

On stability domains of nonconservative systems under small parametric excitation

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Summary. A linear multi-parameter nonconservative system under small periodic parametric excitation is considered. Approximations of the stability domain in the parameter space are derived in the cases, when the corresponding autonomous system has a zero eigenvalue or a pair of complex conjugate imaginary eigenvalues. Formulae of the approximations use information on the unperturbed autonomous system and derivatives of system matrices with respect to parameters. Singularities arising on the stability boundary are analyzed. As a numerical application, stability of a pipe conveying pulsating fluid is studied.

1 Introduction

A wide range of important practical problems in mechanics and physics are modeled by multi-parameter linear systems of ordinary differential equations with periodic coefficients. These problems appear in structural mechanics, acoustics, celestial mechanics, plasma physics etc. Objective of the analysis for such systems is to find the stability domain in the parameter space. Methods of stability analysis of periodic systems can be grouped into three categories: the Floquet method [1], [2], Bolotin's method [3], [4], and analytic methods [5], [6], [7], [8]. More references can be found in the cited papers and books; see also [9], [10], [11], [12]. In many cases small parametric excitation applied to an autonomous system is considered, which means that periodic terms in the differential equations are small compared to the constant ones. In the studies devoted to the stability analysis of systems under small parametric excitation it was usually assumed that the corresponding autonomous system is conservative. In the papers [5], [6] the case of an autonomous oscillatory system including both conservative and circulatory (nondissipative nonconservative) forces was studied.

In this paper a multi-parameter multi-degree-of-freedom system under small parametric excitation is considered. It is assumed that the corresponding autonomous system (without excitation) is nonconservative. In this setting there are two most interesting cases, when the autonomous system has a zero eigenvalue or a pair of complex conjugate imaginary eigenvalues. This means that the autonomous system is subjected to the static or dynamic instability. Stability analysis is performed using the perturbation theory of eigenvalues [13], [14], the versal deformation theory for matrices [15], [16], [17], and explicit expressions for derivatives of the monodromy matrix found recently in [18], [19]. As a result, local approximations of any order for the stability domain are obtained. Formulae of the approximations use only information on the system at initial values of parameters: eigenvalues and eigenvectors of the autonomous system and derivatives of system matrices with respect to parameters. Due to the

multi-parameter formulation of the problem, the method of the paper is useful for studying combined influence of several parameters on stability of the system and, in particular, for the robust stability analysis. This method follows the approach developed in the papers [17], [20], [21], where first-order approximations for stability boundaries of general autonomous and periodic systems were found. For other applications of the versal deformation theory to stability problems see [22], [23].

Approximations derived in the paper describe the geometry of the stability boundary. Using these approximations, it is shown that the stability boundary is typically a smooth surface. Nevertheless, in the case of a special relation of a flutter frequency of the autonomous system with a period of excitation, the stability boundary is nonsmooth (has a singularity). Different types of possible singularities are studied.

As an application, a stability problem for an elastic cantilevered pipe conveying pulsating fluid is considered. Approximations of the stability domain in the regular and singular cases are given. Comparison of the approximations with the stability domains found numerically shows efficiency of the suggested approach. It is shown that singularities of the stability boundary provide a geometric description of typical stability diagrams on the amplitude-frequency plane.

The paper is organized as follows. Section 2 gives a short introduction to the Floquet theory. In Sect. 3 approximations of the stability domain are derived in the regular case. Section 4 studies the singular case, when the stability boundary is not smooth. In Sect. 5, the obtained results are applied to the stability analysis of a pipe conveying pulsating fluid. The Appendix collects formulae needed for calculations.

2 Stability of nonconservative periodic system

A linear multi-degree-of-freedom nonconservative system is governed by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{L}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0, \quad (1)$$

where $\mathbf{q} = (q_1, \dots, q_{m'})^T$ is a real vector of generalized coordinates; \mathbf{M} is a positive definite mass matrix; the nonsymmetric $m' \times m'$ matrices \mathbf{L} and \mathbf{K} determine dissipative, gyroscopic, conservative, and circulatory (nondissipative nonconservative) forces; dot denotes a derivative with respect to time t . Transforming Eq. (1) to the system of first order, we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{L} \end{pmatrix}, \quad (2)$$

where \mathbf{I} is the identity matrix; the real vector \mathbf{x} and the square real matrix \mathbf{A} have dimension $m = 2m'$.

Let us consider autonomous system (2) determined by a matrix $\mathbf{A} = \mathbf{A}_0$. The corresponding eigenvalue problem has the form

$$\mathbf{A}_0\mathbf{u} = \lambda\mathbf{u}, \quad (3)$$

where λ is an eigenvalue and \mathbf{u} is an eigenvector. System (2) is asymptotically stable if and only if all the eigenvalues λ have negative real parts ($\text{Re } \lambda < 0$).

Let us consider an n -parameter periodic perturbation of the autonomous system in the form

$$\dot{\mathbf{x}} = \mathbf{A}(t, \mathbf{p})\mathbf{x}, \quad \mathbf{A}(t, \mathbf{p}_0) \equiv \mathbf{A}_0, \quad (4)$$

where $\mathbf{p} = (p_1, \dots, p_n)^T$ is a vector of real parameters. Equation (4) represents a nonautonomous linear system, which is autonomous at $\mathbf{p}_0 = (p_1^0, \dots, p_n^0)^T$. It is assumed that the matrix $\mathbf{A}(t, \mathbf{p})$ is a smooth function of the parameter vector \mathbf{p} and a periodic continuous function of time, $\mathbf{A}(t+T, \mathbf{p}) = \mathbf{A}(t, \mathbf{p})$, where the period $T = T(\mathbf{p}) > 0$ smoothly depends on \mathbf{p} . In this paper we study stability of system (4) in the neighborhood of the point \mathbf{p}_0 , where periodic terms are small (small parametric excitation).

If the autonomous system at \mathbf{p}_0 is asymptotically stable, then periodic system (4) is stable for the parameter vectors \mathbf{p} from some neighborhood of the point \mathbf{p}_0 . If the autonomous system is unstable and has an eigenvalue with a positive real part, $\text{Re } \lambda > 0$, then perturbed system (4) remains unstable for \mathbf{p} sufficiently close to \mathbf{p}_0 . Hence, the interesting cases, when small periodic terms can affect stability properties, correspond to the matrix \mathbf{A}_0 having eigenvalues on the imaginary axis. In this paper two basic cases are considered: there is the simple zero eigenvalue or there is a pair of complex conjugate simple eigenvalues $\lambda = \pm i\omega$, $\omega \neq 0$, on the imaginary axis. These critical cases describe autonomous systems subjected to the static or dynamic instability, respectively.

Let us fix some value of the parameter vector \mathbf{p} . A *matriciant* of system (4) is an $m \times m$ matrix function $\mathbf{X}(t)$ satisfying the differential equation [12],

$$\dot{\mathbf{X}} = \mathbf{A}(t, \mathbf{p}) \mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}. \quad (5)$$

Equation (5) is equivalent to m Eqs. (4) for the columns of $\mathbf{X}(t)$ with initial conditions being the columns of \mathbf{I} . A value of the matriciant at the period is called a *monodromy matrix* [12]:

$$\mathbf{F} = \mathbf{X}(T). \quad (6)$$

The eigenvalue problem for the monodromy matrix has the form

$$\mathbf{F}\mathbf{u} = \varrho\mathbf{u}, \quad (7)$$

where eigenvalues ϱ are called *multipliers*. System (4) is asymptotically stable if and only if all the multipliers lie inside the unit circle in the complex plane, $|\varrho| < 1$ [12]. If for some multiplier $|\varrho| > 1$, then system (4) is unstable.

The monodromy matrix $\mathbf{F}(\mathbf{p})$ is a smooth function of the parameter vector. Derivatives of the monodromy matrix calculated at \mathbf{p}_0 can be found using the matriciant and derivatives of the functions $\mathbf{A}(t, \mathbf{p})$ and $T(\mathbf{p})$ at \mathbf{p}_0 . Explicit formulae for these derivatives were obtained in [18], [19] and are given in the Appendix; see (51)–(53). In particular, the first-order derivative has the form

$$\frac{\partial \mathbf{F}}{\partial p_i} = \mathbf{F} \int_0^T \mathbf{X}^{-1} \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{X} dt + \mathbf{A}_0 \mathbf{F} \frac{\partial T}{\partial p_i}. \quad (8)$$

Note that though system (4) is autonomous at \mathbf{p}_0 , derivatives of the matrix $\mathbf{A}(t, \mathbf{p})$ at \mathbf{p}_0 are time-dependent periodic functions.

The stability criterion defines the stability and instability domains in the parameter space. A boundary of the stability domain consists of points \mathbf{p} such that the monodromy matrix $\mathbf{F}(\mathbf{p})$ has multipliers lying on the unit circle, $|\varrho| < 1$, while for other multipliers the inequality $|\varrho| < 1$ holds.

Since system (4) is autonomous at \mathbf{p}_0 , the matriciant and the monodromy matrix calculated at \mathbf{p}_0 have the form

$$\mathbf{X}(t, \mathbf{p}_0) = \exp(\mathbf{A}_0 t), \quad \mathbf{F}(\mathbf{p}_0) = \exp(\mathbf{A}_0 T_0), \quad (9)$$

where $T_0 = T(\mathbf{p}_0)$. From the second expression of (9) it follows that the multipliers ϱ of the matrix $\mathbf{F}(\mathbf{p}_0)$ and the eigenvalues λ of the matrix \mathbf{A}_0 are connected by the relation

$$\varrho = \exp(\lambda T_0) \quad (10)$$

and the corresponding eigenvectors are equal. For two types of the matrix \mathbf{A}_0 under consideration the monodromy matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$ has a simple multiplier $\varrho = 1$ or a pair of simple multipliers $\varrho = \exp(\pm i\omega T_0)$ on the unit circle, while other multipliers lie inside the unit circle, $|\varrho| = \exp(\operatorname{Re} \lambda T_0) < 1$. Hence \mathbf{p}_0 is a point of the stability boundary and system (4) can be stable or unstable for different points \mathbf{p} in the neighborhood of \mathbf{p}_0 .

