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Bimodal Bifurcations of Equilibria in Symmetric Potential Systems

A. P. Seyranian[1],* and A. A. Mailybaev

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Some early examples of bimodal bifurcations found in potential systems, characterized by two modes of the loss of stability at the critical point, were reported in [1–3]. A general approach to analysis of the postbuckling behavior of elastic systems was developed by Thompson and Hunt [4]. Bimodal bifurcations and unfolding for symmetric systems with two degrees of freedom were studied in [5–7]. Monograph [8] gave, in particular, a review of investigations into the supercritical behavior of elastic systems. It was established [4, 9–13] that bimodal bifurcations are closely connected with the problems of structural optimization. However, it should be noted that the cited works present neither complete analysis nor a full list of possible bifurcations.

In this study, we present the general theory of bimodal bifurcations in potential systems with one or two symmetries. A complete classification of bifurcations and unfolding caused by changes in the system parameters is given. All formulas are written in explicit mode, in terms of the derivatives of the potential energy of the system with an arbitrary number of degrees of freedom. As an example from mechanics, the loss of stability and the supercritical behavior of an elastic articulated beam loaded by a longitudinal force is investigated. The phenomenon of the loss of stability of a symmetric articulated beam in an asymmetric mode is detected.

1. Let us consider a system with potential forces, described by the generalized coordinate vector $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. The equilibrium of this system is a singular point of the potential energy $V(\mathbf{q})$:

$$\nabla V = 0, \quad \nabla = \left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_n}\right). \tag{1}$$

The equilibrium is stable if it corresponds to a local minimum of the potential $V(\mathbf{q})$. Thus, a sufficient condition for the stability is the positive definiteness of the matrix of the second derivatives

$$\mathbf{C}(\mathbf{q}) = \begin{bmatrix} \dfrac{\partial^2 V}{\partial q_1^2} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & \cdots & \dfrac{\partial^2 V}{\partial q_1 \partial q_n} \\ \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & \dfrac{\partial^2 V}{\partial q_2^2} & \cdots & \dfrac{\partial^2 V}{\partial q_1 \partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 V}{\partial q_2 \partial q_n} & \dfrac{\partial^2 V}{\partial q_2 \partial q_n} & \cdots & \dfrac{\partial^2 V}{\partial q_n^2} \end{bmatrix} > 0, \tag{2}$$

where the derivatives are taken at the equilibrium point $\mathbf{q}$.

The symmetric matrix $\mathbf{C}$ of an elastic system is called the rigidity matrix.

Let us consider the potential $V(\mathbf{q})$, which is symmetric with respect to the coordinate inversion

$$V(\mathbf{q}) = V(-\mathbf{q}). \tag{3}$$

This property is inherent in many systems, such as rectilinear beams, plates, pendulum systems, etc. In this case, there is the trivial equilibrium position $\mathbf{q} = 0$ and the expansion of the potential into Taylor series in the vicinity of $\mathbf{q} = 0$ contains only even-order terms.

Let the system depend smoothly on a single parameter ε, and let the trivial equilibrium $\mathbf{q} = 0$ be stable at $\varepsilon < 0$ and unstable at $\varepsilon > 0$. For example, ε can describe the deviation of a load from its critical value. For $\varepsilon = 0$, the matrix $\mathbf{C}_0 = \mathbf{C}(0)$ is degenerate and positively

[1]This author has also appeared under an alternate spelling as A.P. Seĭranyan.

