Z. angew. Math. Phys. 52 (2001) 669–679
0010-2571/01/040669-11 \$ 1.50+0.20/0

Zeitschrift für angewandte Mathematik und Physik ZAMP

On stability boundaries of conservative systems

Alexander P. Seyranian and Alexei A. Mailybaev

Abstract. Stability boundaries of linear conservative systems smoothly dependent on several parameters are studied. Generic singularities appearing on the stability boundaries are classified. Explicit formulae for the approximations to the stability domain at regular and singular points of the boundary are derived. These formulae use information on the system only at the point under consideration (eigenvectors and derivatives of the stiffness matrix with respect to parameters). As an example a buckling problem of a column loaded by an axial force is considered and discussed in detail.

Mathematics Subject Classification (2000). 70J25.

Keywords. conservative system, stability boundary, singularity, multiple eigenvalue, perturbation.

Introduction

Let us consider a linear autonomous conservative system of the form

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\mathbf{y} = 0, \tag{1}$$

where $\mathbf{y}$ is an m-dimensional vector of generalized coordinates, $\mathbf{M}$ is a positive definite symmetrical mass matrix, $\mathbf{C}$ is a symmetrical stiffness matrix, and dots denote derivatives with respect to time. The matrices $\mathbf{M}$ and $\mathbf{C}$ are assumed to depend smoothly on a vector $\mathbf{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$ of real parameters.

We seek a solution of equation (1) in the form $\mathbf{y} = \mathbf{u}\exp(i\sqrt{\lambda}t)$. Substitution of this expression into equation (1) yields the eigenvalue problem

$$\mathbf{C}\mathbf{u} = \lambda\mathbf{M}\mathbf{u}. \tag{2}$$

Due to the symmetry of the matrices $\mathbf{C}$ and $\mathbf{M}$ all eigenvalues λ are real and semi-simple, that is there are r linearly independent eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ corresponding to the eigenvalue λ with the algebraic multiplicity r. The system is stable if and only if all the eigenvalues are positive $\lambda > 0$. Note that due to positive definiteness of the matrix $\mathbf{M}$ stability of the system is equivalent to positive definiteness of the matrix $\mathbf{C}$. If at least one of the eigenvalues is zero or negative (the matrix $\mathbf{C}$ is not positive definite) the system is unstable. This stability criterion defines stability and instability domains in the parameter space.

Because of continuous dependence of eigenvalues on parameters a boundary of the stability domain is determined by the zero eigenvalue, while all other eigenvalues are positive. The zero eigenvalue can be simple or multiple (semi-simple). In the latter case we denote a corresponding boundary point by $0(r)$, where $r > 1$ is the multiplicity of $\lambda = 0$.

Qualitative and quantitative analysis of the stability domains for conservative systems is a classical subject. It is of great practical importance due to applications to mechanical and civil engineering problems. Stability domains in the case of linear dependence of the matrix $\mathbf{C}$ on parameters have been studied by Papkovich [7]. He has shown that the stability boundary can not have convexity toward the stability domain, and has found a tangent plane to the stability boundary at a regular point, characterized by a simple zero eigenvalue. However the nature of singularities of the stability domains connected with multiple eigenvalues has not been studied and understood in full. Note that an account of Papkovich's work is given in [8, 5].

Using Papkovich's approach Huseyin [5] has shown that the fundamental frequency surface has concavity toward the fundamental region in the case of the matrix $\mathbf{C}$ linearly depending on parameters and constant matrix $\mathbf{M}$.

Singularities that can appear on the stability boundaries or fundamental frequency surfaces are connected with multiple eigenvalues. Arnold [1] has found a number of parameters required for the appearence of eigenvalues of a given multiplicity in the generic (typical) case.

In this paper we consider a general case of smooth dependence of system matrices on parameters. Using the results of Arnold [1] and a perturbation technique for eigenvalues [3, 9], a classification of singularities of stability boundaries in the generic case is obtained. A constructive method for quantitative analysis of singularities is proposed. For regular boundary points a general expression for the approximation of the stability boundary surface up to the second order terms is found. At singular (irregular) points tangent cones (first order approximations) to the stability domain are derived. As an example a buckling problem of a column loaded by an axial force is studied and the corresponding stability domain is analyzed.