3 Approximation of the stability domain in a regular case

Let us consider the first case, when the matrix \mathbf{A}_0 has the simple zero eigenvalue $\lambda_0 = 0$ and the other eigenvalues have negative real parts (the corresponding autonomous system is subjected to the static instability). There is a right and left real eigenvector \mathbf{u}_0 and \mathbf{v}_0 corresponding to $\lambda_0 = 0$ and satisfying the equations

$$\mathbf{A}\mathbf{u}_0 = 0, \quad \mathbf{v}_0^T \mathbf{A}_0 = 0, \quad \mathbf{v}_0^T \mathbf{u}_0 = 1, \quad (11)$$

where the last equality represents the normalization condition.

From relation (10) it follows that the monodromy matrix \mathbf{F}_0 has the simple multiplier $\varrho_0 = 1$ on the unit circle, while the other multipliers lie inside the unit circle. Stability of system (4) depends on the behavior of the multiplier $\varrho_0 = 1$ under the perturbation of the parameter vector \mathbf{p} . Since $\mathbf{F}(\mathbf{p})$ is a smooth function, the simple multiplier $\varrho(\mathbf{p})$ depends smoothly on \mathbf{p} [14] and can be approximated in the neighborhood of \mathbf{p}_0 by the Taylor series

$$\varrho(\mathbf{p}) = \varrho_0 + \sum_{i=1}^n \frac{\partial \varrho}{\partial p_i} \Delta p_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \varrho}{\partial p_i \partial p_j} \Delta p_i \Delta p_j + \dots, \quad \Delta p_i = p_i - p_i^0. \quad (12)$$

Derivatives of the multiplier $\varrho(\mathbf{p})$ at \mathbf{p}_0 can be calculated using explicit formulae (59), (60) given in the Appendix. For this purpose only the eigenvectors \mathbf{u}_0 , \mathbf{v}_0 and derivatives of the matrix $\mathbf{F}(\mathbf{p})$ at \mathbf{p}_0 are needed. For example, the first derivative of $\varrho(\mathbf{p})$ has the form

$$\frac{\partial \varrho}{\partial p_i} = \mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_i} \mathbf{u}_0, \quad (13)$$

where the first derivative of $\mathbf{F}(\mathbf{p})$ is determined by Eq. (8).

Stability of the system in the neighborhood of \mathbf{p}_0 is determined by the inequality $|\varrho(\mathbf{p})| < 1$ for multiplier (12). Since $\varrho(\mathbf{p})$ is real in the neighborhood of \mathbf{p}_0 , the stability condition can be written with the use of (12) in the form

$$\varrho(\mathbf{p}) - 1 = \sum_{i=1}^n \frac{\partial \varrho}{\partial p_i} \Delta p_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \varrho}{\partial p_i \partial p_j} \Delta p_i \Delta p_j + \dots < 0. \quad (14)$$

Calculating derivatives of $\varrho(\mathbf{p})$ at \mathbf{p}_0 , we obtain approximation (14) of the stability domain with the accuracy up to the terms of any order. Taking the equality sign in (14), we get the approximation of the stability boundary.

According to (13), (14), the first approximation of the stability domain has the form

$$(\mathbf{g}, \Delta \mathbf{p}) < 0, \quad \Delta \mathbf{p} = \mathbf{p} - \mathbf{p}_0, \quad (15)$$

where $(\mathbf{g}, \Delta\mathbf{p})$ is the scalar product in R^n ; the vector \mathbf{g} is

$$\mathbf{g} = \left(\frac{\partial \varrho}{\partial p_1}, \dots, \frac{\partial \varrho}{\partial p_n} \right)^T = \left(\mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_1} \mathbf{u}_0, \dots, \mathbf{v}_0^T \frac{\partial \mathbf{F}}{\partial p_n} \mathbf{u}_0 \right)^T. \quad (16)$$

If $\mathbf{g} \neq 0$, then the stability boundary is a smooth surface in the neighborhood of \mathbf{p}_0 , and \mathbf{g} is the normal vector to the stability boundary, directed into the instability domain. Note that formulae (16), (8) for the normal vector were found earlier in [21]. The stability boundary is associated with the simple multiplier $\varrho = 1$ and, according to the traditional terminology, represents the harmonic parametric resonance boundary.

Let us consider the second case, when the matrix \mathbf{A}_0 has a pair of simple imaginary eigenvalues $\lambda_0, \bar{\lambda}_0 = \pm i\omega$, $\omega \neq 0$, while for other eigenvalues the inequality $\text{Re } \lambda < 0$ holds. This means that the autonomous system is subjected to the dynamic instability (flutter). There is a right and left complex eigenvector \mathbf{u}_0 and \mathbf{v}_0 corresponding to λ_0 and satisfying the equations

$$\mathbf{A}\mathbf{u}_0 = \lambda_0\mathbf{u}_0, \quad \mathbf{v}_0^T \mathbf{A} = \lambda_0\mathbf{v}_0^T, \quad \mathbf{v}_0^T \mathbf{u}_0 = 1, \quad (17)$$

where the last equality represents the normalization condition. From (10) it follows that the monodromy matrix \mathbf{F}_0 has the multipliers $\varrho_0, \bar{\varrho}_0 = \exp(\pm i\omega T_0)$ lying on the unit circle, and for other multipliers $|\varrho| < 1$. In this section we assume that $\omega T_0 \neq \pi k$ for any integer k (a regular case), which means that the multipliers ϱ_0 and $\bar{\varrho}_0$ are distinct and complex.

The simple multipliers $\varrho_0, \bar{\varrho}_0$ are smooth functions of the parameters in the neighborhood of the point \mathbf{p}_0 . Stability of the system depends on absolute values of these multipliers. Since they are complex conjugate, it is sufficient to study behavior of one multiplier ϱ_0 . Then the stability condition takes the form

$$|\varrho(\mathbf{p})| = \left| \varrho_0 + \sum_{i=1}^n \frac{\partial \varrho}{\partial p_i} \Delta p_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \varrho}{\partial p_i \partial p_j} \Delta p_i \Delta p_j + \dots \right| < 1. \quad (18)$$

Calculating derivatives of $\varrho(\mathbf{p})$ at \mathbf{p}_0 up to the order s by formulae (59), (60) given in the Appendix, we find approximation of the stability domain (18) up to small terms of order s . Note that these formulae need only the eigenvectors $\mathbf{u}_0, \mathbf{v}_0$ and derivatives of the functions $\mathbf{A}(t, \mathbf{p})$ and $T(\mathbf{p})$ at \mathbf{p}_0 . The stability boundary in this case is the combination resonance boundary associated with a pair of complex conjugate multipliers $\varrho, \bar{\varrho}$ lying on the unit circle.

Using expression (13) for the first derivative of $\varrho(\mathbf{p})$ and the relations $|\varrho(\mathbf{p})| = |\bar{\varrho}_0 \varrho(\mathbf{p})|$, $\varrho_0 = \cos(\omega T_0) + i \sin(\omega T_0)$ in the stability condition (18), we find the first approximation of the stability domain in the form

$$(\mathbf{n}, \Delta\mathbf{p}) < 0. \quad (19)$$

Here the vector \mathbf{n} is

$$\mathbf{n} = \text{Re}(\bar{\varrho}_0 \mathbf{g}) = \cos(\omega T_0) \text{Re } \mathbf{g} + \sin(\omega T_0) \text{Im } \mathbf{g}, \quad (20)$$

where the vector \mathbf{g} is defined by (16). If $\mathbf{n} \neq 0$, then the stability boundary is a smooth surface in the vicinity of the point \mathbf{p}_0 , and \mathbf{n} is a normal vector to the stability boundary directed into the instability domain. Note that the normal vector (20) was found earlier in [21].

For the numerical construction of the stability domain we can directly use approximations (14) or (18), depending on the case, together with formulae (51)–(53), (59), (60) given in the Appendix. Alternatively, we can employ standard methods for manipulation with Taylor series to express one of the parameters p_i in terms of the other parameters from (14) or (18). This would give us an approximation for the stability boundary surface in explicit form.

4 Local analysis of the stability domain in a singular case

In the previous section the stability domain had a smooth boundary in the neighborhood of \mathbf{p}_0 . In this section a singular case, when the stability boundary is not smooth at \mathbf{p}_0 is investigated.