Institute of Mechanics, Moscow State University, Moscow, 119192 Russia
** e-mail: seyran@imec.msu.ru*

semidefinite: $\mathbf{C}_0 \geq 0$. In this study, we consider the case of a bimodal bifurcation in which the matrix $\mathbf{C}_0$ is associated with two linearly independent eigenvectors, $\mathbf{u}_1$ and $\mathbf{u}_2$, so that

$$\mathbf{C}_0\mathbf{u}_1 = 0, \quad \mathbf{C}_0\mathbf{u}_2 = 0. \tag{4}$$

Expanding the potential V in the Taylor series in the vicinity of the point ($\mathbf{q} = 0$, $\varepsilon = 0$), we obtain from Eq. (1) in the first approximation

$$\nabla V = \mathbf{C}_0\mathbf{q} + \ldots = 0. \tag{5}$$

Therefore, for small $\mathbf{q}$ and ε, nontrivial equilibrium positions take the asymptotic mode

$$\mathbf{q}(\varepsilon) \approx \alpha\mathbf{u}_1 + \beta\mathbf{u}_2, \tag{6}$$

where α and β are unknown functions of ε. To determine these functions, we must consider the next-order terms in Eq. (1). Writing these terms with account of Eqs. (4) and (6) and multiplying scalarly the relations thus obtained by $\mathbf{u}_1$ and $\mathbf{u}_2$, we eventually arrive at a system of two equations with respect to α and β:

$$\begin{aligned} &(v_{11\varepsilon}\alpha + v_{12\varepsilon}\beta)\varepsilon + \frac{v_{1111}}{6}\alpha^3 + \frac{v_{1112}}{2}\alpha^2\beta \\ &\quad + \frac{v_{1122}}{2}\alpha\beta^2 + \frac{v_{1222}}{6}\beta^3 = 0, \\ &(v_{12\varepsilon}\alpha + v_{22\varepsilon}\beta)\varepsilon + \frac{v_{1112}}{6}\alpha^3 + \frac{v_{1122}}{2}\alpha^2\beta \\ &\quad + \frac{v_{1222}}{2}\alpha\beta^2 + \frac{v_{2222}}{6}\beta^3 = 0. \end{aligned} \tag{7}$$

Here, the following designations are used:

$$\begin{aligned} v_{abcd} &= (\mathbf{u}_a\cdot\nabla)(\mathbf{u}_b\cdot\nabla)(\mathbf{u}_c\cdot\nabla)(\mathbf{u}_d\cdot\nabla)V, \\ v_{ab\varepsilon} &= (\mathbf{u}_a\cdot\nabla)(\mathbf{u}_b\cdot\nabla)\frac{\partial V}{\partial\varepsilon}, \end{aligned} \tag{8}$$

where $\mathbf{u}_a \cdot \nabla$ is the derivative at point ($\mathbf{q} = 0$, $\varepsilon = 0$) in the direction of the fixed vector $\mathbf{u}_a = (u_{a1}, u_{a2}, \ldots, u_{an})$, for example

$$\begin{aligned} v_{1112} &\equiv (\mathbf{u}_1\cdot\nabla)^3(\mathbf{u}_2\cdot\nabla)V \\ &= \sum_{i,j,k,l=1}^{n}\frac{\partial^4 V}{\partial q_i\partial q_j\partial q_k\partial q_l}u_{1i}u_{1j}u_{1k}u_{2l}. \end{aligned} \tag{9}$$

It can readily be checked that the solution of the system of equations (7) can be written as $\varepsilon = c\beta^2$ and $\alpha = \gamma\beta$, where

$$\begin{aligned} c &= -\frac{v_{1111}\gamma^3 + 3v_{1112}\gamma^2 + 3v_{1122}\gamma + v_{1222}}{6(v_{11\varepsilon}\gamma + v_{12\varepsilon})} \\ &= -\frac{v_{1112}\gamma^3 + 3v_{1122}\gamma^2 + 3v_{1222}\gamma + v_{2222}}{6(v_{12\varepsilon}\gamma + v_{22\varepsilon})}, \end{aligned} \tag{10}$$

and γ is determined from the fourth-order polynomial equation

$$c_4\gamma^4 + c_3\gamma^3 + c_2\gamma^2 + c_1\gamma + c_0 = 0 \tag{11}$$

with the coefficients

$$\begin{aligned} c_0 &= v_{1222}v_{22\varepsilon} - v_{2222}v_{12\varepsilon}, \\ c_1 &= 3v_{1122}v_{22\varepsilon} - 2v_{1222}v_{12\varepsilon} - v_{2222}v_{11\varepsilon}, \\ c_2 &= 3v_{1112}v_{22\varepsilon} - 3v_{1222}v_{11\varepsilon}, \\ c_3 &= v_{1111}v_{22\varepsilon} + 2v_{1112}v_{12\varepsilon} - 3v_{1122}v_{11\varepsilon}, \\ c_4 &= v_{1111}v_{12\varepsilon} - v_{1112}v_{11\varepsilon}. \end{aligned} \tag{12}$$

It can be shown that Eq. (11) has two or four real roots.

