

Uncontrollability Sets of Linear Systems Depending on Parameters

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We consider the linear system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector of state variables; $\mathbf{u} \in \mathbb{R}^m$ is the vector of control variables; and $\mathbf{A}(\mathbf{p})$ and $\mathbf{B}(\mathbf{p})$ are constant real matrices of sizes $n \times n$ and $n \times m$, respectively, smoothly depending on the parameter vector $\mathbf{p} \in \mathbb{R}^k$; the dot denotes the derivative with respect to time t . System (1) is called controllable if it can be transferred from any initial state $\mathbf{x}(0) = \mathbf{x}_0$ to any finite state $\mathbf{x}(T) = \mathbf{x}_1$ in a finite time T by choosing a suitable control law $\mathbf{u}(t)$ [1, 2]. As is known, a system is controllable if and only if the rank of the matrix $[\mathbf{B}, \mathbf{A}\mathbf{B}, \dots, \mathbf{A}^{n-1}\mathbf{B}]$ equals n . The numerical study of the controllability of system (1) is impeded by the structural instability of an uncontrollable system. An arbitrarily small change in the matrices \mathbf{A} and \mathbf{B} (e.g., resulting from an inaccurate specification of the parameter vector \mathbf{p}) may make an uncontrollable system controllable, but controlling such a system requires high energy expenses and is very sensitive to various perturbations.

This paper studies the uncontrollability set \mathcal{N} in the parameter space; this is the set of values of the parameters vector \mathbf{p} for which system (1) is uncontrollable. We give a qualitative description of the structure of this set and develop a theory of perturbations that makes it possible to calculate an approximation of the uncontrollability set in a neighborhood of its regular points. Using the method developed in this paper, we can numerically construct uncontrollability sets in parameter spaces. The obtained results have both theoretical and practical importance for an analysis of the controllability of multiparameter systems.

1. Let \mathbf{P} and \mathbf{Q} be nonsingular real matrices of sizes $n \times n$ and $m \times m$, respectively, and let \mathbf{R} be a real $m \times n$ matrix. Changing a basis in the spaces of state and control variables and introducing the linear feedback $\mathbf{x} =$

$\mathbf{P}\mathbf{x}'$ and $\mathbf{u} = \mathbf{Q}\mathbf{u}' + \mathbf{R}\mathbf{x}'$, we reduce the matrices of system (1) to the form

$$\mathbf{A}' = \mathbf{P}^{-1}(\mathbf{A}\mathbf{P} + \mathbf{B}\mathbf{R}), \quad \mathbf{B}' = \mathbf{P}^{-1}\mathbf{B}\mathbf{Q}. \quad (2)$$

Transformation (2) does not change the controllability properties of the system. By means of such a transformation, any system can be reduced to the normal (Brunovsky's) form [3]

$$\mathbf{A}' = \begin{pmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{pmatrix}, \quad \mathbf{B}' = \begin{pmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where \mathbf{J} is the Jordan matrix (or its real equivalent); $\mathbf{N} = \text{diag}(\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_r)$ and $\mathbf{E} = \text{diag}(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_r)$. Here, \mathbf{N}_i are the Jordan blocks of sizes k_i with zero eigenvalues and $\mathbf{E}_i = (0, 0, \dots, 0, 1)^T$ are the column k_i -vectors. The numbers $k_1 \geq \dots \geq k_r > 0$ are called the indices of controllability; the matrices \mathbf{N} and \mathbf{J} are the controllable and uncontrollable components of the Brunovsky form. The controllability of system (1) is equivalent to the absence of the uncontrollable component \mathbf{J} in the Brunovsky form. The Brunovsky form of system (1) is unique up to a permutation of the diagonal blocks in the Jordan matrix \mathbf{J} .