Let us consider the case, when the matrix \mathbf{A}_0 has a pair of complex conjugate simple eigenvalues on the imaginary axis $\lambda_0, \bar{\lambda}_0 = \pm i\omega, \omega \neq 0$, while for other eigenvalues $\operatorname{Re} \lambda < 0$. Let us assume that the frequency of vibrations of the autonomous system ω is related to the period of excitation by the equality

$$\omega T_0 = \pi k \quad (21)$$

for some integer k . Let \mathbf{u}_0 and \mathbf{v}_0 be the right and left complex eigenvectors corresponding to λ_0 (17). Then the right and left eigenvectors, corresponding to the complex conjugate eigenvalue $\bar{\lambda}_0$, have the form $\bar{\mathbf{u}}_0$ and $\bar{\mathbf{v}}_0$, respectively. From (10) it follows that the monodromy matrix \mathbf{F}_0 has a double multiplier $\varrho_0 = (-1)^k$. Both \mathbf{u}_0 and $\bar{\mathbf{u}}_0$ are the eigenvectors corresponding to ϱ_0 . This means that the multiplier ϱ_0 is semi-simple (the number of linearly independent eigenvectors is equal to the algebraic multiplicity of the multiplier). Since ϱ_0 is real, we can choose two real eigenvectors as follows:

$$\mathbf{u}_1 = \operatorname{Re} \mathbf{u}_0, \quad \mathbf{u}_2 = \operatorname{Im} \mathbf{u}_0 \quad (22)$$

and define an $m \times 2$ matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2]$. Two left real eigenvectors \mathbf{v}_1 and \mathbf{v}_2 can be taken in the form

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2] = [\operatorname{Re} \mathbf{v}_0, \operatorname{Im} \mathbf{v}_0] \begin{pmatrix} \operatorname{Re} \mathbf{v}_0^T \mathbf{u}_1 & \operatorname{Im} \mathbf{v}_0^T \mathbf{u}_1 \\ \operatorname{Re} \mathbf{v}_0^T \mathbf{u}_2 & \operatorname{Im} \mathbf{v}_0^T \mathbf{u}_2 \end{pmatrix}^{-1}. \quad (23)$$

Expression (23) gives the left eigenvectors satisfying the following normalization condition

$$\mathbf{V}^T \mathbf{U} = \mathbf{I}, \quad (24)$$

where \mathbf{I} is the 2×2 identity matrix.

For the vectors \mathbf{p} in the vicinity of \mathbf{p}_0 the double multiplier ϱ_0 splits into two simple multipliers $\varrho_1(\mathbf{p})$ and $\varrho_2(\mathbf{p})$. Stability of the trivial solution of periodic system (4) depends on the absolute values $|\varrho_{1,2}(\mathbf{p})|$. The multipliers $\varrho_{1,2}(\mathbf{p})$ are generally nonsmooth functions of the vector \mathbf{p} at \mathbf{p}_0 [14]. This makes it difficult to use the perturbation theory of eigenvalues for the stability analysis, especially, for finding high order approximations of the stability domain. In this case it is convenient to apply the versal deformation theory, which allows considering the matrix operator $\mathbf{F}(\mathbf{p})$ restricted to the invariant subspace of the multipliers $\varrho_{1,2}(\mathbf{p})$. This means that instead of analyzing the nonsmooth multipliers $\varrho_{1,2}(\mathbf{p})$ of the $m \times m$ matrix $\mathbf{F}(\mathbf{p})$ we consider a 2×2 matrix $\mathbf{F}'(\mathbf{p})$. This matrix is a smooth function of \mathbf{p} and its eigenvalues are $\varrho_{1,2}(\mathbf{p})$. As a result, the stability analysis becomes similar to the analysis of the regular case considered in the previous section.

4.1 Versal deformation

According to the versal deformation theory, there is a smooth 2×2 matrix function $\mathbf{F}'(\mathbf{p})$ determined in the neighborhood of \mathbf{p}_0 by the equations

$$\mathbf{F}(\mathbf{p}) \mathbf{C}(\mathbf{p}) = \mathbf{C}(\mathbf{p}) \mathbf{F}'(\mathbf{p}), \quad \mathbf{F}'(\mathbf{p}_0) = \varrho_0 \mathbf{I}, \quad \mathbf{C}(\mathbf{p}_0) = \mathbf{U}, \quad (25)$$

where $\mathbf{C}(\mathbf{p})$ is an $m \times 2$ matrix smoothly dependent on \mathbf{p} [15], [16]. The matrix $\mathbf{F}'(\mathbf{p})$ represents a block of a so-called *versal deformation*. Note that the matrix functions $\mathbf{F}'(\mathbf{p})$ and $\mathbf{C}(\mathbf{p})$ are not uniquely determined.

Eigenvalues of the matrix $\mathbf{F}'(\mathbf{p})$ are equal to the multipliers $\varrho_{1,2}(\mathbf{p})$ of the matrix $\mathbf{F}(\mathbf{p})$. In the neighborhood of \mathbf{p}_0 the matrix function $\mathbf{F}'(\mathbf{p})$ can be approximated by the Taylor series

$$\mathbf{F}'(\mathbf{p}) = \varrho_0 \mathbf{I} + \sum_{i=1}^n \frac{\partial \mathbf{F}'}{\partial p_i} \Delta p_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \mathbf{F}'}{\partial p_i \partial p_j} \Delta p_i \Delta p_j + \dots \quad (26)$$

Explicit formulae for derivatives of $\mathbf{F}'(\mathbf{p})$ are given in the Appendix; see (64). These formulae use derivatives of the monodromy matrix $\mathbf{F}(\mathbf{p})$ at \mathbf{p}_0 and the matrices \mathbf{U} , \mathbf{V} . In particular, the first derivative of $\mathbf{F}'(\mathbf{p})$ has the form

$$\frac{\partial \mathbf{F}'}{\partial p_i} = \mathbf{V}^T \frac{\partial \mathbf{F}}{\partial p_i} \mathbf{U}. \quad (27)$$

Note that Eqs. (25) and (7) for the matrix \mathbf{F}' and the multiplier ϱ look very similar, as well as Eqs. (27) and (13) for their derivatives.

4.2 Approximation of the stability domain

The stability domain in the neighborhood of \mathbf{p}_0 is determined by the condition $|\varrho_{1,2}(\mathbf{p})| < 1$ for both eigenvalues of $\mathbf{F}'(\mathbf{p})$. To write this condition in more convenient form, we can take a 2×2 matrix $\mathbf{H}(\mathbf{p}) = \log(\varrho_0 \mathbf{F}'(\mathbf{p}))$, where the logarithm of the matrix is defined as follows [24]

$$\log(\mathbf{I} + \mathbf{D}) = \sum_{i=1}^{\infty} (-1)^{i+1} \mathbf{D}^i / i. \quad (28)$$

Since $\varrho_0 = (-1)^k$ and $\varrho_0 \mathbf{F}'(\mathbf{p}_0) = \mathbf{I}$, we have $\mathbf{H}(\mathbf{p}_0) = 0$. The inequality $|\varrho_{1,2}(\mathbf{p})| < 1$ is equivalent to the condition $\operatorname{Re} \mu_{1,2}(\mathbf{p}) < 0$ for the eigenvalues $\mu_{1,2}(\mathbf{p}) = \log(\varrho_0 \varrho_{1,2}(\mathbf{p}))$ of the matrix $\mathbf{H}(\mathbf{p})$. By means of the Routh-Hurwitz conditions [27], the stability criterion $\operatorname{Re} \mu_{1,2}(\mathbf{p}) < 0$ for both eigenvalues of $\mathbf{H}(\mathbf{p})$ can be written in the form

$$\begin{cases} h_{11}(\mathbf{p}) + h_{22}(\mathbf{p}) < 0, \\ h_{12}(\mathbf{p}) h_{21}(\mathbf{p}) - h_{11}(\mathbf{p}) h_{22}(\mathbf{p}) < 0, \end{cases} \quad (29)$$

where $h_{ij}(\mathbf{p})$ are elements of the matrix $\mathbf{H}(\mathbf{p})$. Using Taylor expansions (26) and (28) for the matrix $\mathbf{F}'(\mathbf{p})$ and the logarithm function, we find the Taylor series for the matrix $\mathbf{H}(\mathbf{p})$ as follows

$$\begin{aligned} \mathbf{H}(\mathbf{p}) &= \begin{pmatrix} h_{11}(\mathbf{p}) & h_{12}(\mathbf{p}) \\ h_{21}(\mathbf{p}) & h_{22}(\mathbf{p}) \end{pmatrix} = \log(\varrho_0 \mathbf{F}'(\mathbf{p})) \\ &= \varrho_0 \sum_{i=1}^n \frac{\partial \mathbf{F}'}{\partial p_i} \Delta p_i + \frac{\varrho_0}{2} \sum_{i,j=1}^n \left[\frac{\partial^2 \mathbf{F}'}{\partial p_i \partial p_j} - \varrho_0 \frac{\partial \mathbf{F}'}{\partial p_i} \frac{\partial \mathbf{F}'}{\partial p_j} \right] \Delta p_i \Delta p_j + \dots, \end{aligned} \quad (30)$$

where any number of terms in the series can be found explicitly using symbolic computation software. Thus, evaluating derivatives of $\mathbf{F}'(\mathbf{p})$ at \mathbf{p}_0 by formulae (64) given in the Appendix and substituting them into expression (30), we find the approximation of the stability domain (29) up to small terms of arbitrary order.

For the first-order approximation of the stability domain expressions (29) yield

$$\begin{cases} (\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta \mathbf{p}) < 0, \\ (\mathbf{Q} \Delta \mathbf{p}, \Delta \mathbf{p}) < 0, \end{cases} \quad (31)$$

where the vectors \mathbf{h}_{ij} are gradients of the functions $h_{ij}(\mathbf{p})$ at \mathbf{p}_0 ; the symmetric $n \times n$ matrix \mathbf{Q} has the form

$$\mathbf{Q} = \frac{1}{2}(\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}^T), \quad \tilde{\mathbf{Q}} = \mathbf{h}_{12} \mathbf{h}_{21}^T - \mathbf{h}_{11} \mathbf{h}_{22}^T. \quad (32)$$

Using expressions (27), (30), the vectors \mathbf{h}_{ij} can be found as follows

$$\mathbf{h}_{ij} = \varrho_0 \left(\mathbf{v}_i^T \frac{\partial \mathbf{F}}{\partial p_1} \mathbf{u}_j, \dots, \mathbf{v}_i^T \frac{\partial \mathbf{F}}{\partial p_n} \mathbf{u}_j \right)^T. \quad (33)$$

We found approximation of the stability domain (29), (30). Since the stability condition consists of two equations, the stability boundary is generally nonsmooth at \mathbf{p}_0 . In the following subsections we analyze the geometry of the stability boundary and classify its singularities using first approximation (31).