The problem of the stability of the nontrivial equilibrium (6) reduces to the determination of the condition of positive definiteness of the matrix $\mathbf{C}(\mathbf{q})$. Due to the positive semidefiniteness of the matrix $\mathbf{C}_0$, it is sufficient to check this condition for small ε for the directions $\delta\mathbf{q} \sim a\mathbf{u}_1 + b\mathbf{u}_2$ with arbitrary coefficients a and b in the space of states. This leads to a simple condition: the equilibrium is stable in the case of positive definiteness of the matrix

$$\begin{pmatrix} v_{11\varepsilon}\varepsilon + \frac{v_{1111}}{2}\alpha^2 + v_{1112}\alpha\beta + \frac{v_{1122}}{2}\beta^2 & v_{12\varepsilon}\varepsilon + \frac{v_{1112}}{2}\alpha^2 + v_{1122}\alpha\beta + \frac{v_{1222}}{2}\beta^2 \\ v_{12\varepsilon}\varepsilon + \frac{v_{1112}}{2}\alpha^2 + v_{1122}\alpha\beta + \frac{v_{1222}}{2}\beta^2 & v_{22\varepsilon}\varepsilon + \frac{v_{1122}}{2}\alpha^2 + v_{1222}\alpha\beta + \frac{v_{2222}}{2}\beta^2 \end{pmatrix} > 0. \tag{13}$$

Since the trivial equilibrium position ($\alpha = \beta = 0$) was assumed to be stable for $\varepsilon < 0$, condition (13) leads to the following inequalities:

$$v_{11\varepsilon} < 0, \quad v_{22\varepsilon} < 0, \quad v_{11\varepsilon}v_{22\varepsilon} - v_{12\varepsilon}^2 > 0. \tag{14}$$

2. Let us consider the case where the potential of the system possesses an additional symmetry:

$$V(\mathbf{q}) = V(\mathcal{S}(\mathbf{q})), \tag{15}$$

where $\mathcal{S}(\mathbf{q})$ is a linear map such that $\mathcal{S}(\mathcal{S}(\mathbf{q})) = \mathbf{q}$. Such maps are often used to describe the spatial symmetry (e.g., for a beam of variable cross section, which is symmetric about the center). We will call the vector $\mathbf{q}$ symmetric or antisymmetric provided that $\mathcal{S}(\mathbf{q}) = \mathbf{q}$ or

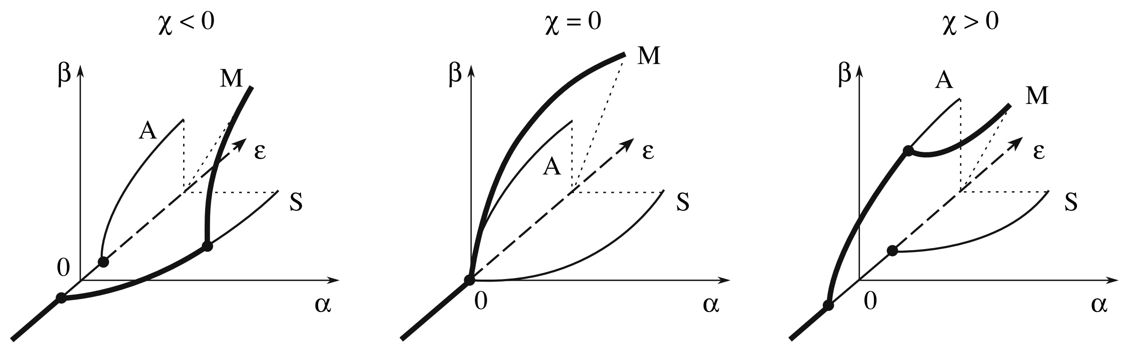

Fig. 1. Rearrangement of the bimodal bifurcation, case 6 of Table 1.