Suppose that $\mathbf{p} \in \mathcal{N}$, i.e., system (1) is uncontrollable, and its Brunovsky form has an uncontrollable component of form $\mathbf{J}_\sigma = (\sigma)$ or

$$\mathbf{J}_{\sigma \pm i\omega} = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}, \quad \omega > 0.$$

We say that such a system has type \mathbf{J}_σ or $\mathbf{J}_{\sigma \pm i\omega}$, respectively. The matrix \mathbf{J}_σ ($\mathbf{J}_{\sigma \pm i\omega}$) corresponds to the one-dimensional aperiodic (two-dimensional oscillatory) behavior of the system in the uncontrollable part of the state space. The quantities σ and $\sigma \pm i\omega$ are called generalized eigenvalues.

Theorem 1. *In the case of general position, the uncontrollability set of system (1) has a regular part formed by smooth surfaces of codimensions m and $2m$, which correspond to systems of types \mathbf{J}_σ and $\mathbf{J}_{\sigma \pm i\omega}$,*

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respectively. The other points in the uncontrollability set belong to the boundary of the regular part.

Theorem 1 describes a typical structure of an uncontrollability set. Its proof uses versal deformations of matrix pairs; their explicit forms are given in [4].

Example 1. Consider system (1) depending on the parameter vector $\mathbf{p} = (p_1, p_2, p_3)$, where

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 1 \\ p_2 & 0 & p_3 & 0 \end{pmatrix}, \quad \mathbf{B}(\mathbf{p}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Using the uncontrollability condition $\det[\mathbf{B}, \mathbf{A}\mathbf{B}, \mathbf{A}^2\mathbf{B}, \mathbf{A}^3\mathbf{B}] = 0$, we obtain

$$\mathcal{N} = \{\mathbf{p} \mid p_1^2 p_3 - p_2^2 = 0\}. \tag{3}$$

The uncontrollability set (3) is a well-known surface, namely, the Whitney umbrella [5] (Fig. 1). Its regular part is formed by the two smooth surfaces $p_1^2 p_3 - p_2^2 = 0, p_3 \geq 0, p_1 < 0$ or $p_1 > 0$, which have codimension $m = 1$ and type \mathbf{J}_σ , and the half-line $p_1 = p_2 = 0, p_3 < 0$ having codimension $2m = 2$ and type $\mathbf{J}_{\sigma \pm i\omega}$. The remaining points in set (3) lie on the half-line $p_1 = p_2 = 0, p_3 \geq 0$ and have a different type from \mathbf{J}_σ and $\mathbf{J}_{\sigma \pm i\omega}$; they are singularities on the boundary of the regular part.

2. Let \mathbf{p} be a point in the regular part of the uncontrollability set, and let \mathbf{P}, \mathbf{Q} , and \mathbf{R} be the matrices reducing system (1) to the Brunovsky form. In the case of $\mathbf{J} = \mathbf{J}_\sigma$, we introduce the real vectors $\mathbf{f}_i = (f_i^1, f_i^2, \dots, f_i^k)$ with $i = 1, 2, \dots, m$ and $\mathbf{f}_\sigma = (f_\sigma^1, f_\sigma^2, \dots, f_\sigma^k)$ with the components

$$f_i^j = \mathbf{P}^{-1}(n, :) \left[\frac{\partial \mathbf{A}}{\partial p_j} \sum_{s=1}^{k_i} \sigma^{s-1} \mathbf{P}(:, K_i + s) + \frac{\partial \mathbf{B}}{\partial p_j} \left(\sum_{s=1}^{k_i} \sigma^{s-1} \mathbf{R}(:, K_i + s) + \sigma^{k_i} \mathbf{Q}(:, i) \right) \right], \tag{4}$$

$$i = 1, 2, \dots, r;$$

$$f_i^j = \mathbf{P}^{-1}(n, :) \frac{\partial \mathbf{B}}{\partial p_j} \mathbf{Q}(:, i), \quad i = r + 1, r + 2, \dots, m;$$

$$f_\sigma^j = \mathbf{P}^{-1}(n, :) \left[\frac{\partial \mathbf{A}}{\partial p_j} \mathbf{P}(:, n) + \frac{\partial \mathbf{B}}{\partial p_j} \mathbf{R}(:, n) \right],$$

where $K_1 = 0, K_i = k_1 + k_2 + \dots + k_{i-1}$; $\mathbf{P}^{-1}(n, :)$ and $\mathbf{Q}(:, i)$ denote the n th row of the matrix \mathbf{P}^{-1} and the i th column of the matrix \mathbf{Q} , respectively; and the derivatives are evaluated at the point \mathbf{p} .