4.3 Two-parameter case

The first inequality of (31) defines a half-plane in the parameter space $\mathbf{p} = (p_1, p_2)$. The second inequality of (31) gives different solutions for $\Delta \mathbf{p}$ depending on the type of the matrix \mathbf{Q} . In the case $\mathbf{Q} > 0$ (positive definite) the second inequality of (31) is not satisfied for all $\Delta \mathbf{p}$. If $\mathbf{Q} < 0$ (negative definite), then the second inequality of (31) is satisfied for all $\Delta \mathbf{p}$. Finally, in the case of a nondefinite matrix \mathbf{Q} ($\det \mathbf{Q} < 0$) this inequality gives two domains lying between two intersecting lines $(\mathbf{Q} \Delta \mathbf{p}, \Delta \mathbf{p}) = 0$. Equations of these lines can be written in the form

$$q_{11} \Delta p_1 + \left(q_{12} \pm \sqrt{q_{12}^2 - q_{11} q_{22}} \right) \Delta p_2 = 0, \quad (34)$$

where q_{ij} are elements of the matrix \mathbf{Q} . Expression (34) is found by solving the quadratic equation $(\mathbf{Q} \Delta \mathbf{p}, \Delta \mathbf{p}) = 0$ with respect to Δp_1 .

The first approximation of the stability boundary is the intersection of the domains defined by two inequalities (31). For the case of the nondefinite matrix \mathbf{Q} the geometry of the stability domain depends on the mutual position of these domains. There are two typical cases: when the line $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta \mathbf{p}) = 0$ lies inside or outside the domain $(\mathbf{Q} \Delta \mathbf{p}, \Delta \mathbf{p}) < 0$. This corresponds to the inequalities $(\mathbf{Q} \mathbf{t}, \mathbf{t}) < 0$ or $(\mathbf{Q} \mathbf{t}, \mathbf{t}) > 0$ respectively, where \mathbf{t} is a nonzero vector satisfying the equation $(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{t}) = 0$.

The general result can be formulated as follows. The stability domain in the neighborhood of the point \mathbf{p}_0 has four typical forms corresponding to the cases: a) $\mathbf{Q} < 0$, b) $\mathbf{Q} > 0$, c) \mathbf{Q} is nondefinite, $(\mathbf{Q} \mathbf{t}, \mathbf{t}) > 0$, and d) \mathbf{Q} is nondefinite, $(\mathbf{Q} \mathbf{t}, \mathbf{t}) < 0$. In the case a) the stability boundary is a smooth curve; in the case b) the system is unstable for all \mathbf{p} near \mathbf{p}_0 ; in the cases c) and d) the stability domain consists of one and two angles respectively with the vertices at \mathbf{p}_0 ; see Fig. 1. In Fig. 1 the curves $h_{12}(\mathbf{p}) h_{21}(\mathbf{p}) - h_{11}(\mathbf{p}) h_{22}(\mathbf{p}) = (\mathbf{Q} \Delta \mathbf{p}, \Delta \mathbf{p}) + o(\|\Delta \mathbf{p}\|^2) = 0$ are denoted by “ α ” and the curve $h_{11}(\mathbf{p}) + h_{22}(\mathbf{p}) = (\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta \mathbf{p}) + o(\|\Delta \mathbf{p}\|) = 0$ is denoted by “ β ”; the stability domain is denoted by the character “S”. Since the multipliers $\varrho_{1,2} = \varrho_0 \exp \mu_{1,2}$, where $\mu_{1,2}$ are eigenvalues of $\mathbf{H}(\mathbf{p})$, the lines “ α ” represent the parametric

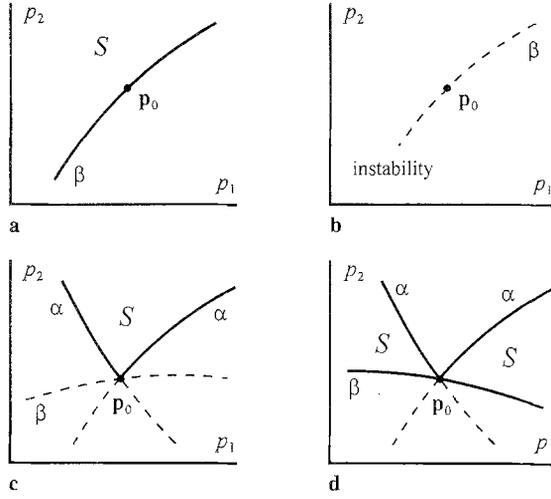


Fig. 1. Forms of the stability domain in the two-parameter case

resonance boundary (corresponding to the multiplier $\varrho = (-1)^k$, while the lines “ β ” are the combination resonance boundaries (determined by a pair of complex conjugate multipliers on the unit circle).

Other (degenerate) cases are realized, when the matrix \mathbf{Q} is singular, or $\mathbf{h}_{11} + \mathbf{h}_{22} = 0$, or $(\mathbf{Q}\mathbf{t}, \mathbf{t}) = 0$ (if \mathbf{Q} is nondefinite). Then higher order approximations for the functions $h_{ij}(\mathbf{p})$ should be used to determine the form of the stability domain.

4.4 Three-parameter case

In this case \mathbf{Q} and \mathbf{h}_{ij} are the real matrix and the real vectors of dimension 3 respectively. It is easy to show that the matrix \mathbf{Q} is always nondefinite. Indeed, there exists a nonzero vector \mathbf{e} satisfying the equations $(\mathbf{h}_{11}, \mathbf{e}) = (\mathbf{h}_{12}, \mathbf{e}) = 0$ such that $(\mathbf{Q}\mathbf{e}, \mathbf{e}) = \mathbf{e}^T[\mathbf{h}_{12}\mathbf{h}_{21}^T - \mathbf{h}_{11}\mathbf{h}_{22}^T]\mathbf{e} = 0$. Therefore, for the nonsingular \mathbf{Q} equation $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = 0$ defines a cone surface [27]. Depending on the sign of $\det \mathbf{Q}$ the domain $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) < 0$ is placed inside or outside the cone (inside for $\det \mathbf{Q} < 0$). There exists a 3×3 nonsingular real matrix $\mathbf{W} = [\mathbf{u}, \mathbf{v}, \mathbf{w}]$ transforming \mathbf{Q} to the diagonal form [27]

$$\mathbf{Q} = \mathbf{W}\mathbf{D}\mathbf{W}^T, \quad (35)$$

where $\mathbf{D} = \text{diag}(-1, -1, 1)$ or $\mathbf{D} = \text{diag}(1, 1, -1)$. Then, the equation for the cone surface $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) = 0$ can be written in the form

$$(\mathbf{u}, \Delta\mathbf{p})^2 + (\mathbf{v}, \Delta\mathbf{p})^2 = (\mathbf{w}, \Delta\mathbf{p})^2. \quad (36)$$

The cone (36) can be written in the parametrized form as follows:

$$\begin{aligned} \Delta\mathbf{p} &= s(\mathbf{a} + \mathbf{b} \cos \alpha + \mathbf{c} \sin \alpha), \quad s \in R, \quad 0 \leq \alpha < 2\pi \\ \mathbf{a} &= \mathbf{u} \times \mathbf{v}, \quad \mathbf{b} = \mathbf{w} \times \mathbf{u}, \quad \mathbf{c} = \mathbf{v} \times \mathbf{w}, \end{aligned} \quad (37)$$

where the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} describe geometry of the cone as shown in Fig. 2. Expression (37) can be checked by substitution into (36).

The form of the stability domain is determined by the mutual position of the cone $(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) < 0$ and the half-space $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta\mathbf{p}) < 0$. There are two typical cases: when the

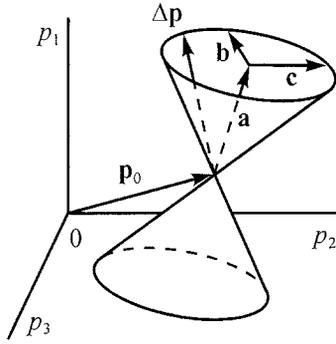


Fig. 2. Parametrization of the cone surface

plane $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta \mathbf{p}) = 0$ intersects the cone and when the only joint point of the cone and the plane is $\Delta \mathbf{p} = 0$. The first case occurs if there exists a nonzero vector $\Delta \mathbf{p}$, satisfying both equations $(\mathbf{Q} \Delta \mathbf{p}, \Delta \mathbf{p}) = 0$ and $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta \mathbf{p}) = 0$. Substituting $\Delta \mathbf{p}$ from (37), which is the solution of the first equation, into the second one, we obtain

$$\begin{aligned} (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{a} + \mathbf{b} \cos \alpha + \mathbf{c} \sin \alpha) &= 0, \\ (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{b}) \cos \alpha + (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{c}) \sin \alpha &= -(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{a}), \\ \sin(\alpha + \varphi_0) &= \xi, \end{aligned} \quad (38)$$

where

$$\begin{aligned} \xi &= -(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{a}), / \sqrt{(\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{b})^2 + (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{c})^2}, \\ \tan \varphi_0 &= (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{b}) / (\mathbf{h}_{11} + \mathbf{h}_{22}, \mathbf{c}). \end{aligned} \quad (39)$$

Thus, two cases, corresponding to the plane $(\mathbf{h}_{11} + \mathbf{h}_{22}, \Delta \mathbf{p}) = 0$ intersecting or not intersecting the cone, are determined by the inequalities $|\xi| < 1$ and $|\xi| > 1$ respectively. If $|\xi| < 1$, then two different roots α of (38) after substitution into (37) give the first-order approximations of the intersection curves (edges of the stability boundary).