1. Regular points of the stability boundary

Consider a point $\mathbf{p} = \mathbf{p}_0$ on the stability boundary corresponding to the simple zero eigenvalue $\lambda = 0$. We denote $\mathbf{M}_0 = \mathbf{M}(\mathbf{p}_0)$, $\mathbf{C}_0 = \mathbf{C}(\mathbf{p}_0)$. Let $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\varepsilon \geq 0$ be an arbitrary smooth curve in the parameter space starting at the point $\mathbf{p}(0) = \mathbf{p}_0$:

$$\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \epsilon\mathbf{e} + \varepsilon^2\mathbf{d} + o(\varepsilon^2) \tag{3}$$

where ε is a small parameter, $\mathbf{e} = d\mathbf{p}/d\varepsilon$, $\mathbf{d} = \frac{1}{2}d^2\mathbf{p}/d\varepsilon^2$ (these derivatives are evaluated at $\varepsilon = 0$). Then along this curve the eigenvalue λ and the corresponding eigenvector can be represented as series of integer powers of ε. The perturbed eigenvalue λ takes the form [3, 9]

$$\lambda = \left[\mathbf{u}^{\mathrm{T}}\frac{d\mathbf{C}}{d\varepsilon}\mathbf{u}\right]\varepsilon + o(\varepsilon) = \left[\mathbf{u}^{\mathrm{T}}\sum_{i=1}^{n}\frac{\partial\mathbf{C}}{\partial p_i}\frac{dp_i}{d\varepsilon}\mathbf{u}\right]\varepsilon + o(\varepsilon) = (\mathbf{g},\mathbf{e})\varepsilon + o(\varepsilon),$$
$$\mathbf{g} = \left(\mathbf{u}^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial p_1}\mathbf{u},\ldots,\mathbf{u}^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial p_n}\mathbf{u}\right)^{\mathrm{T}}, \tag{4}$$

where $\mathbf{u}$ is an eigenvector corresponding to the zero eigenvalue ($\mathbf{C}_0\mathbf{u} = 0$) and satisfying the normalization condition $\mathbf{u}^{\mathrm{T}}\mathbf{M}_0\mathbf{u} = 1$; $\mathbf{g}$ is a gradient vector of the eigenvalue; $(\mathbf{a},\mathbf{b})$ is the scalar product in the parameter space. If $(\mathbf{g},\mathbf{e}) = 0$, then we get [3]

$$\lambda = \left[\frac{1}{2}\mathbf{u}^{\mathrm{T}}\frac{d^2\mathbf{C}}{d\varepsilon^2}\mathbf{u} + \mathbf{v}^{\mathrm{T}}\frac{d\mathbf{C}}{d\varepsilon}\mathbf{u}\right]\varepsilon^2 + o(\varepsilon^2)$$
$$= \left[\frac{1}{2}\mathbf{u}^{\mathrm{T}}\left(\sum_{i=1}^{n}\frac{\partial\mathbf{C}}{\partial p_i}\frac{d^2p_i}{d\varepsilon^2} + \sum_{i,j=1}^{n}\frac{\partial^2\mathbf{C}}{\partial p_i\partial p_j}\frac{dp_i}{d\varepsilon}\frac{dp_j}{d\varepsilon}\right)\mathbf{u} + \mathbf{v}^{\mathrm{T}}\sum_{i=1}^{n}\frac{\partial\mathbf{C}}{\partial p_i}\frac{dp_i}{d\varepsilon}\mathbf{u}\right]\varepsilon^2 \tag{5}$$
$$+ o(\varepsilon^2) = [(\mathbf{g},\mathbf{d}) + (\mathbf{Ge},\mathbf{e}) + \eta]\,\varepsilon^2 + o(\varepsilon^2).$$

The matrix $\mathbf{G}$ and the scalar quantity η are given by

$$\mathbf{G} = \left[\frac{1}{2}\mathbf{u}^{\mathrm{T}}\frac{\partial^2\mathbf{C}}{\partial p_i\partial p_j}\mathbf{u}\right], \quad i,j = 1,\ldots,n,$$
$$\eta = (\mathbf{f},\mathbf{e}), \quad \mathbf{f} = \left(\mathbf{v}^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial p_1}\mathbf{u},\ldots,\mathbf{v}^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial p_n}\mathbf{u}\right)^{\mathrm{T}}, \tag{6}$$

where $\mathbf{v}$ is a vector satisfying the equation

$$\mathbf{C}_0\mathbf{v} = -\sum_{i=1}^{n}\frac{\partial\mathbf{C}}{\partial p_i}\mathbf{u}\,e_i. \tag{7}$$

Note that $\mathbf{u}+\varepsilon\mathbf{v}$ is the first order approximation of the eigenvector corresponding to the eigenvalue λ [3].