$\mathcal{S}(\mathbf{q}) = -\mathbf{q}$, respectively. The case of $\mathcal{S}(\mathbf{q}) \neq \pm\mathbf{q}$ will be referred to as mixed.

The eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ can always be chosen symmetric or antisymmetric. Let $\mathbf{u}_1$ be a symmetric vector and $\mathbf{u}_2$ be an antisymmetric vector. In this case, a complete analytical investigation can be carried out. From condition (15) it follows that coefficients (8) remain invariant when $\mathbf{u}_1$ and $\mathbf{u}_2$ are changed for $\mathcal{S}(\mathbf{u}_1) = \mathbf{u}_1$ and $\mathcal{S}(\mathbf{u}_2) = -\mathbf{u}_2$, respectively. Therefore, we have

$$v_{12\varepsilon} = v_{1112} = v_{1222} = 0, \tag{16}$$

since these coefficients change sign as a result of the above substitution. For the sake of convenience, let us introduce the normalization conditions for the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that

$$v_{11\varepsilon} = -1, \quad v_{22\varepsilon} = -1. \tag{17}$$

This is always possible, since, in accordance with Eq. (14), $v_{11\varepsilon} < 0$ and $v_{22\varepsilon} < 0$.

Equations (7) with allowance for Eqs. (16) and (17) can be solved in an explicit mode that yields nontrivial equilibria of three types:

$$\alpha^2 = \frac{6\varepsilon}{v_{1111}}, \quad \beta = 0; \tag{18}$$

$$\alpha = 0, \quad \beta^2 = \frac{6\varepsilon}{v_{2222}}; \tag{19}$$

$$\alpha^2 = \frac{v_{2222} - 3v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2}6\varepsilon, \qquad \beta^2 = \frac{v_{1111} - 3v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2}6\varepsilon. \tag{20}$$

Solutions (18), (19), and (20) with different signs of α and β determine two symmetric, two antisymmetric, and four mixed equilibria (6), respectively. The equilibria of the mixed type exist if the quantities $v_{2222} - 3v_{1122}$ and $v_{1111} - 3v_{1122}$ are of the same sign.

Using condition (13), the condition of stability for the symmetric equilibria (18) can be written as

$$v_{1111} > 0, \quad 3v_{1122} - v_{1111} > 0. \tag{21}$$

For the antisymmetric equilibria (19), the conditions of stability take the mode

$$v_{2222} > 0, \quad 3v_{1122} - v_{2222} > 0. \tag{22}$$

Finally, for the mixed equilibria (20), we obtain the following conditions of stability:

$$v_{1111} > 0, \quad v_{2222} > 0, \quad v_{1111}v_{2222} - 9v_{1122}^2 > 0. \tag{23}$$

Using the above results, it is possible to classify the main types of bimodal bifurcations. There are 16 types of such bifurcations, which are presented in Table 1. These bifurcations differ in the mode of the nontrivial equilibria given in columns 2 to 4 and are characterized by the signs of the coefficients indicated in columns 5 to 8. Columns 2 to 4 (S, A, and M) describe the symmetric, antisymmetric, and mixed types of nontrivial equilibria, respectively. Here, the leftward and rightward arrows indicate that the equilibria exist for $\varepsilon > 0$ and $\varepsilon < 0$, respectively. Numbers under the arrows indicate the instability degree (the number of negative eigenvalues of the rigidity matrix $\mathbf{C}$) for the given equilibrium; stable equilibria are determined by the instability degree equal to zero. The case in which no equilibrium of the mixed type can exist is designated by the symbol $\emptyset$.