The general result for the three-parameter case can be formulated as follows: the form of the stability domain in the neighborhood of \mathbf{p}_0 has four typical forms. These forms are determined by the conditions: a) $\det \mathbf{Q} < 0$, $|\xi| > 1$, b) $\det \mathbf{Q} > 0$, $|\xi| > 1$, c) $\det \mathbf{Q} < 0$, $|\xi| < 1$, and d) $\det \mathbf{Q} > 0$, $|\xi| < 1$, and shown in Fig. 3 (the stability domain is denoted by “S”). A part of the stability boundary corresponding to the cone is the parametric resonance boundary (harmonic or subharmonic depending on the sign of $\varrho_0 = (-1)^k$), while the other part represents the combination resonance boundary.

Note that in all cases a)–d) the instability domain consists of two parts, where one part is the combination resonance domain (determined by complex multipliers lying outside the unit circle) and another part is the parametric resonance domain (determined by real multipliers $|\varrho| > 1$). A boundary between these domains is characterized by the existence of a double multiplier $\varrho_1(\mathbf{p}) = \varrho_2(\mathbf{p})$. This happens, when $\mu_1(\mathbf{p}) = \mu_2(\mathbf{p})$, which means that the discriminant of the characteristic equation for $\mathbf{H}(\mathbf{p})$ is equal to zero:

$$\begin{aligned} (h_{11}(\mathbf{p}) - h_{22}(\mathbf{p}))^2 + 4h_{12}(\mathbf{p})h_{21}(\mathbf{p}) \\ = (\mathbf{h}_{11} - \mathbf{h}_{22}, \Delta \mathbf{p})^2 + 4(\mathbf{h}_{12}, \Delta \mathbf{p})(\mathbf{h}_{21}, \Delta \mathbf{p}) + o(\|\Delta \mathbf{p}\|^2) = 0. \end{aligned} \quad (40)$$

Equation (40) defines a cone surface in the first approximation, except the degenerate case, when a 3×3 matrix $(\mathbf{h}_{11} - \mathbf{h}_{22})(\mathbf{h}_{11} - \mathbf{h}_{22})^T + 4\mathbf{h}_{12}\mathbf{h}_{21}^T$ is singular. A part of this surface

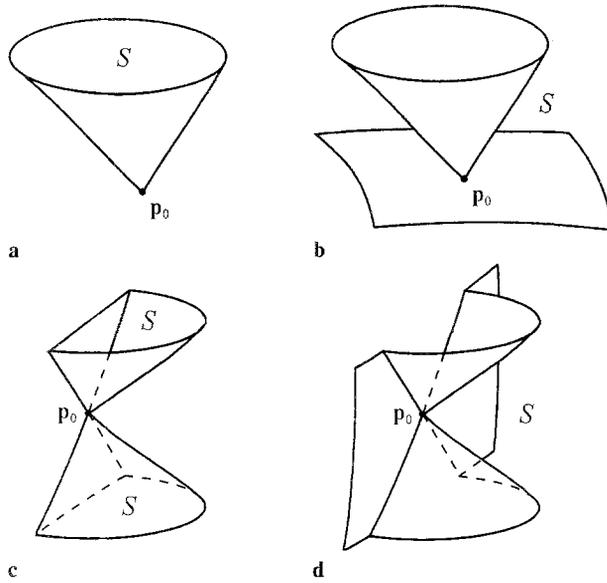


Fig. 3. Forms of the stability domain in the three-parameter case

belonging to the instability domain divides the combination and parametric resonance domains.

To determine the form of the stability domain in degenerate cases $\det \mathbf{Q} = 0$ or $|\xi| = 1$, we need to use higher order approximations of the functions $h_{ij}(\mathbf{p})$ in formulae (29).

4.5 General case $n \geq 4$

Let us consider the case of four or more parameters. If the vectors \mathbf{h}_{ij} , $i, j = 1, 2$ (gradients of the functions $h_{ij}(\mathbf{p})$ at \mathbf{p}_0) are linearly independent, then the stability domain (29), after a nonsingular smooth change of parameters $\mathbf{p}' = \mathbf{p}'(\mathbf{p})$, $\mathbf{p}'(\mathbf{p}_0) = 0$, in the vicinity of \mathbf{p}_0 , takes the form

$$\begin{cases} p_1' + p_2' < 0, \\ p_3' p_4' - p_1' p_2' < 0, \\ p_5', \dots, p_n' \in \mathbb{R}, \end{cases} \quad (41)$$

where $p_1' = h_{11}(\mathbf{p})$, $p_2' = h_{22}(\mathbf{p})$, $p_3' = h_{12}(\mathbf{p})$, $p_4' = h_{21}(\mathbf{p})$, and other functions $p_i' = p_i'(\mathbf{p})$, $i = 5 \dots, n$, are chosen such that the Jacobi matrix $[d\mathbf{p}'/d\mathbf{p}]$ at $\mathbf{p} = \mathbf{p}_0$ is nonsingular. Thus, there is one typical form (41) of the stability domain in the case $n \geq 4$. The case, when the vectors \mathbf{h}_{ij} are linearly dependent, can be studied using higher-order approximations of the functions $h_{ij}(\mathbf{p})$.

5 Stability of a pipe conveying pulsating fluid

As an application, let us consider a uniform flexible cantilevered pipe of length L , mass per unit length m , and flexural rigidity EI , conveying an incompressible fluid. Let the mass of the fluid per unit length be M and the flow velocity be $U(t)$. The pipe hangs down vertically and

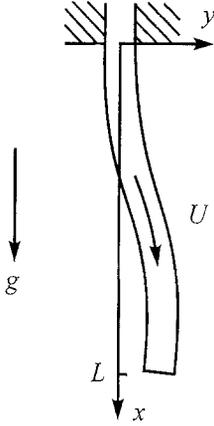


Fig. 4. Elastic pipe conveying pulsating fluid

the undeformed pipe axis coincides with the x -axis; see Fig. 4. We consider small lateral motions of the pipe $y(x, t)$ in the (x, y) -plane, where t is the time.

The dimensionless equation of motion of the pipe has the form [25]

$$\alpha \frac{\partial^5 \eta}{\partial \xi^4 \partial \tau} + \frac{\partial^4 \eta}{\partial \xi^4} + \left[u^2 + \left(\beta^{1/2} \frac{du}{d\tau} - \gamma \right) (1 - \xi) \right] \frac{\partial^2 \eta}{\partial \xi^2} + 2\beta^{1/2} u \frac{\partial^2 \eta}{\partial \xi \partial \tau} + \gamma \frac{\partial \eta}{\partial \xi} + \frac{\partial^2 \eta}{\partial \tau^2} = 0, \quad (42)$$

and the boundary conditions are

$$\eta = \frac{\partial \eta}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0; \quad \frac{\partial^2 \eta}{\partial \xi^2} = \frac{\partial^3 \eta}{\partial \xi^3} = 0 \quad \text{at} \quad \xi = 1. \quad (43)$$

The dimensionless variables and parameters are given by

$$\begin{aligned} \xi &= \frac{x}{L}, & \eta &= \frac{y}{L}, & \tau &= \left(\frac{EI}{M+m} \right)^{1/2} \frac{t}{L^2}, & \alpha &= \left(\frac{I}{E(M+m)} \right)^{1/2} \frac{E^*}{L^2}, \\ u &= \left(\frac{M}{EI} \right)^{1/2} UL, & \beta &= \frac{M}{M+m}, & \gamma &= \frac{M+m}{EI} L^3 g, \end{aligned} \quad (44)$$

where E^* is a coefficient of internal dissipation and g is the acceleration due to gravity. An external viscous damping coefficient is assumed to be zero (the parameter $\chi = 0$ in [25]).

Let us study the case of pulsating flow velocity $u(t) = u_0(1 + \mu \cos(\Omega t))$, where $u_0 = (M/EI)^{1/2} U_0 L$, μ , and $\Omega = [(M+m)/EI]^{1/2} \hat{\Omega} L^2$ are dimensionless parameters; U_0 , μ , and $\hat{\Omega}$ are the flow mean velocity, the amplitude, and the frequency of pulsations respectively.

