0, & k > l, \quad s = m_i^l, \end{cases} \tag{A.3}$$

$$s'(r,s) = \max(1, s-r+2), \; r'(r,s) = \min(m_i^k, r-s+m_i^l-1).$$

The matrix $\mathbf{X}$ given by (A.3) is denoted by $\mathbf{M}(\mathbf{A}, \mathbf{Y})$. Note that another algorithm for finding a particular solution of (A.1) for the Jordan normal form $\mathbf{A}$ was suggested in [28].

The general solution of (A.1) is a sum $\mathbf{X} = \mathbf{M}(\mathbf{A}, \mathbf{Y}) + \mathbf{Z}$ of the particular solution and a general solution of the homogeneous equation $\mathbf{Z} \in \mathrm{Cent}(\mathbf{A}) = \{\mathbf{Z} \in \mathrm{gl}(m, \mathbb{C})\colon\; [\mathbf{A}, \mathbf{Z}] = 0\}$. The general form of the matrix $\mathbf{Z}$, commuting with the Jordan normal form $\mathbf{A}$, is $\mathbf{Z} = \mathrm{diag}(\mathbf{Z}_1, \ldots, \mathbf{Z}_s)$, where each block $\mathbf{Z}_i$ has the form shown in Fig. 1 [14]. Blocks in Fig. 1 correspond to the Jordan structure of $\mathbf{J}_i$; each slanted segment is filled by equal complex numbers and blank places are zeros.

For an arbitrary matrix $\widetilde{\mathbf{A}}$ the particular solution of equation $[\widetilde{\mathbf{A}}, \mathbf{X}] = \mathbf{Y}$ can be found in the form $\mathbf{X} = \mathbf{M}(\widetilde{\mathbf{A}}, \mathbf{Y}) = \mathbf{C}\,\mathbf{M}(\mathbf{A}, \mathbf{C}^{-1}\mathbf{Y}\mathbf{C})\,\mathbf{C}^{-1}$, where $\mathbf{A}$ is the Jordan normal form of $\widetilde{\mathbf{A}} = \mathbf{C}\mathbf{A}\mathbf{C}^{-1}$. Similarly, the general form of a matrix $\widetilde{\mathbf{Z}}$ commuting with $\widetilde{\mathbf{A}}$ is $\widetilde{\mathbf{Z}} = \mathbf{C}\mathbf{Z}\mathbf{C}^{-1}$, where $\mathbf{Z}$ is a matrix commuting with the Jordan normal form $\mathbf{A}$.

The same formulae for the particular solution $\mathbf{M}(\widetilde{\mathbf{A}}, \mathbf{Y})$ can be used in the real and quaternionic cases. Here it can be shown that the matrix $\mathbf{M}(\widetilde{\mathbf{A}}, \mathbf{Y})$ is real for real matrices $\widetilde{\mathbf{A}}$ and $\mathbf{Y}$, if we choose the columns of $\mathbf{C}$ corresponding to complex conjugate eigenvalues to be complex conjugate [18]. In the case $\mathbb{D} = \mathbb{H}$ we should use the representation of quaternions by 2×2 complex matrices [10]. Then the matrix $\widetilde{\mathbf{A}} \in \mathrm{gl}(m, \mathbb{H})$ is substituted by the corresponding matrix from $\mathrm{gl}(2m, \mathbb{C})$ and the Jordan normal form transformation $\widetilde{\mathbf{A}} = \mathbf{C}\mathbf{A}\mathbf{C}^{-1}$ is chosen such that the obtained $2m \times 2m$ complex matrix $\mathbf{M}(\widetilde{\mathbf{A}}, \mathbf{Y})$ keeps the “quaternionic” structure.

A.2. Equation $\widetilde{\mathbf{A}}\mathbf{X} - \mathbf{X}\mathbf{A}' = \mathbf{Y}$

Let us consider the equation

$$\widetilde{\mathbf{A}}\mathbf{X} - \mathbf{X}\mathbf{J}_\lambda(s) = \mathbf{Y}, \tag{A.4}$$

where $\widetilde{\mathbf{A}}$ and $\mathbf{Y}$ are given complex matrices of dimensions $m \times m$ and $m \times s$; $\mathbf{X}$ is an unknown complex matrix of dimension $m \times s$; $\mathbf{J}_\lambda(s)$ is the Jordan block of dimension s corresponding to the eigenvalue λ of the matrix $\widetilde{\mathbf{A}}$. Eq. (A.4) can be written in the form

$$\begin{aligned} (\widetilde{\mathbf{A}} - \lambda\mathbf{I})\mathbf{x}_1 &= \mathbf{y}_1, \\ (\widetilde{\mathbf{A}} - \lambda\mathbf{I})\mathbf{x}_2 &= \mathbf{x}_1 + \mathbf{y}_2, \\ &\vdots \\ (\widetilde{\mathbf{A}} - \lambda\mathbf{I})\mathbf{x}_s &= \mathbf{x}_{s-1} + \mathbf{y}_s, \end{aligned} \tag{A.5}$$