Equation (7) can be solved in several manners. One way is to find the vector $\mathbf{v}$ as a linear combination of all eigenvectors of the matrix $\mathbf{C}_0$. Then the expression for η takes the form [3]

$$\eta = -\sum_{i=2}^{m} (\mathbf{f}_i, \mathbf{e})^2/\lambda_i = (\mathbf{F}\mathbf{e}, \mathbf{e}),$$
$$\mathbf{f}_i = \left(\mathbf{u}_i^{\mathrm{T}} \frac{\partial \mathbf{C}}{\partial p_1} \mathbf{u}, \ldots, \mathbf{u}_i^{\mathrm{T}} \frac{\partial \mathbf{C}}{\partial p_n} \mathbf{u} \right)^{\mathrm{T}}, \quad \mathbf{F} = -\sum_{i=2}^{m} \mathbf{f}_i \mathbf{f}_i^{\mathrm{T}}/\lambda_i, \tag{8}$$

where $\mathbf{u}_i$, $i = 2, \ldots, m$ are eigenvectors corresponding to the positive eigenvalues $0 < \lambda_2 \le \cdots \le \lambda_m$. Derivatives are taken at $\mathbf{p} = \mathbf{p}_0$.

To use expression (8) we need to find all eigenvalues and eigenvectors of the matrix $\mathbf{C}_0$. But we can avoid these calculations by solving equation (7) with the method described in [12, 11]. Then we get a solution to (7) in the form

$$\mathbf{v} = -\mathbf{A}_0^{-1} \sum_{i=1}^{n} \frac{\partial \mathbf{C}}{\partial p_i} \mathbf{u}\, e_i, \quad \mathbf{A}_0 = \mathbf{C}_0 - \mathbf{u}\mathbf{u}^{\mathrm{T}}, \tag{9}$$

where $\mathbf{A}_0$ is the nonsingular matrix. Substituting (9) into (6) we obtain

$$\eta = (\mathbf{F}\mathbf{e}, \mathbf{e}), \quad \mathbf{F} = \left[-\mathbf{u}^{\mathrm{T}} \frac{\partial \mathbf{C}}{\partial p_i} \mathbf{A}_0^{-1} \frac{\partial \mathbf{C}}{\partial p_j} \mathbf{u} \right], \; i, j = 1, \ldots, n. \tag{10}$$

Using equation (10) in (5) and taking into account $(\mathbf{g}, \mathbf{e}) = 0$ we find the stability boundary ($\lambda = 0$) up to the second order terms

$$(\mathbf{g}, \Delta\mathbf{p}) + ((\mathbf{F} + \mathbf{G})\Delta\mathbf{p}, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|^2) = 0, \quad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0. \tag{11}$$

Thus, the stability boundary is a smooth surface in the vicinity of the point $\mathbf{p} = \mathbf{p}_0$, the gradient $\mathbf{g}$ being the normal vector to the stability boundary directed into the stability domain, see Fig. 1.

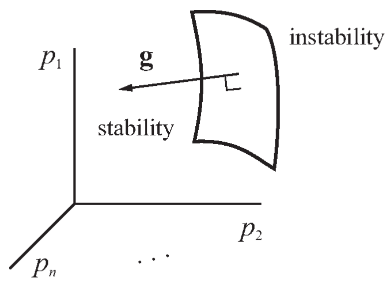

Figure 1. The normal vector $\mathbf{g}$ to the stability boundary lying in the stability domain.

The tangent plane to the stability boundary at $\mathbf{p}_0$ is determined by the equation $(\mathbf{g}, \Delta\mathbf{p}) = 0$. Considering curves lying in the tangent plane ($(\mathbf{g}, \mathbf{e}) = 0$, $(\mathbf{g}, \mathbf{d}) = 0$) and using (5), (10) we get $\lambda = ((\mathbf{F} + \mathbf{G})\mathbf{e}, \mathbf{e})\varepsilon^2 + o(\varepsilon^2)$. Hence, the stability domain is convex at $\mathbf{p} = \mathbf{p}_0$ ($\lambda < 0$ on the tangent plane) if the matrix $\mathbf{F} + \mathbf{G}$ is negative definite, and concave if $\mathbf{F} + \mathbf{G}$ is positive definite.