For the numerical analysis of system (42), (43) we use Galerkin's method. For this purpose the solution $\eta(\xi, \tau)$ is expressed as a linear combination of m' normalized coordinate functions $\varphi_j(\xi)$ with coefficients $q_j(\tau)$, where the functions $\varphi_j(\xi)$ are the free-vibration eigenfunctions of a uniform cantilevered beam. Application of Galerkin's method to Eq. (42) gives [25]

$$\begin{aligned} \ddot{\mathbf{q}} + \mathbf{L}(\tau) \dot{\mathbf{q}} + \mathbf{K}(\tau) \mathbf{q} &= 0, \\ \mathbf{L}(\tau) &= \alpha \mathbf{A} + 2\beta^{1/2} u \mathbf{B}, & \mathbf{K}(\tau) &= \mathbf{A} + (u^2 + \beta^{1/2} \dot{u} - \gamma) \mathbf{C} + (\gamma - \beta^{1/2} \dot{u}) \mathbf{D} + \gamma \mathbf{B}, \end{aligned} \quad (45)$$

where $\mathbf{q} = (q_1, \dots, q_{m'})^T$. Elements λ_{sr} , b_{sr} , c_{sr} , and d_{sr} of the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} are determined by the expressions

$$\lambda_{sr} = \int_0^1 \varphi_s \varphi_r^{(4)} d\xi, \quad b_{sr} = \int_0^1 \varphi_s \varphi_r' d\xi, \quad c_{sr} = \int_0^1 \varphi_s \varphi_r'' d\xi, \quad d_{sr} = \int_0^1 \xi \varphi_s \varphi_r'' d\xi. \quad (46)$$

Values of integrals (46) are given in [25]. Equation (45) can be transformed to the form

$$\dot{\mathbf{x}} = \mathbf{A}(\tau) \mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{A}(\tau) = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{K}(\tau) & -\mathbf{L}(\tau) \end{pmatrix}. \quad (47)$$

The matrix $\mathbf{A}(\tau)$ of dimension $m = 2m'$ is periodic with the period $T = 2\pi/\Omega$. Introducing a new time $\tau' = \Omega\tau$, system (47) can be transformed to the form $d\mathbf{x}/d\tau' = \mathbf{A}'(\tau') \mathbf{x}$, where $\mathbf{A}'(\tau') = \mathbf{A}(\tau'/\Omega)/\Omega$ is a periodic matrix with the constant period $T' = 2\pi$. This form is more convenient for computation of derivatives of the monodromy matrix. In what follows, $m' = 6$ functions in the Galerkin's method are used. Hence, the matrix \mathbf{A} has dimension $m = 12$.

Let us consider the pipe with the parameters $\gamma = 10$, $\beta = 1/2$, $\mu = \alpha = 0$. This means that the mass of the fluid in the pipe is equal to the mass of the pipe and there is no pulsations and internal dissipation. Then system (47) is autonomous. Increasing the dimensionless flow velocity u_0 starting from zero and checking numerically the stability condition $\text{Re } \lambda < 0$ for all eigenvalues of the matrix \mathbf{A} , we find the critical velocity of the fluid $u_0^c = 9.84$. At this velocity the autonomous system is subjected to the dynamic instability (flutter) associated with the third mode. The flutter frequency is equal to $\omega = 28.11$ and the matrix \mathbf{A}_0 at $u_0 = u_0^c$ has simple eigenvalues $\lambda = \pm i\omega$ on the imaginary axis.

5.1 Stability domain in the regular case

Let us study the change of the critical flow velocity in the presence of small pulsations with the frequency $\Omega = 21$ and small internal damping $\alpha \geq 0$. To this end, we need to find the stability boundary in the three-parameter space $\mathbf{p} = (\mu, \alpha, u_0)^T$ in the neighborhood of the point $\mathbf{p}_0 = (0, 0, u_0^c)^T$. In this case $\omega T_0 = 2\pi\omega/\Omega = 2.677\pi \neq \pi k$ for any integer k . Using results of Sect. 3, we conclude that the stability boundary is a smooth surface being a boundary of the combination resonance domain.

Using formulae (51)–(53), (59), (60) given in the Appendix for derivatives of the monodromy matrix $\mathbf{F}(\mathbf{p})$ and the simple multiplier $\varrho(\mathbf{p})$, we can find the approximation (18) of the stability boundary up to small terms of any order. Accuracy of the approximation can be estimated numerically by comparison of approximations of different orders. In Fig. 5 the third- and fifth-order approximations of the stability boundary are shown. It can be concluded from Fig. 5 that pulsations stabilize the system (increase the critical mean velocity of the flow), while the internal damping has destabilizing effect, which becomes stronger for higher amplitudes of pulsations.

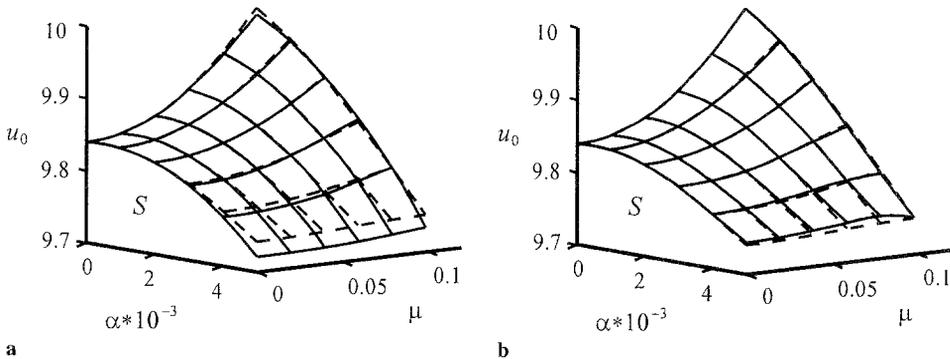


Fig. 5. The third- **a**, and fifth- **b** order approximations of the stability boundary

The dotted line in Fig. 5 shows the exact stability boundary. It was found by the calculation of the monodromy matrix at different values of the parameters, where differential Eq. (5) was solved using the Runge-Kutta method. It can be seen that the fifth-order approximation of the stability boundary is almost identical to the exact one for the range of parameters under consideration; see Fig. 5b. Note that the time spent for calculation of the third- and fifth-order approximations and exact form of the stability domain relates as 1 : 6 : 360. This shows the efficiency of the developed method for numerical analysis.

5.2 Stability domain in the singular case

Let us study the influence of pulsations with different amplitudes μ and frequencies Ω on the stability of the pipe. We consider the point $\mathbf{p}_0 = (2\omega, 0, u_0^c)^T$ in the parameter space $\mathbf{p} = (\Omega, \mu, u_0)^T$. In this case $T_0 = 2\pi/\Omega_0 = \pi/\omega$ and $\omega T_0 = k\pi$, $k = 1$. Hence, this is the singular case, when the stability boundary has a singularity.

The monodromy matrix $\mathbf{F}(\mathbf{p}_0) = \exp(\mathbf{A}_0 T_0)$ has a semisimple double multiplier $\varrho_0 = (-1)^k = -1$. The first-order approximation of the stability domain in the neighborhood of \mathbf{p}_0 is given by Eq. (31), where the vectors \mathbf{h}_{ij} and the matrix \mathbf{Q} calculated by expressions (8), (32), (33) are as follows:

$$\begin{aligned} \mathbf{h}_{11} &= (0, 1.63, 0.74)^T, & \mathbf{h}_{12} &= (-0.056, 0.24, 0.27)^T, \\ \mathbf{h}_{21} &= (0.056, 0.24, -0.27)^T, & \mathbf{h}_{22} &= (0, -1.63, 0.74)^T, \\ \mathbf{Q} &= \begin{pmatrix} -0.0031 & 0 & 0.015 \\ 0 & 2.72 & 0 \\ 0.015 & 0 & -0.62 \end{pmatrix}. \end{aligned} \quad (48)$$

Using (48), the first inequality of (31) takes the form

$$u_0 < u_0^c. \quad (49)$$

Since $\det \mathbf{Q} = 0.0047 > 0$, the second inequality of (31) defines the external part of cone (37) in the form

$$\begin{aligned} \Delta \mathbf{p} &= s(\mathbf{a} + \mathbf{b} \cos \alpha + \mathbf{c} \sin \alpha), \quad s \in R, \quad 0 \leq \alpha < 2\pi, \\ \mathbf{a} &= (0, -0.042, 0)^T, \quad \mathbf{b} = (0.0021, 0, -0.087)^T, \quad \mathbf{c} = (-1.3, 0, 0.031)^T. \end{aligned} \quad (50)$$

Intersection of the half-space (49) with the external part of the cone (50) gives the first order approximation of the stability domain; see Fig. 6. In Fig. 6 only the half-space $\mu \geq 0$ is shown (the other part $\mu \leq 0$ is symmetric with respect to the $\mu = 0$ plane). The stability

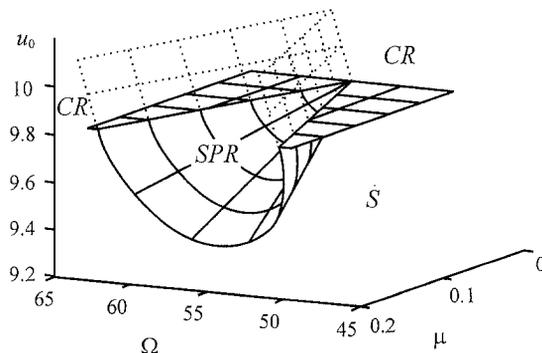


Fig. 6. The first-order approximation of the stability (S), subharmonic parametric resonance (SPR), and combination resonance (CR) domains

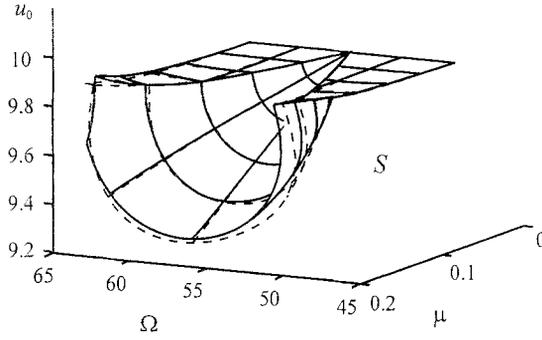


Fig. 7. The fourth-order approximation of the stability domain

boundary has a singularity at \mathbf{p}_0 of the type d); see Fig. 3d. The instability domain consists of the subharmonic parametric resonance and combination resonance domains separated by the boundary (40) shown in Fig. 6 by a point line.

