Bifurcations of Equilibria in Potential Systems at Bimodal Critical Points

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Bifurcations of equilibria at bimodal branching points in potential systems are investigated. General formulas describing postbuckling paths and conditions for their stability are derived in terms of the original potential energy. Formulas describing unfolding of bimodal branching points due to a change of system parameters are given. A full list of possible cases for postbuckling paths, their stability, and unfolding depending on three system coefficients is presented. In order to calculate these coefficients, one needs the derivatives of the potential energy and eigenvectors of the linearized problem taken at the bifurcation point. The presented theory is illustrated by a mechanical example on stability and postbuckling behavior of an articulated elastic column having four degrees of freedom and depending on three problem parameters (stiffness coefficients at the hinges). For some of the bimodal critical points, numerical results are obtained illustrating influence of parameters on postbuckling paths, their stability, and unfolding. A surprising phenomenon that a symmetric bimodal column loaded by an axial force can buckle with a stable asymmetric mode is recognized. An example with a constrained sum of the stiffnesses of the articulated column shows that the maximum critical load (optimal design) is attained at the bimodal point. [DOI: 10.1115/1.2793136]

1 Introduction

This paper is devoted to analysis of bimodal branching points of stable trivial equilibrium in multiple degrees-of-freedom potential systems with the symmetry. These points were studied in a number of books and papers [1–5]. In the books [1,2], a rather general method how to analyze postbuckling paths and their stability is presented. This method involves diagonalization procedure of the potential energy and elimination of passive coordinates, i.e., some transformations of the original potential energy are needed. References [3,4] deal with the unfolding of bimodal branching points of general two degrees-of-freedom systems with symmetry. The bimodal critical points and their unfolding for two degrees-of-freedom systems with double symmetry were studied in Chap. X of the well-known book [5] on bifurcation theory. However, in these works, the full list of possible bifurcations was not given.

Some early examples on bimodal critical points were presented in Refs. [6–8]. It turns out that bimodal branching points are closely related to structural optimization problems [1]. Bimodal optimal columns (in continuous formulation) were recognized in Ref. [9]. Since that time, bi- and multimodality (multiplicity of eigenmodes at the same critical load) became a popular topic in structural optimization under stability constraints [10–14].

In this paper, we intend to give a complete theory of bimodal bifurcations in potential systems with symmetries. We present the full classification of possible cases for postbuckling paths and their stability depending on three coefficients. It is important that all the formulas derived in this paper are given in terms of the original potential energy of the system with multiple degrees of freedom. Then, we study unfolding of bimodal branching points due to change of problem parameters. Our approach is straightforward, explicit, and practical allowing to analyze bifurcations and stability of postbuckling paths, as well as their unfolding, based on calculation of the derivatives of the potential energy and eigenvectors of the linearized problem, taken at the bifurcation point. The presented theory is illustrated by a mechanical example on stability and postbuckling behavior of an articulated bimodal elastic column having four degrees of freedom and depending on three parameters.

2 Potential Systems

Consider a potential system with a state vector $\mathbf{q}=(q_1,q_2,\ldots,q_n)$. Equilibria of such a system are determined by critical points of the potential energy function $V(\mathbf{q})$ at which first variation of the potential energy with respect to the state vector is zero:

$$\delta V(\mathbf{q})=0 \tag{1}$$

An equilibrium is stable if it is a minimum of the potential. The sufficient stability condition is that the second variation of the potential is positive for all small variations $\delta\mathbf{q}$:

$$\delta^2 V(\mathbf{q}) > 0 \tag{2}$$

with the unstrict inequality $\delta^2 V(\mathbf{q}) \geqslant 0$ giving the necessary condition. The equilibrium condition (1) can be written in the form

$$\boldsymbol{\nabla} V=0 \quad \nabla=\left(\frac{\partial}{\partial q_1},\frac{\partial}{\partial q_2},\ldots,\frac{\partial}{\partial q_n}\right) \tag{3}$$

The stability condition (2) requires positive definiteness of the Hessian matrix

$$\mathbf{C}(\mathbf{q})=\begin{bmatrix} \partial^2 V/\partial q_1^2 & \partial^2 V/\partial q_1\partial q_2 & \cdots & \partial^2 V/\partial q_1\partial q_n \\ \partial^2 V/\partial q_1\partial q_2 & \partial^2 V/\partial q_2^2 & \cdots & \partial^2 V/\partial q_2\partial q_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2 V/\partial q_1\partial q_n & \partial^2 V/\partial q_2\partial q_n & \cdots & \partial^2 V/\partial q_n^2 \end{bmatrix} > 0 \tag{4}$$

with the second derivatives taken at the equilibrium point $\mathbf{q}$. For the second variation of the potential, one has $\delta^2 V=\frac{1}{2}\mathbf{C}\,\delta\mathbf{q}\cdot\delta\mathbf{q}$, where a dot denotes the inner product in $\mathbb{R}^n$. The symmetric matrix $\mathbf{C}$ is called the stiffness matrix for elastic systems.