If the second order derivatives of the matrix $\mathbf{C}$ with respect to parameters are all zeros then $\mathbf{G} = 0$, and according to (8) we have $\mathbf{F} \leq 0$. Hence the stability domain is convex, this result agreeing with Papkovich's theorem on convexity of the stability domain of a conservative system linearly dependent on parameters [7, 5].

2. Singularities of the stability boundary

Multiple zero eigenvalues determine singularities of the stability boundary (points where the boundary surface is not smooth). One of the basic concepts of singularity and bifurcation theory, allowing constructive study of singularities, is the notion of general position. According to it for a fixed dimension n of the parameter space only some singularities of the type $0(r)$, $r > 1$ are generic (typical). Other (nongeneric) singularities though artificially created in specific examples, disappear if we take an arbitrarily small perturbation of the family $\mathbf{C} + \delta\mathbf{C}$ (caused for example by numerical errors) [2]. Thus, when studying the stability boundary the generic singularities are most interesting. Arnold [1] has shown that in the case of general position the singularity $0(r)$ appears if the number of parameters n is equal or greater than $r(r+1)/2$. In particular, this means that in the case of two parameters there are no singularities and the stability boundary is a smooth curve with the normal $\mathbf{g}$ directed into the stability domain.

The simplest singularity $0(2)$ can appear if we have $r(r+1)/2 = 3$ (or more) parameters. In this case $r = 2$, i.e. there are two linearly independent eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, corresponding to the zero eigenvalue, which can be chosen satisfying the normalization conditions

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_j = \delta_{ij}, \quad i,j = 1,2, \tag{12}$$

where δ_{ij} is the Kronecker delta. It should be noted that normalization conditions can be taken in different ways. They do not influence the final result since the stability doesn't depend on $\mathbf{M}$.

Let $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\varepsilon \geq 0$ be an arbitrary smooth curve in the parameter space starting at the point $\mathbf{p}(0) = \mathbf{p}_0$ with a direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$ (the derivative is taken at $\varepsilon = 0$). Then a semi-simple double zero eigenvalue splits into two simple eigenvalues $\lambda = \varepsilon\mu + o(\varepsilon)$, where two different values of μ are found from the quadratic equation [10]

$$\det[(\mathbf{r}_{ij}, \mathbf{e}) - \mu\delta_{ij}] = 0, \quad i,j = 1,2.$$

The real vectors $\mathbf{r}_{ij} \in \mathrm{R}^n$, $i,j = 1,2$ are determined by the expression

$$\mathbf{r}_{ij} = \left(\mathbf{u}_i^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial p_1}\mathbf{u}_j, \ldots, \mathbf{u}_i^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial p_n}\mathbf{u}_j\right)^{\mathrm{T}}. \tag{13}$$

Due to the symmetry of the matrix $\mathbf{C}$ we have $\mathbf{r}_{12} = \mathbf{r}_{21}$. For stability we need both roots μ to be nonnegative. This means that the 2×2 matrix $[(\mathbf{r}_{ij}, \mathbf{e})]$ should be positive semi-definite. Using Sylvester's conditions we obtain

$$(\mathbf{r}_{11} + \mathbf{r}_{22}, \mathbf{e}) \geq 0, \quad (\mathbf{r}_{11}, \mathbf{e})(\mathbf{r}_{22}, \mathbf{e}) - (\mathbf{r}_{12}, \mathbf{e})^2 \geq 0. \tag{14}$$

In the case of three parameters ($n = 3$) these inequalities determine a cone in the vicinity of the point $\mathbf{p}_0$ in the parameter space. In fact, after introduction of new variables

$$x = (\mathbf{r}_2, \mathbf{e}), \quad y = (\mathbf{r}_{12}, \mathbf{e}), \quad z = (\mathbf{r}_1, \mathbf{e}), \tag{15}$$

where

$$\mathbf{r}_1 = \frac{\mathbf{r}_{11} + \mathbf{r}_{22}}{2}, \quad \mathbf{r}_2 = \frac{\mathbf{r}_{22} - \mathbf{r}_{11}}{2}, \tag{16}$$

the approximation to the stability domain (14) takes the form of the cone $x^2+y^2 \leq z^2$, $z \geq 0$, see Fig. 2a.