where $\mathbf{x}_i$ and $\mathbf{y}_i$, $i = 1, \ldots, s$, are columns of the $m \times s$ matrices $\mathbf{X}$ and $\mathbf{Y}$. Since λ is the eigenvalue of $\widetilde{\mathbf{A}}$, the matrix $\widetilde{\mathbf{A}} - \lambda\mathbf{I}$ is singular. Let $\mathbf{q}_1, \ldots, \mathbf{q}_t$ be a basis of the null space of the matrix $\widetilde{\mathbf{A}} - \lambda\mathbf{I}$ and $\mathbf{r}_1, \ldots, \mathbf{r}_t$ be a basis of the null space of the matrix $\overline{(\widetilde{\mathbf{A}} - \lambda\mathbf{I})}^{\mathrm{T}}$.

We assume that the solution $\mathbf{X}$ of (A.4) exists. In the paper this condition is fulfilled automatically by the construction of the matrix $\mathbf{Y}$. We refer the reader interested in necessary and sufficient conditions for the existence of solution of (A.4) to [5] and references therein. Using (A.5), a particular solution of (A.4) can be found in the form $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_s]$, where

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{P}^{-1}\mathbf{y}_1, \\ \mathbf{x}_2 &= \mathbf{P}^{-1}(\mathbf{x}_1 + \mathbf{y}_2), \\ &\vdots \\ \mathbf{x}_s &= \mathbf{P}^{-1}(\mathbf{x}_{s-1} + \mathbf{y}_s). \end{aligned} \tag{A.6}$$

Here $\mathbf{P} = \widetilde{\mathbf{A}} - \lambda\mathbf{I} - \mathbf{R}\overline{\mathbf{Q}}^{\mathrm{T}}$ ($\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_t]$ and $\mathbf{R} = [\mathbf{r}_1, \ldots, \mathbf{r}_t]$) is a nonsingular matrix [31]. Note that if λ is not an eigenvalue of $\widetilde{\mathbf{A}}$, then $\mathbf{P} = \widetilde{\mathbf{A}} - \lambda\mathbf{I}$. Let us denote the obtained particular solution by $\mathbf{M}'(\widetilde{\mathbf{A}}, \mathbf{J}_\lambda(s), \mathbf{Y}) = [\mathbf{x}_1, \ldots, \mathbf{x}_s]$.

Let us consider the equation

$$\widetilde{\mathbf{A}}\mathbf{X} - \mathbf{X}\mathbf{J} = \mathbf{Y}, \quad \mathbf{J} = \operatorname{diag}(\mathbf{J}_{\lambda_1}(s_1), \ldots, \mathbf{J}_{\lambda_q}(s_q)), \tag{A.7}$$

where $\mathbf{J}$ is the Jordan normal form of dimension $s = s_1 + \cdots + s_q$ with the blocks $\mathbf{J}_{\lambda_i}(s_i)$; λ_i are not necessarily different eigenvalues. A particular solution of (A.7) can be found in the form $\mathbf{M}'(\widetilde{\mathbf{A}}, \mathbf{J}, \mathbf{Y}) = [\mathbf{M}'(\widetilde{\mathbf{A}}, \mathbf{J}_{\lambda_1}(s_1), \mathbf{Y}_1), \ldots, \mathbf{M}'(\widetilde{\mathbf{A}}, \mathbf{J}_{\lambda_q}(s_q), \mathbf{Y}_q)]$, where $\mathbf{Y}_i$ is the $s \times s_i$ part of the matrix $\mathbf{Y} = [\mathbf{Y}_1, \ldots, \mathbf{Y}_q]$.

If $\mathbf{A}'$ is an arbitrary $s \times s$ matrix, then a particular solution of the equation $\widetilde{\mathbf{A}}\mathbf{X} - \mathbf{X}\mathbf{A}' = \mathbf{Y}$ has the form

$$\mathbf{M}'(\widetilde{\mathbf{A}}, \mathbf{A}', \mathbf{Y}) = \mathbf{M}'(\widetilde{\mathbf{A}}, \mathbf{J}, \mathbf{Y}\mathbf{W})\,\mathbf{W}^{-1}, \tag{A.8}$$

where $\mathbf{A}' = \mathbf{W}\mathbf{J}\mathbf{W}^{-1}$ and $\mathbf{J}$ is the Jordan normal form of the matrix $\mathbf{A}'$.

The same formulae can be used for the particular solution $\mathbf{M}'(\widetilde{\mathbf{A}}, \mathbf{A}', \mathbf{Y})$ in the real and quaternionic cases. Here we should appropriately choose the matrix $\mathbf{W}$ in the case $\mathbb{D} = \mathbb{R}$ (columns of $\mathbf{W}$ corresponding to complex conjugate eigenvalues should be complex conjugate) and apply the 2×2 matrix representation of quaternions in the case $\mathbb{D} = \mathbb{H}$.