We consider systems with the potential $V(\mathbf{q})$ having the property

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$$V(\mathbf{q}) = V(-\mathbf{q}) \tag{5}$$

This means that the system is symmetric under the inversion of the state vector $\mathbf{q} \to -\mathbf{q}$ (pendulum systems, straight beams, plates, etc.). Clearly, $\mathbf{q}=0$ is an equilibrium for such systems. Moreover, the Taylor expansion of the potential $V(\mathbf{q})$ in the neighborhood of $\mathbf{q}=0$ contains only even order terms.

3 Unimodal (Pitchfork) Bifurcation

Consider a system with the potential smoothly dependent on a parameter ϵ such that the trivial equilibrium $\mathbf{q}=0$ is stable for $\epsilon < 0$ and unstable for $\epsilon > 0$. For example, ϵ is a deviation of the loading parameter from a critical value. At $\epsilon=0$, the stability condition (4) is violated, and the stiffness matrix $\mathbf{C}_0=\mathbf{C}(0)$ becomes singular and positive semidefinite ($\mathbf{C}_0 \geq 0$). In the case of unimodal (pitchfork) bifurcation, there is only one eigenvector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ satisfying the equation

$$\mathbf{C}_0\mathbf{u} = 0 \tag{6}$$

This eigenvector $\mathbf{u}$ corresponds to the zero eigenvalue of the matrix $\mathbf{C}_0$ and is defined up to an arbitrary nonzero scalar factor. Properties of the unimodal bifurcation are well known. However, in this section, we provide the derivation that facilitates the further analysis of the bimodal case.

For small $\mathbf{q}$ and ϵ, the potential is given by the Taylor expansion

$$V = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j + \frac{1}{4!}\sum_{i,j,k,l=1}^{n} \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} q_i q_j q_k q_l + \cdots + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} q_i q_j \epsilon + \cdots \tag{7}$$

where all the derivatives are taken at $\mathbf{q}=0$ and $\epsilon=0$. Here, we used condition (5) implying that all terms of odd order in $\mathbf{q}$ vanish; an arbitrary constant term of the potential is taken to be zero. The main (second order) term in expansion (7) can be represented as $V=\frac{1}{2}\mathbf{C}_0\mathbf{q}\cdot\mathbf{q}+\cdots$. By using this expression in the equation for equilibria (3), we find

$$\nabla V = \mathbf{C}_0\mathbf{q} + \cdots = 0 \tag{8}$$

Hence, according to Eq. (6), nontrivial equilibria for small ϵ are given asymptotically by

$$\mathbf{q}(\epsilon) \approx \alpha\mathbf{u} \tag{9}$$

where α is an unknown function of ϵ.

In order to find α, consider the equation $\mathbf{u}\cdot\nabla V=0$ following directly from Eq. (3). By using expansion (7), we obtain

$$\mathbf{u}\cdot\nabla V = \frac{1}{3!}\sum_{i,j,k,l=1}^{n} \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} q_i q_j q_k u_l + \sum_{i,j=1}^{n} \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} q_i u_j \epsilon + \cdots = 0 \tag{10}$$

In Eq. (10), the two lowest order terms are presented, and the term $\mathbf{u}\cdot\mathbf{C}_0\mathbf{q}=\mathbf{C}_0\mathbf{u}\cdot\mathbf{q}$ vanishes due to condition (6). Substituting Eq. (9) into Eq. (10) and neglecting higher order terms, we obtain the equation for α as

$$\frac{v_{1111}}{6}\alpha^3 + v_{11\epsilon}\alpha\epsilon = 0 \tag{11}$$

where the coefficients v_{1111} and $v_{11\epsilon}$ are

$$v_{1111} \equiv (\mathbf{u}\cdot\nabla)^4 V = \sum_{i,j,k,l=1}^{n} \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} u_i u_j u_k u_l$$

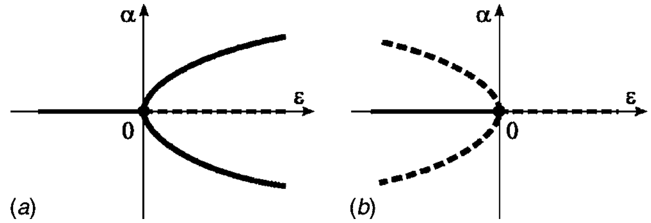

Fig. 1 Pitchfork bifurcation: (*a*) supercritical ($v_{1111}>0$) and (*b*) subcritical ($v_{1111}<0$). Stable equilibria are shown by solid lines.

$$v_{11\epsilon} \equiv (\mathbf{u}\cdot\nabla)^2 \frac{\partial V}{\partial \epsilon} = \sum_{i,j=1}^{n} \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} u_i u_j \tag{12}$$

(here, $\mathbf{u}\cdot\nabla = \sum_{i=1}^{n} u_i(\partial/\partial q_i)$ is the derivative along the direction $\mathbf{u}$ in state space).