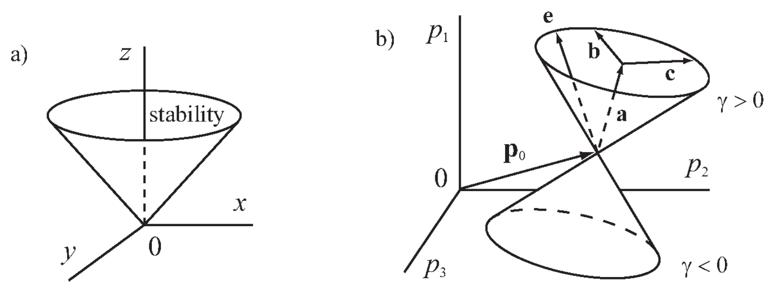

Figure 2. Singularity 0(2) of the stability domain boundary (the cone): a) in the parameter space (x, y, z) ; b) in the parameter space $\mathbf{p}$ (upper part of the cone for $\gamma > 0$ and lower part for $\gamma < 0$).

The conical surface $x^2+y^2 = z^2$, $z \geq 0$, approximating the stability boundary, can be parameterized as follows $x = z\cos\alpha$, $y = z\sin\alpha$, $z \geq 0$. Using (15) we get the equations

$$(\mathbf{r}_2 - \mathbf{r}_1\cos\alpha, \mathbf{e}) = 0, \quad (\mathbf{r}_{12} - \mathbf{r}_1\sin\alpha, \mathbf{e}) = 0, \quad (\mathbf{r}_1, \mathbf{e}) \geq 0. \tag{17}$$

Then the vector $\mathbf{e}$ can be represented in the form

$$\mathbf{e} = t(\mathbf{r}_2 - \mathbf{r}_1\cos\alpha) \times (\mathbf{r}_{12} - \mathbf{r}_1\sin\alpha) =$$
$$= t(\mathbf{r}_2 \times \mathbf{r}_{12} - \mathbf{r}_2 \times \mathbf{r}_1\sin\alpha - \mathbf{r}_1 \times \mathbf{r}_{12}\cos\alpha),$$

where t and α are real parameters. From the third expression of (17) we obtain the inequality $t(\mathbf{r}_1, \mathbf{r}_2 \times \mathbf{r}_{12}) \geq 0$, which means that $t \geq 0$ or $t \leq 0$ depending on the sign of $(\mathbf{r}_1, \mathbf{r}_2 \times \mathbf{r}_{12})$.

Thus, the cone (14) can be written in the parameterized form

$$\mathrm{K} = \{\, \mathbf{e} : \ \mathbf{e} = t[\mathbf{a} + d(\mathbf{b}\sin\alpha + \mathbf{c}\cos\alpha)], \ \gamma t > 0, \ d \in [0,1], \ \alpha \in [0, 2\pi] \,\}, \tag{18}$$

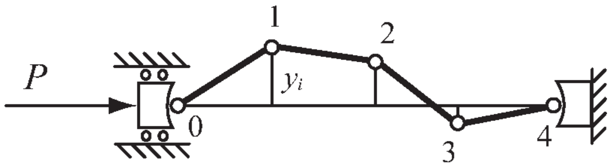

Figure 3. Simple model of a column loaded by an axial force.

where the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and the constant γ are defined as follows

$$\mathbf{a} = \mathbf{r}_2 \times \mathbf{r}_{12},\ \mathbf{b} = \mathbf{r}_1 \times \mathbf{r}_2,\ \mathbf{c} = \mathbf{r}_{12} \times \mathbf{r}_1,\ \gamma = (\mathbf{r}_1, \mathbf{r}_2 \times \mathbf{r}_{12}). \tag{19}$$

The vector $\mathbf{e}$ runs through the cone surface, when $d = 1$ and t, α are changed. The cone surface approximating the stability boundary near $\mathbf{p} = \mathbf{p}_0$ is shown in Fig. 2b, where depending on the sign of γ its upper of lower part is taken (stability domain is inside the corresponding cone part). Note that the normalization conditions (12) that are used in the calculations are not important since the stability does not depend on the mass matrix $\mathbf{M}$. Taking eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ not satisfying (12) we get other vectors $\mathbf{r}_{11}$, $\mathbf{r}_{12}$, $\mathbf{r}_{22}$ (and also $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$), but the cone determined by expressions (14) or (18) remains unchanged.