For cases 6 and 12, the bifurcations are qualitatively shown in Figs. 1 and 2 ($\chi = 0$), respectively. Stable equilibria are indicated by bold lines, while thin solid and dashed lines indicate the unstable equilibria with the instability degrees 1 and 2, respectively.

From Table 1, it can be seen that nontrivial stable equilibria exist in cases 1 to 3, 6, 11, and 12 and the equilibrium of any symmetry type can be stable. All stable nontrivial equilibria are supercritical; that is, they exist for $\varepsilon > 0$. It should be noted that cases 1, 6, 7, 11, and 12 were qualitatively described in [7].

3. Let us consider a potential obeying the symmetry properties (3) and (15) and smoothly dependent on *m*

Table 1. Classification of bimodal bifurcations

No.	S	A	M	v_{1111}	v_{2222}	$v_{1111}v_{2222} - 9v_{1122}^2$	$v_{1111} - 3v_{1122}$	$v_{2222} - 3v_{1122}$
1	$\vec{0}$	$\vec{0}$	$\vec{1}$	+	+	–	–	–
2	$\vec{0}$	$\overleftarrow{1}$	$\vec{1}$	+	–	–	–	–
3	$\overleftarrow{1}$	$\vec{0}$	$\vec{1}$	–	+	–	–	–
4	$\overleftarrow{1}$	$\overleftarrow{1}$	$\vec{1}$	–	–	–	–	–
5	$\overleftarrow{1}$	$\overleftarrow{1}$	$\overleftarrow{2}$	–	–	+	–	–
6	$\vec{1}$	$\vec{1}$	$\vec{0}$	+	+	+	+	+
7	$\vec{1}$	$\vec{1}$	$\overleftarrow{1}$	+	+	–	+	+
8	$\vec{1}$	$\overleftarrow{2}$	$\overleftarrow{1}$	+	–	–	+	+
9	$\overleftarrow{2}$	$\vec{1}$	$\overleftarrow{1}$	–	+	–	+	+
10	$\overleftarrow{2}$	$\overleftarrow{2}$	$\overleftarrow{1}$	–	–	–	+	+
11	$\vec{0}$	$\vec{1}$	$\varnothing$	+	+		–	+
12	$\vec{1}$	$\vec{0}$	$\varnothing$	+	+		+	–
13	$\vec{1}$	$\overleftarrow{1}$	$\varnothing$	+	–		+	–
14	$\overleftarrow{1}$	$\vec{1}$	$\varnothing$	–	+		–	+
15	$\overleftarrow{1}$	$\overleftarrow{2}$	$\varnothing$	–	–		–	+
16	$\overleftarrow{2}$	$\overleftarrow{1}$	$\varnothing$	–	–		+	–

real parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$. Let the values $\varepsilon_1 = \varepsilon_2 = \ldots \varepsilon_m = 0$ correspond to the bimodal bifurcation point. The bifurcation pattern for fixed $\varepsilon_2 = \ldots = \varepsilon_m = 0$, and the variable parameter $\varepsilon = \varepsilon_1$ is described above (see Section 2). In this section, we will consider the rearrangement of a bimodal bifurcation upon a change to $\varepsilon = \varepsilon_1$ for nonzero (but small and fixed) values of the parameters $\varepsilon_2, \ldots, \varepsilon_m$.

In the case under consideration, the nontrivial equilibria are also described by the asymptotic formula (6). The unknown coefficients α and β corresponding to the