Note that $\mathbf{M}'(\mathbf{A}, \mathbf{A}, \mathbf{Y})$ is a particular solution of (A.1), but generally $\mathbf{M}'(\mathbf{A}, \mathbf{A}, \mathbf{Y}) \neq \mathbf{M}(\mathbf{A}, \mathbf{Y})$.

References

[1] V.I. Arnold, On matrices depending on parameters, Russian Math. Surveys 26 (1971) 29–43.
[2] V.I. Arnold, Lectures on bifurcations and versal families, Russian Math. Surveys 27 (1972) 54–123.
[3] V.I. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations, Springer, New York, 1983.
[4] M.B. Sevryuk (Ed.), Arnold's Problems, PHASIS, Moscow, 2000 (in Russian).
[5] M.A. Beitia, J.M. Gracia, Sylvester matrix equation for matrix pencils, Linear Algebra Appl. 232 (1996) 155–197.
[6] J.M. Berg, H.G. Kwatny, A canonical parametrization of the Kronecker form of a matrix pencil, Automatica 31 (1995) 669–680.
[7] J.V. Burke, M.L. Overton, Stable perturbations of nonsymmetric matrices, Linear Algebra Appl. 171 (1992) 249–273.
[8] C. Chevalley, Theory of Lie Groups. I, Princeton University Press, Princeton, 1946.
[9] R. Cushman, A. Kelley, H. Koçak, Versal normal form at the Lagrange equilibrium L_4, J. Differential Equations 64 (1986) 340–374.
[10] D.Z. Djoković, J. Patera, P. Winternitz, H. Zassenhaus, Normal forms of elements of classical real and complex Lie and Jordan algebras, J. Math. Phys. 24 (1983) 1363–1374.
[11] A. Edelman, E. Elmroth, B. Kågström, A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations, SIAM J. Matrix Anal. Appl. 18 (1997) 653–692.
[12] D.M. Galin, On real matrices depending on parameters, Uspekhi Mat. Nauk 27 (1972) 241–242 (in Russian).
[13] D.M. Galin, Versal deformations of linear Hamiltonian systems, Amer. Math. Soc. Transl. 118 (1982) 1–12.
[14] F.R. Gantmacher, The Theory of Matrices, vol. I, Chelsea, New York, 1959.
[15] M.I. García-Planas, D. Magret, Deformation and stability of triples of matrices, Linear Algebra Appl. 254 (1997) 159–192.
[16] M.I. García-Planas, V.V. Sergeichuk, Simplest miniversal deformations of matrices, matrix pencils, and contragredient matrix pencils, Linear Algebra Appl. 302–303 (1999) 45–61.
[17] A.A. Mailybaev, On tangent cones to the stability domain of a family of real matrices, Moscow Univ. Mech. Bulletin 53 (1998) 20–22.
[18] A.A. Mailybaev, Transformation of families of matrices to normal forms and its application to stability theory, SIAM J. Matrix Anal. Appl. 21 (2000) 396–417.
[19] A.A. Mailybaev, A.P. Seiranyan, The stability domains of Hamiltonian systems, J. Appl. Math. Mech. 63 (1999) 545–555.
[20] A.A. Mailybaev, A.P. Seyranian, On singularities of a boundary of the stability domain, SIAM J. Matrix Anal. Appl. 21 (2000) 106–128.
[21] J. Patera, C. Rousseau, Complex orthogonal and symplectic matrices depending on parameters, J. Math. Phys. 23 (1982) 705–714.
[22] J. Patera, C. Rousseau, Versal deformations of elements of classical Jordan algebras, J. Math. Phys. 24 (1983) 1375–1380.

[23] J. Patera, C. Rousseau, D. Schlomiuk, Versal deformations of elements of real classical Lie algebras, J. Phys. A: Math. Gen. 15 (1982) 1063–1086.
[24] M. Postnikov, Lie Groups and Lie Algebras, Mir, Moscow, 1986.
[25] D.S. Schmidt, Transformations to versal normal form, in: K.R. Meyer, D.S. Schmidt (Eds.), Computer Aided Proofs in Analysis, IMA Vol. Math. Appl., vol. 28, Springer, New York, 1989, pp. 235–240.
[26] D. Schmidt, Versal normal form of the Hamiltonian function of the restricted problem of three bodies near L_4, J. Comput. Appl. Math. 52 (1994) 155–176.
[27] M.B. Sevryuk, Reversible linear systems and their versal deformations, J. Soviet Math. 60 (1992) 1663–1680.
[28] L. Stolovitch, On the computation of a versal family of matrices, Numer. Algorithms 4 (1993) 25–46.
[29] J.M.T. Thompson, Instabilities and Catastrophes in Science and Engineering, Wiley, New York, 1982.
[30] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.
[31] V.A. Yakubovich, V.M. Starzhinskii, Parametric Resonance in Linear Systems, Nauka, Moscow, 1987 (in Russian).