3. Buckling problem of a column loaded by an axial force

Let us consider a finite dimensional model of a column, consisted of four equal links of the length l, and loaded by an axial force P, see Fig. 3. A bending moment in the i-th node is proportional to $a_i^2\theta_i$, where a_i is the cross-section area of the column at the i-th node, θ_i is the angle between links in the i-th node. Taking into account boundary conditions $y_0 = y_4 = 0$ the system has three degrees of freedom determined by components of the vector of generalized coordinates $\mathbf{y} = (y_1, y_2, y_3)^{\mathrm{T}}$, where y_i is a deflection of the i-th node. The stiffness matrix $\mathbf{C}$ of the system in nondimensional coordinates takes the form [4]

$$\mathbf{C} = \begin{pmatrix} a_0^2 + 4a_1^2 + a_2^2 - 2\mathrm{P} & -2a_1^2 - 2a_2^2 + \mathrm{P} & a_2^2 \\ -2a_1^2 - 2a_2^2 + \mathrm{P} & a_1^2 + 4a_2^2 + a_3^2 - 2\mathrm{P} & -2a_2^2 - 2a_3^2 + \mathrm{P} \\ a_2^2 & -2a_2^2 - 2a_3^2 + \mathrm{P} & a_2^2 + 4a_3^2 + a_4^2 - 2\mathrm{P} \end{pmatrix}. \tag{20}$$

We assume that the cross-section areas at the ends of the column are equal to $a_0 = a_4 = \sqrt{3}$ and the total volume is fixed $a_1 + a_2 + a_3 = 7/2$. Then the stiffness matrix $\mathbf{C}$ depends on three parameters $\mathbf{p} = (a_1, a_3, \mathrm{P})^{\mathrm{T}}$, and the following natural conditions $a_1 > 0$, $a_3 > 0$ and $a_2 = 7/2 - a_1 - a_3 > 0$ are implied.

Let us consider stability of the system in the vicinity of the point $\mathbf{p}_0 =$

$(1, 1, 7/2)$. At this point the matrix $\mathbf{C}$ takes the form

$$\mathbf{C}_0 = \begin{pmatrix} 9/4 & -3 & 9/4 \\ -3 & 4 & -3 \\ 9/4 & -3 & 9/4 \end{pmatrix}.$$

The matrix $\mathbf{C}_0$ possesses the double zero eigenvalue $\lambda = 0$ and the simple eigenvalue $\lambda = 17/2$. Thus, at $\mathbf{p} = \mathbf{p}_0$ we have the singularity $0(2)$ (the cone). The vectors $\mathbf{r}_{11}$, $\mathbf{r}_{12}$, $\mathbf{r}_{22}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and the constant γ can be calculated using formulae (13), (16), (19)

$$\mathbf{r}_{11} = (-2.5, -2.5, -2.5)^{\mathrm{T}},\ \mathbf{r}_{12} = (4, -4, 0)^{\mathrm{T}},\ \mathbf{r}_{22} = (32, 32, -16)^{\mathrm{T}},$$
$$\mathbf{a} = (-27, -27, -138)^{\mathrm{T}},\ \mathbf{b} = (60, -60, 0)^{\mathrm{T}},\ \mathbf{c} = (37, 37, 118)^{\mathrm{T}},$$
$$\gamma = 480 > 0.$$

Then the cone (18), which is the first order approximation of the stability domain at the singular point $\mathbf{p}_0$, can be written as

$$\begin{aligned} \mathrm{K} = \{\, \mathbf{e} :\ \mathbf{e} = (&-27t + 60td\sin\alpha + 37td\cos\alpha, -27t - 60td\sin\alpha + \\ &+37td\cos\alpha, -138t + 118td\cos\alpha)^{\mathrm{T}},\ t > 0,\ d \in [0,1],\ \alpha \in [0, 2\pi]\,\}. \end{aligned} \tag{21}$$