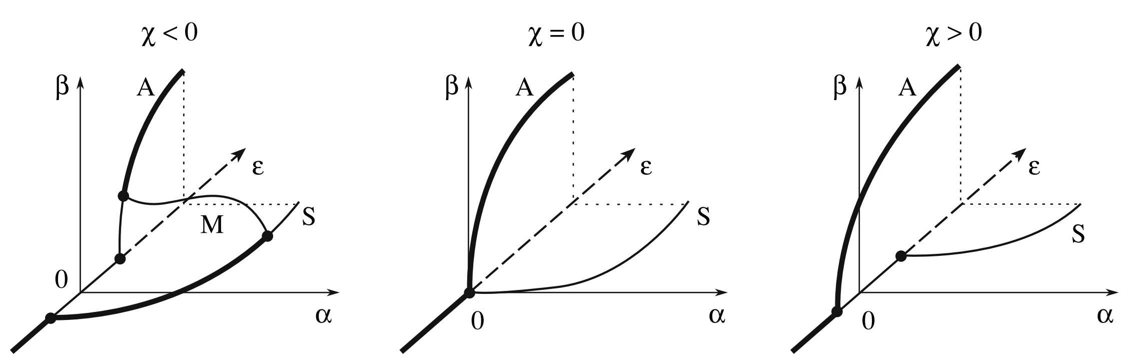

Fig. 2. Rearrangement of the bimodal bifurcation, case 12 of Table 1.

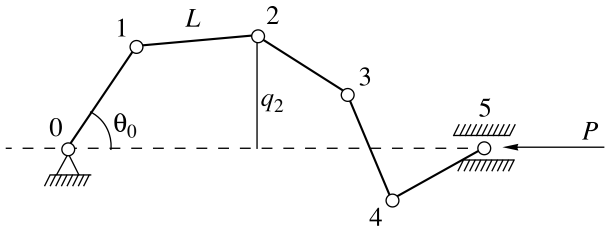

Fig. 3. Elastic articulated beam losing stability under the action of an axial force.

symmetric, antisymmetric, and mixed types of equilibria are obtained in the following mode:

$$\alpha^2 = \frac{6(\varepsilon - \tilde{v}_{11\varepsilon})}{v_{1111}}, \quad \beta = 0; \tag{24}$$

$$\alpha = 0, \quad \beta^2 = \frac{6(\varepsilon - \tilde{v}_{22\varepsilon})}{v_{2222}}; \tag{25}$$

$$\begin{aligned} \alpha^2 &= 6\frac{(\varepsilon - \tilde{v}_{11\varepsilon})v_{2222} - 3(\varepsilon - \tilde{v}_{22\varepsilon})v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2}, \\ \beta^2 &= 6\frac{(\varepsilon - \tilde{v}_{22\varepsilon})v_{1111} - 3(\varepsilon - \tilde{v}_{11\varepsilon})v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2}, \end{aligned} \tag{26}$$

where

$$\tilde{v}_{11\varepsilon} = \sum_{k=2}^{n} (\mathbf{u}_1 \cdot \nabla)^2 \frac{\partial V}{\partial \varepsilon_k}\varepsilon_k, \quad \tilde{v}_{22\varepsilon} = \sum_{k=2}^{n} (\mathbf{u}_2 \cdot \nabla)^2 \frac{\partial V}{\partial \varepsilon_k}\varepsilon_k. \tag{27}$$

The bimodal critical point splits into two unimodal points: $\varepsilon = \tilde{v}_{11\varepsilon}$ and $\varepsilon = \tilde{v}_{22\varepsilon}$. In this case, equilibria of the mixed type occur at the points of secondary bifurcations, namely, the bifurcations of symmetric equilibria for

$$\begin{aligned} (\mathrm{S})\colon \alpha_s^2 &= -\frac{6(\tilde{v}_{11\varepsilon} - \tilde{v}_{22\varepsilon})}{v_{1111} - 3v_{1122}}, \quad \beta_s = 0, \\ \varepsilon_s &= \frac{\tilde{v}_{22\varepsilon}v_{1111} - 3\tilde{v}_{11\varepsilon}v_{1122}}{v_{1111} - 3v_{1122}} \end{aligned} \tag{28}$$

or the bifurcations of antisymmetric equilibria for

$$\begin{aligned} (\mathrm{A})\colon \alpha_a &= 0, \quad \beta_a^2 = \frac{6(\tilde{v}_{11\varepsilon} - \tilde{v}_{22\varepsilon})}{v_{2222} - 3v_{1122}}, \\ \varepsilon_a &= \frac{\tilde{v}_{11\varepsilon}v_{2222} - 3\tilde{v}_{22\varepsilon}v_{1122}}{v_{2222} - 3v_{1122}}. \end{aligned} \tag{29}$$