Nonzero solutions of Eq. (11) are

$$\alpha = \pm\sqrt{-\frac{v_{11\epsilon}}{v_{1111}}6\epsilon} \tag{13}$$

Nontrivial equilibria exist only if the expression under the square root is positive. Thus, if $v_{11\epsilon}/v_{1111}<0$, then two nontrivial solutions exist for $\epsilon>0$. If $v_{11\epsilon}/v_{1111}>0$, then two nontrivial solutions exist for $\epsilon<0$. These two cases are called supercritical and subcritical bifurcations, respectively [5].

Let us study stability of the equilibria $\mathbf{q}=\alpha\mathbf{u}$ for small ϵ. The equilibrium is stable if the stiffness matrix $\mathbf{C}$ is positive definite or, equivalently, the second variation $\delta^2 V=\frac{1}{2}\mathbf{C}\delta\mathbf{q}\cdot\delta\mathbf{q}$ is positive for all $\delta\mathbf{q}$. Here, the stiffness matrix $\mathbf{C}(\mathbf{q})$ is evaluated at $\mathbf{q}=\alpha\mathbf{u}$. Up to zero order terms, $\mathbf{C}(\mathbf{q})=\mathbf{C}_0+\cdots$ with the positive semidefinite matrix $\mathbf{C}_0$ such that $\mathbf{C}_0\delta\mathbf{q}\cdot\delta\mathbf{q}=0$ only for $\delta\mathbf{q}\sim\mathbf{u}$. Hence, the stability condition must be checked only along the degenerate direction $\delta\mathbf{q}=\mathbf{u}$:

$$\delta^2 V = \frac{1}{2}\sum_{i,j=1}^{n} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\mathbf{q}=\alpha\mathbf{u},\epsilon} \delta q_i \delta q_j \approx \frac{1}{4}\sum_{i,j,k,l=1}^{n} \left.\frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l}\right|_{\mathbf{q}=0,\epsilon=0} u_i u_j (\alpha u_k)(\alpha u_l) + \frac{1}{2}\sum_{i,j=1}^{n} \left.\frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon}\right|_{\mathbf{q}=0,\epsilon=0} u_i u_j \epsilon \tag{14}$$

where we used the expansion similar to Eq. (7) and neglected higher order terms. By using notation (12), we write the stability condition $\delta^2 V>0$ in the form

$$\frac{v_{1111}}{2}\alpha^2 + v_{11\epsilon}\epsilon > 0 \tag{15}$$

For the trivial equilibrium ($\alpha=0$), the stability condition yields $v_{11\epsilon}\epsilon>0$. Since we assumed that the trivial equilibrium is stable for $\epsilon<0$, one obtains

$$v_{11\epsilon} < 0 \tag{16}$$

For the nontrivial solutions (13), we substitute $\epsilon=-v_{1111}\alpha/(6v_{11\epsilon})$ into Eq. (15) and get

$$\frac{v_{1111}}{3}\alpha^2 > 0 \tag{17}$$

This gives the well-known property of the pitchfork bifurcation [1,5]: The nontrivial equilibrium is stable in supercritical bifurcations ($v_{1111}>0$) and unstable in subcritical bifurcations ($v_{1111}<0$), see Fig. 1.

4 Bifurcation at a Bimodal Critical Point

The goal of this paper is to study bifurcation at a so-called *bimodal* critical point, when there are two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ (unstable modes) satisfying Eq. (6). For small $\mathbf{q}$ and ϵ, Eq. (8) gives asymptotic form of the bifurcating equilibria. Hence, $\mathbf{q}$ is a null-space vector of $\mathbf{C}_0$ given by an arbitrary linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathbf{q}(\epsilon) \approx \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 \tag{18}$$

where α and β are unknown functions of ϵ. Consider the equations $\mathbf{u}_1 \cdot \nabla V=0$ and $\mathbf{u}_2 \cdot \nabla V=0$ following directly from Eq. (3). Similar to the unimodal case, by using expressions (10) and (18) and neglecting higher order terms, we obtain the equations for α and β as

$$\begin{gathered}(v_{11\epsilon}\alpha + v_{12\epsilon}\beta)\epsilon + \frac{v_{1111}}{6}\alpha^3 + \frac{v_{1112}}{2}\alpha^2\beta + \frac{v_{1122}}{2}\alpha\beta^2 + \frac{v_{1222}}{6}\beta^3 = 0 \\ (v_{12\epsilon}\alpha + v_{22\epsilon}\beta)\epsilon + \frac{v_{1112}}{6}\alpha^3 + \frac{v_{1122}}{2}\alpha^2\beta + \frac{v_{1222}}{2}\alpha\beta^2 + \frac{v_{2222}}{6}\beta^3 = 0\end{gathered} \tag{19}$$