The third component $e_{\mathrm{P}} = -138t + 118td\cos\alpha$ of the vector $\mathbf{e} = (e_1, e_3, e_{\mathrm{P}})^{\mathrm{T}}$ is negative for all $t > 0$, $d \in [0, 1]$. Hence an increment of the load $\Delta\mathrm{P} = \varepsilon e_{\mathrm{P}} + o(\varepsilon)$ inside the stability domain is negative for all small perturbations of the parameters $\Delta a_1 = \varepsilon e_1$, $\Delta a_3 = \varepsilon e_3$, and the critical load P_{cr} attains its maximum at the point $a_1 = a_3 = 1$. The optimal design of the column $a_1 = a_3 = 1$ is called bimodal since there are two eigenvectors (modes) corresponding to P_{cr} [10]. The stability domain boundary calculated numerically is shown in Fig. 4. Numerical analysis confirms existence of the singularity $0(2)$ (the cone). The numerical results are in a good agreement with the first order approximation (21) to the stability domain at $\mathbf{p} = \mathbf{p}_0$.

4. General case

In this section we study singularities $0(r)$ determined by $\lambda = 0$ with an arbitrary multiplicity $r > 1$. Recall that in the generic case such singularities appear if we have $r(r+1)/2$ or more parameters. Let $\mathbf{u}_1, \ldots, \mathbf{u}_r$ be the linear independent eigenvectors corresponding to the zero eigenvalue and satisfying the normalization conditions $\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_j = \delta_{ij}$ for $i, j = 1, \ldots, r$. We define the vectors $\mathbf{r}_{ij}$, $i, j = 1, \ldots, r$ by formula (13). Due to the symmetry of $\mathbf{C}$ we have $\mathbf{r}_{ij} = \mathbf{r}_{ji}$, $i, j = 1, \ldots, r$. As above, along the curve $\mathbf{p} = \mathbf{p}(\varepsilon)$, $\mathbf{p}(0) = \mathbf{p}_0$ with the direction $\mathbf{e} = d\mathbf{p}/d\varepsilon$ a perturbation of the semi-simple eigenvalue $\lambda = 0$ has the form $\lambda = \varepsilon\mu + o(\varepsilon)$. The r different values of μ are the eigenvalues of the symmetrical $r \times r$ matrix $\mathbf{R} = [(\mathbf{r}_{ij}, \mathbf{e})]$, $i, j = 1, \ldots, r$. Stable perturbations are determined

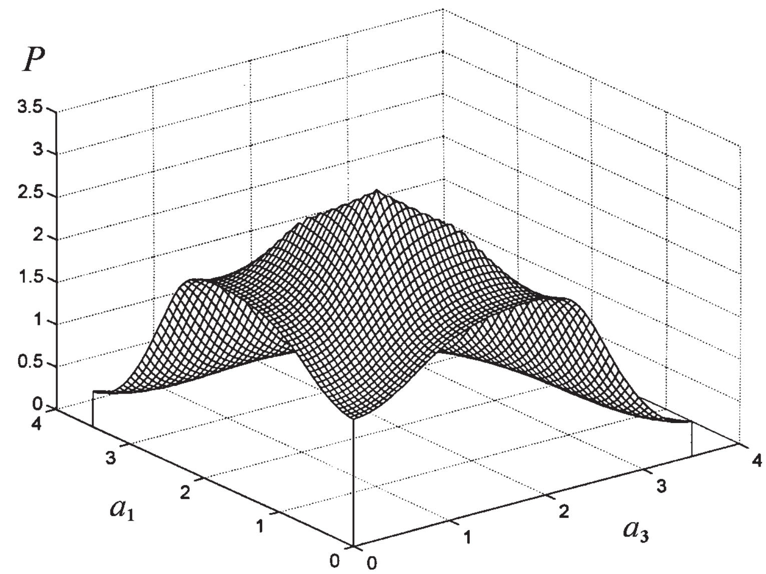

Figure 4. Stability boundary for the column in the parameter space $\mathbf{p}$.

by the condition $\mu \geq 0$ for all eigenvalues of $\mathbf{R}$ or, equivalently, by positive semi-definite matrices $\mathbf{R} \geq 0$. The condition $\mathbf{R} > 0$ defines direction vectors $\mathbf{e}$ in the parameter space penetrating into the stability domain. The vectors $\mathbf{e}$ such that the matrix $\mathbf{R}$ is singular and positive semi-definite are tangent to the stability boundary.