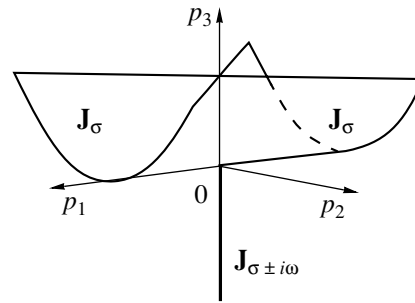


Fig. 1.

In the case of $\mathbf{J} = \mathbf{J}_{\sigma \pm i\omega}$, we define the real vectors $\mathbf{g}_s = (g_s^1, g_s^2, \dots, g_s^k)$ with $s = 1, 2, \dots, 2m$; $\mathbf{g}_\sigma = (g_\sigma^1, g_\sigma^2, \dots, g_\sigma^k)$; and $\mathbf{g}_\omega = (g_\omega^1, g_\omega^2, \dots, g_\omega^k)$ with the components

$$g_{2s-1}^j + i g_{2s}^j = \sum_{z=0}^1 i^{1-z} \mathbf{P}^{-1}(n-z, :) \times \left[\frac{\partial \mathbf{A}}{\partial p_j} \sum_{v=1}^{k_s} (\sigma - i\omega)^{v-1} \mathbf{P}(:, K_s + v) + \frac{\partial \mathbf{B}}{\partial p_j} \times \left(\sum_{v=1}^{k_s} (\sigma - i\omega)^{v-1} \mathbf{R}(:, K_s + v) + (\sigma - i\omega)^{k_s} \mathbf{Q}(:, s) \right) \right], \tag{5}$$

$$s = 1, 2, \dots, r;$$

$$g_{2s-1}^j + i g_{2s}^j = \sum_{z=0}^1 i^{1-z} \mathbf{P}^{-1}(n-z, :) \frac{\partial \mathbf{B}}{\partial p_j} \mathbf{Q}(:, s),$$

$$s = r + 1, r + 2, \dots, m;$$

$$g_\sigma^j + i g_\omega^j = \frac{1}{2} \sum_{z=0}^1 \sum_{v=0}^1 i^{z-v} \mathbf{P}^{-1}(n-z, :) \times \left[\frac{\partial \mathbf{A}}{\partial p_j} \mathbf{P}(:, n-v) + \frac{\partial \mathbf{B}}{\partial p_j} \mathbf{R}(:, n-v) \right];$$

where i is the imaginary unit.

Theorem 2. (a) If \mathbf{p} is a point in the uncontrollability set corresponding to a system of type \mathbf{J}_σ and the vectors $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m$ are linearly independent, then, in a neighborhood of \mathbf{p} , the uncontrollability set is a smooth surface of codimension m . The tangent plane to this surface at the point \mathbf{p} has the form

$$(\mathbf{f}_1, \Delta \mathbf{p}) = (\mathbf{f}_2, \Delta \mathbf{p}) = \dots = (\mathbf{f}_m, \Delta \mathbf{p}) = 0, \tag{6}$$

where $\Delta \mathbf{p}$ is the perturbation of the parameter vector and $(\mathbf{f}_i, \Delta \mathbf{p})$ is the scalar product in \mathbb{R}^k . The perturbation of the generalized eigenvalue σ on the uncontrollability set is specified by

$$\Delta \sigma = (\mathbf{f}_\sigma, \Delta \mathbf{p}) + o(\|\Delta \mathbf{p}\|). \tag{7}$$

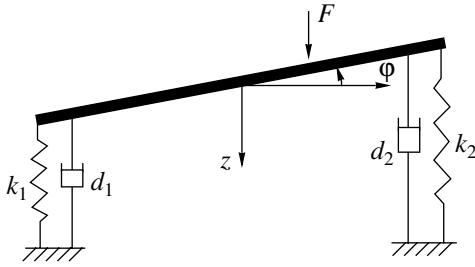


Fig. 2.