Here, we introduced the notation

$$v_{abcd} = (\mathbf{u}_a \cdot \nabla)(\mathbf{u}_b \cdot \nabla)(\mathbf{u}_c \cdot \nabla)(\mathbf{u}_d \cdot \nabla)V \quad v_{ab\epsilon} = (\mathbf{u}_a \cdot \nabla)(\mathbf{u}_b \cdot \nabla)\frac{\partial V}{\partial \epsilon} \tag{20}$$

with the derivatives evaluated at $\mathbf{q}=0$ and $\epsilon=0$ (for comparison, see Eq. (12)).

Equations (19) can be solved as follows. Expressing ϵ from either of Eq. (19), we find

$$\epsilon = c\beta^2 \tag{21}$$

where

$$\begin{aligned} c &= -\frac{v_{1111}\gamma^3 + 3v_{1112}\gamma^2 + 3v_{1122}\gamma + v_{1222}}{6(v_{11\epsilon}\gamma + v_{12\epsilon})} \\ &= -\frac{v_{1112}\gamma^3 + 3v_{1122}\gamma^2 + 3v_{1222}\gamma + v_{2222}}{6(v_{12\epsilon}\gamma + v_{22\epsilon})}\end{aligned} \tag{22}$$

and $\gamma=\alpha/\beta$. It is also possible to express ϵ through α from Eqs. (19) with a coefficient depending on the inverse ratio $1/\gamma=\beta/\alpha$. The second equality in Eq. (22) yields the quartic equation for γ as

$$c_4\gamma^4 + c_3\gamma^3 + c_2\gamma^2 + c_1\gamma + c_0 = 0 \tag{23}$$

with the coefficients

$$\begin{gathered} c_0 = v_{1222}v_{22\epsilon} - v_{2222}v_{12\epsilon} \quad c_1 = 3v_{1122}v_{22\epsilon} - 2v_{1222}v_{12\epsilon} - v_{2222}v_{11\epsilon} \\ c_2 = 3v_{1112}v_{22\epsilon} - 3v_{1222}v_{11\epsilon} \\ c_3 = v_{1111}v_{22\epsilon} + 2v_{1112}v_{12\epsilon} - 3v_{1122}v_{11\epsilon} \\ c_4 = v_{1111}v_{12\epsilon} - v_{1112}v_{11\epsilon}\end{gathered} \tag{24}$$

Equation (23) has two or four real roots, see Sec. 5 for the proof that the situation when all four roots are complex is impossible, i.e., isola point does not exist. The vanishing leading coefficient ($c_4=0$) corresponds to $1/\gamma=\beta/\alpha=0$, which yields $\beta=0$.

The obtained results can be summarized as follows.

THEOREM 1. *Nontrivial equilibria near a bimodal critical point* $\epsilon=0$ *have the asymptotic form* $\mathbf{q}(\epsilon)\approx\alpha\mathbf{u}_1+\beta\mathbf{u}_2$, *with* $\alpha=\gamma\beta$ *and* $\beta=\pm\sqrt{\epsilon/c}$. *There exist two or four branches of nontrivial equilibria given by two or four real solutions* γ *of quartic equation (23), and c given by expression (22). Each branch determines two symmetric equilibria, which differ by the sign; the branch is subcritical if* $c<0$ *(equilibria appear for* $\epsilon<0$*) and supercritical if* $c>0$ *(equilibria appear for* $\epsilon>0$*).*

We note that the maximum number of postbuckling paths was counted [5] but formulas for the coefficients (21)–(24) are new.