As a result, for an arbitrary singularity $0(r)$ we obtain the set

$$\mathrm{K} = \{\, \mathbf{e} : \ \mathbf{R} = [(\mathbf{r}_{ij}, \mathbf{e})] \geq 0,\ i, j = 1, \ldots, r \,\}, \tag{22}$$

which is the tangent cone to the stability domain at the singular point (a linear approximation). The condition of positive semi-definiteness of the matrix $\mathbf{R}$ can be written in the form of inequalities called Sylvester's conditions [6].

5. Conclusion

In this paper we study boundaries of the stability domains of conservative systems in the parameter space. For generic (typical) cases we classify singularities of the stability boundary depending on multiplicity of the zero eigenvalue, responsible for the loss of stability. It is shown that in the generic case the stability boundary is a smooth curve in two-dimensional parameter space; in the case of three parameters the stability boundary is a smooth surface the only singularity of which is a cone. A full description of the cone being a first order approximation to the stability boundary is given. It is shown that calculation of the cone needs only information at the singular point: first order derivatives of the matrix $\mathbf{C}$ with respect to parameters and eigenvectors corresponding to the zero eigenvalue. For regular points the second order approximation to the stability boundary is obtained. We emphasize that for its calculation only first and second order derivatives of the matrix $\mathbf{C}$ with respect to parameters and the eigenvector are needed. In a general case of the singularity with r-multiple zero eigenvalue we obtain the tangent cone (the first order approximation) to the stability domain at the singular point as a set of vectors $\mathbf{e}$ satisfying conditions for semi-positive-definiteness of the matrix $[(\mathbf{r}_{ij}, \mathbf{e})]$, $i, j = 1, \ldots, r$. Approximations to the stability domain at regular and singular points obtained in this paper have simple and constructive form. They can be used with other mechanical problems.

The method proposed in the paper is general and allows to study not only singularities of stability domains connected with zero eigenvalues, but also to analyze multiple eigenfrequencies and singularities arising on frequency surfaces for conservative systems depending on parameters.

The example given in section 3 shows close connection of singularities of stability boundaries with the bimodal solutions in optimization problems for elastic systems [10]. In particular, it turns out that multimodal optimal solutions are generic (typical). One can say that singular (multimodal) points "attract" optimal solutions, as happens in the case of the singularity "cone" studied in sections 2, 3.

References

[1] V.I. Arnold, *Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York 1978.

[2] V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer-Verlag, New York and Berlin 1983.

[3] A.S. Bratus and A.P. Seyranian, Sufficient conditions for an extremum in eigenvalue optimization problems, *J. Appl. Maths Mechs* **48** (1984), 466-474.

[4] K.K. Choi and E.J. Haug, Optimization of structures with repeated eigenvalues. In: *Optimization of Distributed Parameter Structures*, E.J. Haug and J. Cea, eds, Sijthoff and Noordhoff, Alphen aan den Rijn 1981, 219–277.

[5] K. Huseyin, *Vibration and Stability of Multiple Parameter Systems.* Noordhoff, Alphen aan den Rijn 1978.

[6] G.A. Korn and T.M. Korn, *Mathematical Handbook for Scientists and Engineers.* McGraw-Hill, New York 1968.

[7] P.F. Papkovich, *Works on the Structural Mechanics of Ships. Vol. 4.* Sudpromgiz, Leningrad 1963 (in Russian).

[8] J.D. Renton, *Thin-Walled Structures*, A.H. Chilver, ed., Chatto and Windus, London 1967, 1–59.

[9] A.P. Seyranian, Sensitivity analysis of multiple eigenvalues, *Mechanics of Structures and Machines* **21** (1993), 261–284.

[10] A.P. Seyranian, E. Lund and N. Olhoff, Multiple eigenvalues in structural optimization problems, *Structural Optimization* **8** (1994), 207–227.

[11] A.P. Seyranian, F. Solem and P. Pedersen, Stability analysis for multi-parameter linear periodic systems, *Archive of Appl. Mech.* **69** (1999), 160–180.

[12] V.A. Yakubovitch and V.M. Starzhinskii, *Parametric Resonance in Linear Systems*. Nauka, Moscow 1987 (in Russian).

Alexander P. Seyranian and Alexei A. Mailybaev
Institute of Mechanics
Moscow State Lomonosov University
11719 Moscow
Russian Federation

(Received: July 8, 1999)