(b) If \mathbf{p} is a point in the uncontrollability set corresponding to a system of type $\mathbf{J}_{\sigma \pm i\omega}$ and the vectors $\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_{2m}$ are linearly independent, then, in a neighborhood of \mathbf{p} , the uncontrollability set is a smooth surface of codimension $2m$. The tangent plane to this surface at the point \mathbf{p} has the form

$$(\mathbf{g}_1, \Delta\mathbf{p}) = (\mathbf{g}_2, \Delta\mathbf{p}) = \dots = (\mathbf{g}_{2m}, \Delta\mathbf{p}) = 0. \quad (8)$$

The perturbation of the generalized eigenvalues $\sigma \pm i\omega$ on the uncontrollability set is specified by

$$\Delta\sigma \pm i\Delta\omega = (\mathbf{g}_\sigma, \Delta\mathbf{p}) \pm i(\mathbf{g}_\omega, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|). \quad (9)$$

The proof of Theorem 2 uses the reduction of a family of systems (1) to versal deformations [6]. Theorem 2 makes it possible to determine an approximation of the uncontrollability set in a neighborhood of its regular point based on data related to this point, namely, the first derivatives of the matrices \mathbf{A} and \mathbf{B} with respect to the parameters and the matrices of the transformation to the Brunovsky form. The Brunovsky form and the transformation matrices \mathbf{P}, \mathbf{Q} , and \mathbf{R} can be calculated with the use of the software [7] for computing the Kronecker form of the matrix pencil $[\mathbf{A}, \mathbf{B}] - \lambda[\mathbf{I}, 0]$, which differs from the Brunovsky form only in the order of columns.

Note that the linear independence condition of Theorem 2 is a constructive criterion in the case of general position, with which Theorem 1 deals.

Example 2. Consider the oscillatory system consisting of a thin platform of mass m and length $2l$ (Fig. 2). The ends of the platform are fixed on viscoelastic supports with coefficients of elasticity k_1 and k_2 and coefficients of viscous friction d_1 and d_2 . The control is performed by means of a vertical force F applied at the distance ξl from the center of the platform. As generalized coordinates, we take the vertical coordinate z of the center of the platform and the rotation angle ϕ . It is assumed that, in the absence of the force F , the system has the equilibrium $z = 0, \phi = 0$. At $F = 0$, under a small deviation from the equilibrium, the system oscillates with decreasing amplitude and tends to the equilibrium as $t \rightarrow +\infty$. Under the controllability condition, the system can be put into the equilibrium in a finite time.

Let us introduce the dimensionless variables and constants

$$\tau = \frac{t}{\alpha}, \quad f_1 = \frac{(d_1 + d_2)\alpha}{m}, \quad f_2 = \frac{(d_1 - d_2)\alpha}{m},$$

$$c_1 = \frac{(k_1 + k_2)\alpha^2}{m}, \quad c_2 = \frac{(k_1 - k_2)\alpha^2}{m}, \quad u = \frac{\alpha^2}{ml}F,$$

where α is the characteristic time of the system. We take the dimensionless state vector $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ in the form

$$x_1 = \frac{z}{l}, \quad x_2 = \phi, \quad x_3 = \frac{\alpha \dot{z}}{l}, \quad x_4 = \alpha \dot{\phi}.$$

The equations of motion of the system linearized in a neighborhood of the equilibrium point $\mathbf{x} = 0$ have form (1), where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_1 & -c_2 & -f_1 & -f_2 \\ -3c_2 & -3c_1 & -3f_2 & -3f_1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3\xi \end{pmatrix}. \quad (10)$$