Let us study stability of the equilibria $\mathbf{q}=\alpha\mathbf{u}_1+\beta\mathbf{u}_2$ for small ϵ. The equilibrium is stable if the stiffness matrix $\mathbf{C}$ is positive definite or, equivalently, the second variation $\delta^2 V=\frac{1}{2}\mathbf{C}\delta\mathbf{q}\cdot\delta\mathbf{q}$ is positive for all $\delta\mathbf{q}$ with the stiffness matrix $\mathbf{C}(\mathbf{q})$ evaluated at $\mathbf{q}=\alpha\mathbf{u}_1+\beta\mathbf{u}_2$. As in the unimodal case (see Sec. 3), the stability condition must be checked only along the degenerate directions. In the bimodal case, degenerate directions are $\delta\mathbf{q}=a\mathbf{u}_1+b\mathbf{u}_2$ with arbitrary constants a and b. Up to lowest order terms, we have

$$\begin{aligned}\delta^2 V &= \frac{1}{2}\sum_{i,j=1}^{n} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\alpha\mathbf{u}_1+\beta\mathbf{u}_2,\epsilon} \delta q_i \delta q_j \approx \frac{1}{4}\sum_{i,j,k,l=1}^{n} \left.\frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l}\right|_{\mathbf{q}=0,\epsilon=0} (au_{1i}+bu_{2i})(au_{1j}+bu_{2j})(\alpha u_{1k}+\beta u_{2k})(\alpha u_{1l}+\beta u_{2l}) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n} \left.\frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon}\right|_{\mathbf{q}=0,\epsilon=0} (au_{1i}+bu_{2i})(au_{1j}+bu_{2j})\epsilon \\ &= \frac{1}{2}\left(v_{11\epsilon}\epsilon + \frac{v_{1111}}{2}\alpha^2 + v_{1112}\alpha\beta + \frac{v_{1122}}{2}\beta^2\right)a^2 \\ &\quad + \left(v_{12\epsilon}\epsilon + \frac{v_{1112}}{2}\alpha^2 + v_{1122}\alpha\beta + \frac{v_{1222}}{2}\beta^2\right)ab + \frac{1}{2}\left(v_{22\epsilon}\epsilon + \frac{v_{1122}}{2}\alpha^2 + v_{1222}\alpha\beta + \frac{v_{2222}}{2}\beta^2\right)b^2\end{aligned} \tag{25}$$

where we used expansion (7) and notation (20); u_{1i} and u_{2i} are the components of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Then, the stability condition $\delta^2 V>0$ (for arbitrary nonzero a and b) takes the form of positive definiteness of the 2×2 matrix

$$\begin{pmatrix} v_{11\epsilon}\epsilon + (v_{1111}/2)\alpha^2 + v_{1112}\alpha\beta + (v_{1122}/2)\beta^2 & v_{12\epsilon}\epsilon + (v_{1112}/2)\alpha^2 + v_{1122}\alpha\beta + (v_{1222}/2)\beta^2 \\ v_{12\epsilon}\epsilon + (v_{1112}/2)\alpha^2 + v_{1122}\alpha\beta + (v_{1222}/2)\beta^2 & v_{22\epsilon}\epsilon + (v_{1122}/2)\alpha^2 + v_{1222}\alpha\beta + (v_{2222}/2)\beta^2 \end{pmatrix} > 0 \tag{26}$$

It was assumed that the trivial equilibrium ($\alpha=\beta=0$) is stable for $\epsilon<0$. In this case, the stability condition (26) yields the inequalities

$$v_{11\epsilon}<0 \quad v_{22\epsilon}<0 \quad v_{11\epsilon}v_{22\epsilon}-v_{12\epsilon}^2>0 \tag{27}$$

For nontrivial equilibria $\alpha=\gamma\beta$, $\epsilon=c\beta^2$, condition (26) is equivalent to

$$\begin{pmatrix} v_{11\epsilon}c+(v_{1111}/2)\gamma^2+v_{1112}\gamma+(v_{1122}/2) & v_{12\epsilon}c+(v_{1112}/2)\gamma^2+v_{1122}\gamma+(v_{1222}/2) \\ v_{12\epsilon}c+(v_{1112}/2)\gamma^2+v_{1122}\gamma+(v_{1222}/2) & v_{22\epsilon}c+(v_{1122}/2)\gamma^2+v_{1222}\gamma+(v_{2222}/2) \end{pmatrix}>0 \tag{28}$$

Thus, an equilibrium with the branch corresponding to a given γ is stable if the matrix (28) is positive definite. If this matrix has a negative eigenvalue, the equilibrium is unstable.

5 No Isola Point Exists

We show that it is impossible to have four complex roots γ of the quartic equation (23).