The bifurcations at points (28) and (29), as well as the equilibria of the mixed type (26), take place when the quantities α^2 and β^2 in these expressions are positive. Eliminating ε from Eq. (26), we obtain

$$\begin{gathered} (v_{2222} - 3v_{1122})\beta^2 - (v_{1111} - 3v_{1122})\alpha^2 = 6\chi, \\ \chi = \tilde{v}_{11\varepsilon} - \tilde{v}_{22\varepsilon}. \end{gathered} \tag{30}$$

Thus, the equilibrium positions of the mixed type in the (α, β) plane are described by hyperbolas, an ellipse, or an empty set, depending on the signs of coefficients $v_{1111} - 3v_{1122}$, $v_{2222} - 3v_{1122}$, and χ. The points of intersection of these curves with the α and β axes correspond to secondary bifurcations.

Examples of the rearrangement of bimodal bifurcations for cases 6 and 12 of Table 1, obtained from an analysis of Eqs. (24)–(26), are presented in Figs. 1 and 2, respectively. The analysis of stability in the perturbed case ($\chi \neq 0$) is performed using the properties of unimodal bifurcations [7] and the analysis of stability of the bimodal bifurcation described above ($\chi = 0$).

It should be noted that a similar analysis of the bifurcation rearrangement for a system with two degrees of freedom and a double symmetry was performed by Supple [5, 6].

4. As an example, let us consider the problem of the stability of the straight position of an articulated beam loaded by the axial force P (Fig. 3). It is assumed that the beam consists of five rigid members of the same length, connected with elastic hinges with rigidities b_0, b_1, …, and b_5. In the linear formulation, the problem of the stability of this beam was considered in [10]. We will consider a structure symmetric about the center, with symmetric boundary conditions, so that $b_0 = b_5$, $b_1 = b_4$, and $b_2 = b_3$. In the dimensionless variables, the beam deflection is determined by the displacement vector $\mathbf{q} = (q_1, q_2, q_3, q_4)$ whose components are related to the angles of member deflection from the horizontal axis by the formulas

$$q_{i+1} - q_i = \sin\theta_i, \quad i = 0, \ldots, 4, \quad q_0 = q_4 = 0. \tag{31}$$

The potential energy of the articulated beam can be written as

$$\begin{gathered} V = \sum_{i=0}^{5} \left(\frac{b_i}{2}(\theta_i - \theta_{i-1})^2 - P(1 - \cos\theta_i) \right), \\ \theta_{-1} = 0, \quad \theta_5 = 0. \end{gathered} \tag{32}$$

This potential possesses the symmetry properties $V(\mathbf{q}) = V(-\mathbf{q})$ and $V(\mathbf{q}) = V(\mathcal{S}(\mathbf{q}))$, where $\mathcal{S}(\mathbf{q}) = (q_4, q_3, q_2, q_1)$.

Analysis and comparison of the critical loss-of-stability forces of the beam for the symmetric and antisymmetric modes yields the following equation:

$$2b_2 - \frac{2}{5}b_0 + \sqrt{b_0^2 - 2b_0b_2 + 4b_1^2 + b_2^2} = \frac{1}{5}\sqrt{9b_0^2 + 40b_0b_1 - 90b_0b_2 + 100b_1^2 - 200b_1b_2 + 225b_2^2} \tag{33}$$