Calculating higher order derivatives of the matrices $\mathbf{F}(\mathbf{p})$ and $\mathbf{F}'(\mathbf{p})$ at \mathbf{p}_0 with the use of formulae (51)–(53), (64) given in the Appendix, we can find higher order terms in the Taylor series of the functions $h_{ij}(\mathbf{p})$ (30). As a result, we obtain higher order approximations of the stability domain (29). In Fig. 7 the fourth-order approximation of the stability boundary is shown. For comparison, the dotted line in Fig. 7 denotes the exact stability boundary calculated numerically. The computation of the first- and fourth-order approximations took 4 and 150 seconds, respectively, while numerical calculation of the stability boundary by the Floquet method needed 5–6 hours (the calculations were carried out on PC using a standard MATLAB package). Note that even the first-order approximation provides a good qualitative and quantitative description of the stability domain.

It can be seen from Figs. 6 and 7 that small pulsations can destabilize and stabilize the system. For example, pulsations with the amplitude $\mu = 0.2$ and the frequency $\Omega = 2\omega$ change the critical mean velocity of the flow by $u_0 - u_0^c = -0.54$, which leads to about 5.5% decrease of the critical velocity compared with the autonomous system. Therefore, pulsations with the frequency $\Omega \approx 2\omega$ are dangerous for stability of the system. Note that the relation $\Omega = 2\omega$ is similar to the condition for the primary parametric resonance well known in the stability theory of oscillatory periodic systems, where ω represents the natural frequency of the conservative autonomous system (in our problem ω is the flutter frequency of the nonconservative autonomous system).

In the papers devoted to the stability analysis of pipes conveying pulsating fluid the stability domains (stability diagrams) are plotted on the amplitude-frequency plane (μ, Ω) for fixed values of the flow velocity u_0 [25, 26]. Analyzing singularities of the stability boundary in the three parameter space $\mathbf{p} = (\Omega, \mu, u_0)^T$, we can give qualitative description for typical stability diagrams.

It is shown that singularities of the stability boundary arise at the points $\mathbf{p} = (\Omega_0, 0, u_0^c)^T$ with $\Omega_0 = 2\omega/k$, where ω is the flutter frequency of the autonomous system. The stability domain is symmetric with respect to the $\mu = 0$ plane; the upper part of this plane ($\mu = 0, u_0 > u_0^c$) belongs to the instability domain, while the lower part belongs to the stability domain. Using these properties we conclude that all singularities of the stability boundary are of the fourth type, see Fig. 3d, where the cone axis is parallel to the μ -axis and the surface, determining the combination resonance boundary, is tangent to the $u_0 = u_0^c$ plane (see Figs. 6 and 7). Hence, the stability diagrams on the (μ, Ω) -plane for $u_0 < u_0^c$ typically consist of several convex parametric resonance zones at the frequencies $\Omega \approx 2\omega/k$, being cross-sections

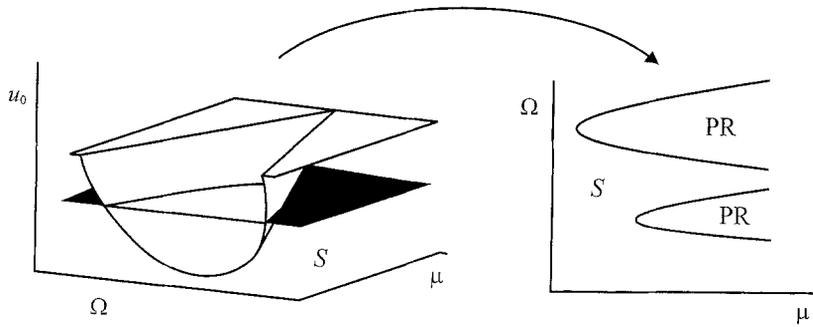


Fig. 8. Singularities of the stability boundary and stability diagrams

of the cones, corresponding to different singular points, by the $u_0 = \text{const}$ plane (see Fig. 8). With decrease of u_0 these domains appear for higher values of the pulsations amplitude μ . Note that for the pipe under consideration similar singularities appear at the supercritical value of the flow velocity $u_0 = 16.9$, when the second mode becomes unstable (the corresponding frequency is $\omega = 59.5$). These singularities lie inside the instability domain, but they also give rise to convex parametric resonance regions on the (μ, Ω) -plane at $u_0 < u_0^c$. Since instability of the autonomous system is associated with the third and second modes, there are no parametric resonance zones corresponding to the first mode. For $u_0 > u_0^c$ the (μ, Ω) -plane consists mostly of the instability domain including both parametric and combination resonance regions. Thus, singularities of the stability boundary provide the geometric description of the stability diagrams for cantilevered pipes conveying pulsating fluid. The described features of the stability diagrams have been observed in the papers [25], [26] for different cantilevered pipes.

6 Conclusion

The results obtained in this paper allow efficient and fast estimation of stabilizing (destabilizing) effect of small parametric excitation on a linear nonconservative system by finding approximations of the stability domain. Two basic cases are studied: when the corresponding autonomous system possesses a simple zero eigenvalue or a pair of complex conjugate eigenvalues on the imaginary axis. Formulae of the approximations need only derivatives of the system matrix with respect to parameters and information on the nonconservative autonomous system. These approximations are useful for both quantitative and qualitative analysis of the stability domain. In particular, they allow to analyze the geometry of the stability boundary and its singularities. Advantages of the method are illustrated on the application to the stability analysis of a cantilevered pipe conveying pulsating fluid.

The approach suggested in this paper is general and can be used for analysis of cases, when the corresponding autonomous system is subjected to a more complex type of instability like, for example, combined static-dynamic instability.

Appendix

A.1 Derivatives of the monodromy matrix

Let $\mathbf{h} = (h(1), \dots, h(n))$ be a vector with integer nonnegative components $h(i) \in \mathbb{Z}_+$. Using this vector we denote the partial derivative with respect to parameters evaluated at \mathbf{p}_0 by

$D^{\mathbf{h}} = \partial^{|\mathbf{h}|} / \partial p_1^{h(1)} \dots p_n^{h(n)}$, where $|\mathbf{h}| = h(1) + \dots + h(n)$ is the order of the derivative (a zero element $h(i)$ means that the derivative with respect to p_i is not taken). Let us introduce a matrix function $\mathbf{B}_{\mathbf{h}}(t)$ as follows:

$$\mathbf{B}_{\mathbf{h}} = \frac{\mathbf{X}^{-1} D^{\mathbf{h}} \mathbf{A} \mathbf{X}}{\mathbf{h}!} = \frac{1}{\mathbf{h}!} \mathbf{X}^{-1} \frac{\partial^{|\mathbf{h}|} \mathbf{A}}{\partial p_1^{h(1)} \dots \partial p_n^{h(n)}} \mathbf{X}, \quad (51)$$

where $\mathbf{X}(t)$ is the matriciant satisfying Eq. (5); $\mathbf{h}! = h(1)! \dots h(n)!$. Then the derivative of the matriciant with respect to parameters has the form [18, 19, 21]

$$D^{\mathbf{h}} \mathbf{X}(t) = \mathbf{h}! \mathbf{X}(t) \sum_{\substack{\mathbf{h}_1 + \dots + \mathbf{h}_s = \mathbf{h} \\ s=1, \dots, |\mathbf{h}|, |\mathbf{h}_i| > 0}} \int_0^t \mathbf{B}_{\mathbf{h}_1}(t_1) \int_0^{t_1} \mathbf{B}_{\mathbf{h}_2}(t_2) \dots \int_0^{t_{s-1}} \mathbf{B}_{\mathbf{h}_s}(t_s) dt_s \dots dt_1, \quad (52)$$

where the sum is taken over all sets of vectors \mathbf{h}_i such that $\mathbf{h}_1 + \dots + \mathbf{h}_s = \mathbf{h}$, $|\mathbf{h}_i| > 0$. Recall that the matriciant $\mathbf{X}(t)$ for the problem considered in this paper has the form (9).

The monodromy matrix is equal to $\mathbf{F}(\mathbf{p}) = \mathbf{X}(T(\mathbf{p}), \mathbf{p})$. If the period T doesn't depend on \mathbf{p} , then the derivative of the monodromy matrix is given by expression (52) evaluated at $t = T$, i.e., $D^{\mathbf{h}} \mathbf{F} = D^{\mathbf{h}} \mathbf{X}(T)$. If $T = T(\mathbf{p})$ is a smooth function of parameters, then

$$D^{\mathbf{h}} \mathbf{F}(\mathbf{p}) = D^{\mathbf{h}} \mathbf{X}(T(\mathbf{p}), \mathbf{p}), \quad (53)$$

and the derivative $D^{\mathbf{h}} \mathbf{F}(\mathbf{p})$ can be found by differentiating $\mathbf{X}(T(\mathbf{p}), \mathbf{p})$ as a composite function with the use of expression (52) and taking into account that $d\mathbf{X}/dt = \mathbf{A}\mathbf{X}$. The first order derivative has the form (8). We can avoid dependence of the period on parameters by making the change of time $t' = t/T(\mathbf{p})$, which doesn't affect the monodromy matrix. Then system (4) takes the form $dx/dt' = \mathbf{A}'\mathbf{x}$, where the matrix $\mathbf{A}'(t', \mathbf{p}) = \mathbf{A}(T(\mathbf{p})t', \mathbf{p}) T'(\mathbf{p})$ is periodic with the constant period $T'(\mathbf{p}) \equiv 1$. Therefore, we can use formula (52) with $t' = 1$ for calculating derivatives of the monodromy matrix.