First, we choose the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that in expressions (19) and (20) $v_{12\epsilon}=0$. With this choice, the 2×2 matrix with the elements $v_{ij\epsilon}$ is reduced to the diagonal form. Then, according to Eq. (24), we have

$$\frac{c_2}{c_4}=3\frac{v_{1222}}{v_{1112}}-3\frac{v_{22\epsilon}}{v_{11\epsilon}} \quad \frac{c_0}{c_4}=-\frac{v_{1222}}{v_{1112}}\frac{v_{22\epsilon}}{v_{11\epsilon}} \tag{29}$$

Now, let us assume that all four roots of the polynomial (23) are complex and equal to $x_1\pm iy_1$, $x_2\pm iy_2$. Then, $c_0/c_4=(x_1^2+y_1^2)(x_2^2+y_2^2)>0$. Hence, from conditions (27) and expressions (29), we obtain $v_{1222}/v_{1112}<0$ and $c_2/c_4<0$. Under these inequalities, it is easy to show that $(c_2/c_4)^2\geqslant 36c_0/c_4$. On the other hand, $c_2/c_4=x_1^2+y_1^2+x_2^2+y_2^2+4x_1x_2$. Since $c_2/c_4<0$, we have $(c_2/c_4)^2\leqslant(4x_1x_2)^2=16x_1^2x_2^2<16c_0/c_4$. But, this contradicts to the inequality $(c_2/c_4)^2\geqslant 36c_0/c_4$ derived above. Therefore, Eq. (23) always has real roots.

This means that there is no isola point, i.e., there exist nontrivial paths bifurcating from the trivial state at the bimodal critical point.

6 Symmetric Systems

In many practical problems, a system possesses an additional symmetry represented by the following invariance condition for the potential:

$$V(\mathbf{q})=V(\mathcal{S}(\mathbf{q})) \tag{30}$$

with a linear map $S(\mathbf{q})$ satisfying the relation $\mathcal{S}(\mathcal{S}(\mathbf{q}))=\mathbf{q}$ (of course, $\mathcal{S}$ is assumed to be different from $\mathbf{q}\to-\mathbf{q}$). This condition may reflect axial or spatial symmetry of the system. For example, consider a beam of variable cross section with the material distribution symmetric with respect to the middle and identical boundary conditions taken at $x=\pm a$, where x is the axial coordinate with the origin at the beam center. Then, $\mathcal{S}(w(x))=w(-x)$, where $\mathbf{q}\equiv w(x)$ is a deflection function of the beam. Keeping in mind this example, we say that $\mathbf{q}$ is a symmetric or antisymmetric form if $\mathcal{S}(\mathbf{q})=\mathbf{q}$ or $\mathcal{S}(\mathbf{q})=-\mathbf{q}$, respectively. If $\mathcal{S}(\mathbf{q})\neq\pm\mathbf{q}$, we say that the form is of mixed type.

In a unimodal (pitchfork) bifurcation, the unstable mode $\mathbf{u}$ must be either symmetric or antisymmetric: $\mathbf{u}=\pm\mathcal{S}(\mathbf{u})$. It cannot be of mixed type since that would automatically provide two linearly independent unstable modes $\mathbf{u}$ and $\mathcal{S}(\mathbf{u})$.

Let us consider a bimodal bifurcation. We can always choose the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ to be symmetric or antisymmetric: $\mathcal{S}(\mathbf{u}_{1,2})=\pm\mathbf{u}_{1,2}$. We assume that $\mathbf{u}_1$ is symmetric, while $\mathbf{u}_2$ is antisymmetric.

According to symmetry condition (30), the coefficients (20) do not change if we substitute $\mathbf{u}_1$ and $\mathbf{u}_2$ by $\mathcal{S}(\mathbf{u}_1)=\mathbf{u}_1$ and $\mathcal{S}(\mathbf{u}_2)=-\mathbf{u}_2$, respectively. The following coefficients vanish:

$$v_{12\epsilon}=v_{1112}=v_{1222}=0 \tag{31}$$

since they change their sign under the substitution $\mathbf{u}_2\to-\mathbf{u}_2$. Note that Eq. (19) with conditions (31) coincides with the corresponding equation for a two degrees-of-freedom system with double symmetry studied earlier [3–5].

For the sake of convenience, we introduce the normalization conditions for the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that

$$v_{11\epsilon}=-1 \quad v_{22\epsilon}=-1 \tag{32}$$

which is possible since according to Eq. (27) $v_{11\epsilon}<0$ and $v_{22\epsilon}<0$.

Solving system (19) with Eqs. (31) and (32) gives the unknown α and β corresponding to three types of nontrivial equilibria:

$$\alpha^2=\frac{6\epsilon}{v_{1111}} \quad \beta=0 \tag{33}$$

$$\alpha=0 \quad \beta^2=\frac{6\epsilon}{v_{2222}} \tag{34}$$

$$\alpha^2=\frac{v_{2222}-3v_{1122}}{v_{1111}v_{2222}-9v_{1122}^2}6\epsilon \quad \beta^2=\frac{v_{1111}-3v_{1122}}{v_{1111}v_{2222}-9v_{1122}^2}6\epsilon \tag{35}$$

Solutions (33), (34), and (35) with different signs of α and β define two symmetric, two antisymmetric, and four mixed-type equilibria (18), respectively. Symmetric equilibria are subcritical or supercritical for negative and positive values of v_{1111}, respectively. The type of antisymmetric equilibria is determined similarly by the sign of v_{2222}. Mixed-type equilibria (35) exist if the quantities $v_{2222}-3v_{1122}$ and $v_{1111}-3v_{1122}$ have the same sign. Under this condition, mixed-type equilibria are subcritical or supercritical for negative and positive signs of the fractional factor in Eq. (35), respectively.