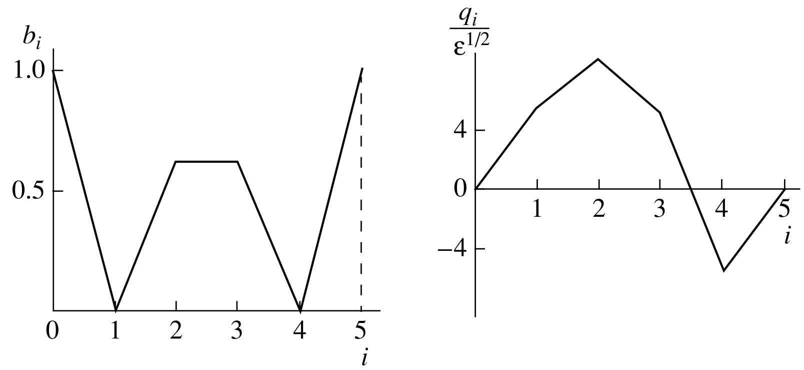

Fig. 4. Rigidities of the articulated beam and the asymmetric mode of the loss of stability.

This equation determines a surface in the three-dimensional space of rigidities (b_0, b_1, b_2). The points on this surface correspond to the beams with bimodal (double) critical forces.

Let us consider the supercritical behavior of the articulated beam with the rigidities $b_0 = 1$, $b_1 = 0$, and $b_2 = 0.6$ satisfying condition (33). In this case, the first hinge has zero rigidity and does not resist bending. The bimodal critical force is $P = 0.6$, and the corresponding modes of the loss of stability (eigenvectors of the rigidity matrix) are $\mathbf{u}_1 = (0, 1, 1, 0)$ and $\mathbf{u}_2 = (3, 1, –1, –3)$. The first mode of the loss of stability is symmetric, while the second is antisymmetric. Note that the symmetric mode of the loss of stability is local.

For the potential energy described by Eqs. (31) and (32), formula (8) yields the coefficients $v_{11\varepsilon} = -2$ and $v_{22\varepsilon} = -30$. We can normalize the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ by dividing them by $\sqrt{-v_{11\varepsilon}}$ and $\sqrt{-v_{22\varepsilon}}$, respectively, so that condition (17) is fulfilled. Using formula (8) with the normalized eigenvectors, we calculate the coefficients to obtain

$$v_{1111} = 0.3, \quad v_{2222} = 0.3, \quad v_{1122} = -0.08. \quad (34)$$

This bifurcation belongs to type 6 in Table 1, for which the symmetric and antisymmetric modes of the loss of stability are supercritical and unstable, whereas the mixed mode is supercritical and stable (Fig. 1, $\chi = 0$). Thus, an interesting phenomenon is found: an articulated beam symmetric about the center, being loaded by an axial force, loses stability in an antisymmetric mode! This effect can also be expected in more complicated symmetric structures such as elastic thin shells.

According to Eqs. (6) and (18)–(20), stable equilibrium states after bifurcation are described by the asymptotic formulas

$$\begin{aligned} \mathbf{q}_{m1} &= \pm\sqrt{\varepsilon}(0, 5.48, 8.90, 5.25, -5.48, 0), \\ \mathbf{q}_{m2} &= \pm\sqrt{\varepsilon}(0, -5.48, 5.25, 8.90, 5.48, 0), \end{aligned} \quad (35)$$

where $\varepsilon = P - 0.6$. The rigidities of the bimodal articulated beam and the mode of the stable nontrivial equilibrium divided by $\sqrt{\varepsilon}$ are presented in Fig. 4. Four stable equilibrium modes (35) are obtained from the mode presented in Fig. 4 by mirror reflections about the beam center and the q_i axis.

Let us investigate the rearrangement of the bimodal bifurcation with increasing rigidity b_1. Setting $\varepsilon_1 = P - 0.6$ and $\varepsilon_2 = b_1$, we obtain the following relation from Eq. (27) with the normalized eigenvectors:

$$\chi = \tilde{v}_{11\varepsilon} - \tilde{v}_{22\varepsilon} = 3.93 b_1. \quad (36)$$

Therefore, in the case of a small rigidity of the first hinge, $b_1 > 0$, the stable antisymmetric mode forks from the trivial equilibrium position for a force smaller than $P = 0.6$. As a result of the secondary bifurcation, this mode becomes unstable, whereas the asymmetric mode is stable. The corresponding rearrangement is qualitatively shown in Fig. 1 ($\chi > 0$).

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