Note that the form of integrals (52) allows evaluating of any number of derivatives in a parallel process, when the variables t_1, \dots, t_s in all the integrals change simultaneously from 0 to t . This way increases the speed of calculations and doesn't require to store much information during the calculations.

A.2 Derivatives of a simple multiplier

Let us consider a simple multiplier ϱ of the monodromy matrix \mathbf{F} :

$$\mathbf{F}\mathbf{u} - \mathbf{u}\varrho = 0. \quad (54)$$

This multiplier and the corresponding (appropriately normalized) eigenvector \mathbf{u} are smooth functions of parameters. Let \mathbf{v} be the left eigenvector of ϱ satisfying the relations

$$\mathbf{v}^T \mathbf{F} = \varrho \mathbf{v}^T, \quad \mathbf{v}^T \mathbf{u} = 1. \quad (55)$$

Let us take the derivative $D^{\mathbf{h}}$ of (54)

$$\sum_{\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h}} \frac{\mathbf{h}!}{\mathbf{h}_1! \mathbf{h}_2!} (D^{\mathbf{h}_1} \mathbf{F} D^{\mathbf{h}_2} \mathbf{u} - D^{\mathbf{h}_2} \mathbf{u} D^{\mathbf{h}_1} \varrho) = 0. \quad (56)$$

Multiplying (56) by \mathbf{v}^T from the left and using (55), we obtain

$$D^{\mathbf{h}} \varrho = \mathbf{v}^T D^{\mathbf{h}} \mathbf{F} \mathbf{u} + \mathbf{v}^T \sum_{\substack{\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h} \\ |\mathbf{h}_1| > 0, |\mathbf{h}_2| > 0}} \frac{\mathbf{h}!}{\mathbf{h}_1! \mathbf{h}_2!} (D^{\mathbf{h}_1} \mathbf{F} D^{\mathbf{h}_2} \mathbf{u} - D^{\mathbf{h}_2} \mathbf{u} D^{\mathbf{h}_1} \varrho). \quad (57)$$

Rearranging terms in Eq. (56) yields

$$(\mathbf{F} - \varrho\mathbf{I}) D^{\mathbf{h}}\mathbf{u} = \sum_{\substack{\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h} \\ |\mathbf{h}_1| > 0}} \frac{\mathbf{h}!}{\mathbf{h}_1! \mathbf{h}_2!} (D^{\mathbf{h}_2}\mathbf{u} D^{\mathbf{h}_1}\varrho - D^{\mathbf{h}_1}\mathbf{F} D^{\mathbf{h}_2}\mathbf{u}). \quad (58)$$

Though $(\mathbf{F} - \varrho\mathbf{I})$ is a singular matrix, equality (57) ensures existence of the solution $D^{\mathbf{h}}\mathbf{u}$ of (58). This solution is determined up to $\gamma\mathbf{u}$, where γ is an arbitrary constant. A particular solution can be found as follows [18], [24]:

$$D^{\mathbf{h}}\mathbf{u} = \mathbf{D} \sum_{\substack{\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h} \\ |\mathbf{h}_1| > 0}} \frac{\mathbf{h}!}{\mathbf{h}_1! \mathbf{h}_2!} (D^{\mathbf{h}_2}\mathbf{u} D^{\mathbf{h}_1}\varrho - D^{\mathbf{h}_1}\mathbf{F} D^{\mathbf{h}_2}\mathbf{u}), \quad (59)$$

where $\mathbf{D} = (\mathbf{F} - \varrho\mathbf{I} - \bar{\mathbf{v}}\mathbf{v}^T)^{-1}$ is a nonsingular matrix. Note that the vector $D^{\mathbf{h}}\mathbf{u}$, $|\mathbf{h}| > 0$, determined by (59) satisfies the condition $\mathbf{v}^T D^{\mathbf{h}}\mathbf{u} = 0$ [18], [24], which simplifies expression (57) as follows:

$$D^{\mathbf{h}}\varrho = \sum_{\substack{\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{h} \\ |\mathbf{h}_1| > 0}} \frac{\mathbf{h}!}{\mathbf{h}_1! \mathbf{h}_2!} \mathbf{v}^T D^{\mathbf{h}_1}\mathbf{F} D^{\mathbf{h}_2}\mathbf{u}. \quad (60)$$

Expressions (60), (59) with the formulae for derivatives of the monodromy matrix allow recurrent evaluating of any order derivatives of the simple multiplier $\varrho(\mathbf{p})$ and its eigenvector $\mathbf{u}(\mathbf{p})$. For the first- and second-order derivatives we have

$$\varrho_i = \mathbf{v}^T \mathbf{F}_i \mathbf{u}, \quad \mathbf{u}_i = \mathbf{D}(\mathbf{u} \varrho_i - \mathbf{F}_i \mathbf{u}), \quad \varrho_{ij} = \mathbf{v}^T (\mathbf{F}_{ij} \mathbf{u} + \mathbf{F}_i \mathbf{u}_j + \mathbf{F}_j \mathbf{u}_i), \quad (61)$$

where for simplicity derivatives are denoted by subscripts. i.e., $\mathbf{F}_{ij} = \partial^2 \mathbf{F} / \partial p_i \partial p_j$.

The given method of finding derivatives of a simple multiplier follows the approach of [13], where perturbations of eigenvalues in the one parameter case were studied. The formulae for the first- and second-order derivatives of a simple multiplier were also given in [18].

A.3 Derivatives of a versal deformation block

Let ϱ_0 be a real semi-simple double multiplier of the matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$. This multiplier generally splits into two simple multipliers $\varrho_1(\mathbf{p})$ and $\varrho_2(\mathbf{p})$ under a change of parameters. These perturbed multipliers are equal to eigenvalues of the 2×2 block of the versal deformation $\mathbf{F}'(\mathbf{p})$, which satisfies the Eqs. [15], [16]

$$\mathbf{F}\mathbf{C}' - \mathbf{C}'\mathbf{F}' = 0, \quad \mathbf{F}_0' = \mathbf{F}'(\mathbf{p}_0) = \varrho_0\mathbf{I}. \quad (62)$$

Here $\mathbf{C}'(\mathbf{p})$ is an $m \times 2$ real matrix smoothly depending on \mathbf{p} such that $D^0\mathbf{C}' = \mathbf{C}'(\mathbf{p}_0) = \mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2]$, where $\mathbf{u}_1, \mathbf{u}_2$ are the real eigenvectors of ϱ_0 . We denote by $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2]$ an $m \times 2$ matrix consisting of left real eigenvectors and satisfying the equations

$$\mathbf{V}^T \mathbf{F}_0 = \mathbf{F}_0' \mathbf{V}^T, \quad \mathbf{V}^T \mathbf{U} = \mathbf{I}. \quad (63)$$

To obtain derivatives $D^{\mathbf{h}}\mathbf{F}'$ and $D^{\mathbf{h}}\mathbf{C}'$ at $\mathbf{p} = \mathbf{p}_0$ we take the derivative $D^{\mathbf{h}}$ of Eq. (62) and multiply the result by \mathbf{V}^T from the left. This procedure is the same as for a simple multiplier, since Eqs. (62), (63) and (54), (55) are formally identical (if \mathbf{u}, \mathbf{v} , and ϱ are substituted by \mathbf{C}' , \mathbf{V} , and \mathbf{F}' , respectively). Thus, the recurrent expressions for calculation of $D^{\mathbf{h}}\mathbf{F}'$ and $D^{\mathbf{h}}\mathbf{C}'$

take the form

$$\begin{aligned} D^h \mathbf{F}' &= \sum_{\substack{h_1+h_2=h \\ |h_1|>0}} \frac{\mathbf{h}!}{\mathbf{h}_1! \mathbf{h}_2!} \mathbf{V}^T D^{h_1} \mathbf{F} D^{h_2} \mathbf{C}', \\ D^h \mathbf{C}' &= \mathbf{D}' \sum_{\substack{h_1+h_2=h \\ |h_1|>0}} \frac{\mathbf{h}!}{\mathbf{h}_1! \mathbf{h}_2!} (D^{h_2} \mathbf{C}' D^{h_1} \mathbf{F}' - D^{h_1} \mathbf{F} D^{h_2} \mathbf{C}'), \end{aligned} \quad (64)$$

where $\mathbf{D}' = (\mathbf{F}_0 - \varrho_0 \mathbf{I} - \bar{\mathbf{V}} \mathbf{V}^T)^{-1}$. In particular, formulae for the first- and second-order derivatives of \mathbf{F}' are

$$\mathbf{F}'_i = \mathbf{V}^T \mathbf{F}_i \mathbf{U}, \quad \mathbf{C}'_i = \mathbf{D}' (\mathbf{U} \mathbf{F}'_i - \mathbf{F}_i \mathbf{U}), \quad \mathbf{F}'_{ij} = \mathbf{V}^T (\mathbf{F}_{ij} \mathbf{U} + \mathbf{F}_i \mathbf{C}'_j + \mathbf{F}_j \mathbf{C}'_i), \quad (65)$$

where subscripts denote derivatives with respect to parameters as in (61).

Note that derivatives of $\mathbf{F}'(\mathbf{p})$ can be obtained by a different method suggested in [17]. But in this case we will need to calculate all the eigenvalues and the corresponding eigenvectors of the matrix \mathbf{F}_0 .

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