The stability condition (26) takes the form

$$\begin{pmatrix} -\epsilon+v_{1111}\alpha^2/2+v_{1122}\beta^2/2 & v_{1122}\alpha\beta \\ v_{1122}\alpha\beta & -\epsilon+v_{1122}\alpha^2/2+v_{2222}\beta^2/2 \end{pmatrix}>0 \tag{36}$$

For symmetric equilibria (33), eigenvalues of the matrix (36) are

$$\lambda_1=\frac{v_{1111}}{3}\alpha^2 \quad \lambda_2=\frac{3v_{1122}-v_{1111}}{6}\alpha^2 \tag{37}$$

By using inequalities (27), we obtain the stability conditions as

$$v_{1111}>0 \quad 3v_{1122}-v_{1111}>0 \tag{38}$$

For antisymmetric equilibria (34), the eigenvalues are

$$\lambda_1=\frac{v_{2222}}{3}\beta^2 \quad \lambda_2=\frac{3v_{1122}-v_{2222}}{6}\beta^2 \tag{39}$$

and the stability conditions become

Table 1 Classification of bifurcations at a bimodal critical point

No.	v_{1111}	v_{2222}	$v_{1111}v_{2222}-9v_{1122}^2$	$v_{1111}-3v_{1122}$	$v_{2222}-3v_{1122}$
1	+	+	−	−	−
2	+	−	−	−	−
3	−	+	−	−	−
4	−	−	−	−	−
5	−	−	+	−	−
6	+	+	+	+	+
7	+	+	−	+	+
8	+	−	−	+	+
9	−	+	−	+	+
10	−	−	−	+	+
11	+	+		−	+
12	+	+		+	−
13	+	−		+	−
14	−	+		−	+
15	−	−		−	+
16	−	−		+	−

$$v_{2222} > 0 \quad 3v_{1122} - v_{2222} > 0 \tag{40}$$

Finally, for mixed-type equilibria (35), we find the eigenvalues

$$\lambda_{1,2} = \frac{v_{1111}\alpha^2 + v_{2222}\beta^2 \pm \sqrt{(v_{1111}\alpha^2 - v_{2222}\beta^2)^2 + 36v_{1122}^2\alpha^2\beta^2}}{6} \tag{41}$$

and the stability conditions

$$v_{1111} > 0 \text{ and } v_{2222} > 0 \quad v_{1111}v_{2222} - 9v_{1122}^2 > 0 \tag{42}$$

The obtained results allow classifying all types of bimodal bifurcations by the signs of specific quantities (depending on derivatives of the potential) evaluated at $\epsilon=0$ and $\mathbf{q}=0$, see Table 1. We note that only three numbers, namely, v_{1111}, v_{2222}, and v_{1122}, govern the postbuckling behavior.

Bifurcation diagrams corresponding to 16 cases of Table 1 are shown in Figs. 2 and 3, see the diagrams corresponding to $\chi=0$. Due to the symmetry with respect to the planes $\alpha=0$ and $\beta=0$, we show only the quarter domain $\alpha,\beta\geqslant 0$ of the (ϵ,α,β) space. In the figures, stable equilibria are shown by thick lines. Thin solid and dashed lines correspond to unstable equilibria with one and two negative eigenvalues of the matrix (36), respectively. *S*, A, and M are abbreviations for symmetric, antisymmetric, and mixed-type equilibria, respectively. Pictures in Figs. 2 and 3 ($\chi=0$) are based on relations (33)–(36), (38), (40), and (42).

One can see from Figs. 2 and 3 ($\chi=0$) that stable nontrivial equilibria exist in six cases (the Cases 1–3, 6, 11, and 12). An equilibrium of any type can be stable: symmetric, antisymmetric, or mixed type. In the remaining ten cases, all nontrivial equilibria are unstable. These cases describe limit points leading to dynamic snaps since beyond these critical points, there is no stable solution. Nontrivial supercritical equilibria can be unstable for bimodal bifurcations, while for unimodal bifurcations, they are always stable. However, stable nontrivial equilibria are always supercritical. If symmetric or antisymmetric equilibrium is stable, mixed-type equilibrium is unstable, and if mixed-type equilibrium is stable, the symmetric and antisymmetric equilibria are unstable.