Let us fix the values $c_1 = \frac{25}{12}$ and $f_1 = 1$ characterizing the total elasticity and viscous friction in the supports, and introduce the parameter vector $\mathbf{p} = (c_2, f_2, \xi)$. Consider the point $\mathbf{p} = 0$ in the parameter space, which corresponds to equal supports and to the application of the controlling force to the center of the platform. Such a system is uncontrollable and has the type $\mathbf{J}_{\sigma \pm i\omega}$ with generalized eigenvalues $\sigma \pm i\omega = -1.5 \pm i \cdot 2$. The matrices reducing the system to the Brunovsky form are

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1.5 & 2 \end{pmatrix}, \quad \mathbf{Q} = (1), \quad (11)$$

$$\mathbf{R} = (25/12 \quad 1 \quad 0 \quad 0).$$

Substituting (10) and (11) in (5) where $r = 1$ and $k_1 = 2$ yields

$$\mathbf{g}_1 = (0, -3, 6), \quad \mathbf{g}_2 = (-1.5, 2.25, 1.75),$$

$$\mathbf{g}_\sigma = (0, 0, 0), \quad \mathbf{g}_\omega = (0, 0, 0). \quad (12)$$

Since the vectors \mathbf{g}_1 and \mathbf{g}_2 are linearly independent, Theorem 2 implies that the uncontrollability set of the system in a neighborhood of the point $\mathbf{p} = 0$ is a smooth curve in the three-dimensional parameter space, and its tangent line is

$$\mathbf{p} = (75, 36, 18)c, \quad c \in \mathbb{R}. \quad (13)$$

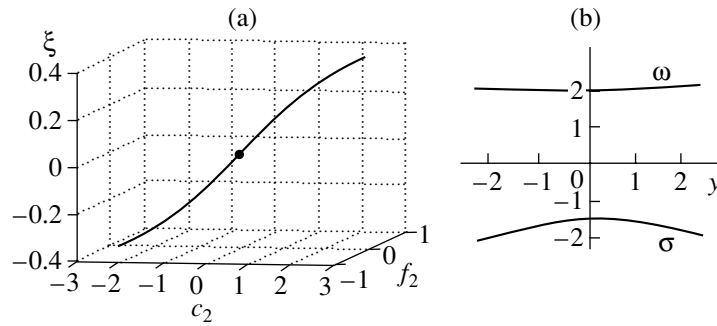


Fig. 3.

The variation of the generalized eigenvalues along this curve has the order $o(\|\mathbf{p}\|)$.

Moving along the tangent line, we can calculate the entire curve of the uncontrollability set in the physical domain of the parameter space where $|c_2| \leq |c_1|$, $|f_2| \leq |f_1|$, and $|\xi| \leq 1$. The result of calculations is shown in Fig. 3a. Figure 3b gives the graphs of the real and imaginary parts of the generalized eigenvalue $\sigma(y) + i\omega(y)$, where y is the length parameter of the curve. It can be shown that, at the parameter values $\mathbf{p} = (c_2, f_2, \xi)$ on the curve \mathcal{N} , one of the modes of the free oscillations of the system is a rotation about a point lying at a distance of ξl from the center of the platform. The application of the force to this point does not affect the form of oscillations under consideration, which results in the uncontrollability of the system. It is seen from Fig. 3a that the uncontrollability takes place when the coefficients of elasticity and viscous friction of one support are smaller than those of the other. The force is applied on the side of the less elastic support.

3. The results obtained in this paper help to avoid the negative effects related to uncontrollability in designing control systems. All these results can be transferred to unobservability sets of parameter-dependent linear

systems of the form $\dot{\mathbf{x}} = -\mathbf{A}^T \mathbf{x}$, $\mathbf{z} = \mathbf{B}^T \mathbf{x}$ with the use of the duality principle [1, 2].

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