Note that Cases 1, 6, 7, 11, and 12 of Table 1 were recognized and qualitatively described [5].

7 Unfolding of Bifurcations at Bimodal Critical Points

Now, let us consider a symmetric system, as in the previous section, with the potential smoothly depending on m parameters $\epsilon_1,\epsilon_2,\ldots,\epsilon_m$. We assume that, for fixed $\epsilon_2=\cdots=\epsilon_m=0$, the bimodal bifurcation takes place in one-parameter system $\epsilon=\epsilon_1$, as described above. For small nonzero (but fixed) values of the parameters $\epsilon_2,\ldots,\epsilon_m$, the system behavior depending on $\epsilon=\epsilon_1$ can change qualitatively, i.e., we can observe unfolding of the bimodal bifurcation. The parameters $\epsilon_2,\ldots,\epsilon_m$ can be treated as imperfections that keep the system symmetry. Nontrivial equilibria in this case are described by the asymptotic formula (18). The unknown coefficients α and β are determined by the equations

$$\begin{aligned}(-\epsilon+\tilde{v}_{11\epsilon})\alpha + v_{1111}\alpha^3/6 + v_{1122}\alpha\beta^2/2 &= 0\\ (-\epsilon+\tilde{v}_{22\epsilon})\beta + v_{1122}\alpha^2\beta/2 + v_{2222}\beta^3/6 &= 0\end{aligned} \tag{43}$$

where

$$\tilde{v}_{11\epsilon} = \sum_{k=2}^{n} (\mathbf{u}_1\cdot\nabla)^2\frac{\partial V}{\partial\epsilon_k}\epsilon_k \quad \tilde{v}_{22\epsilon} = \sum_{k=2}^{n} (\mathbf{u}_2\cdot\nabla)^2\frac{\partial V}{\partial\epsilon_k}\epsilon_k \tag{44}$$

with the derivatives taken at $\epsilon_1=\epsilon_2=\cdots=\epsilon_m=0$ and $\mathbf{q}=0$. Equations (43) differ from Eqs. (19) and (31) only by small constants $\tilde{v}_{11\epsilon}$ and $\tilde{v}_{22\epsilon}$ dependent on the unfolding parameters $\epsilon_2,\ldots,\epsilon_m$. By solving system (43), we find the coefficients α and β corresponding to nontrivial equilibria. Similar to Eqs. (33)–(35), there can be symmetric, antisymmetric, and mixed-type solutions:

$$\alpha^2 = \frac{6(\epsilon-\tilde{v}_{11\epsilon})}{v_{1111}} \quad \beta=0 \tag{45}$$

$$\alpha=0 \quad \beta^2 = \frac{6(\epsilon-\tilde{v}_{22\epsilon})}{v_{2222}} \tag{46}$$

$$\begin{aligned}\alpha^2 &= 6\frac{(\epsilon-\tilde{v}_{11\epsilon})v_{2222} - 3(\epsilon-\tilde{v}_{22\epsilon})v_{1122}}{v_{1111}v_{2222}-9v_{1122}^2}\\ \beta^2 &= 6\frac{(\epsilon-\tilde{v}_{22\epsilon})v_{1111} - 3(\epsilon-\tilde{v}_{11\epsilon})v_{1122}}{v_{1111}v_{2222}-9v_{1122}^2}\end{aligned} \tag{47}$$

If $\tilde{v}_{11\epsilon}\neq\tilde{v}_{22\epsilon}$, the branches of symmetric and antisymmetric equilibria (45) and (46) do not intersect in the space (ϵ,α,β). This means that the bimodality is destroyed. As for the equilibria of mixed type (47), they coincide with the symmetric ones (45) at the points

$$(s):\ \alpha_s^2 = -\frac{6(\tilde{v}_{11\epsilon}-\tilde{v}_{22\epsilon})}{v_{1111}-3v_{1122}} \quad \beta_s=0 \quad \epsilon_s = \frac{\tilde{v}_{22\epsilon}v_{1111} - 3\tilde{v}_{11\epsilon}v_{1122}}{v_{1111}-3v_{1122}} \tag{48}$$

Similarly, mixed-type equilibria (47) coincide with the antisymmetric ones (46) at the points

$$(a):\ \alpha_a=0 \quad \beta_a^2 = \frac{6(\tilde{v}_{11\epsilon}-\tilde{v}_{22\epsilon})}{v_{2222}-3v_{1122}}, \quad \epsilon_a = \frac{\tilde{v}_{11\epsilon}v_{2222} - 3\tilde{v}_{22\epsilon}v_{1122}}{v_{2222}-3v_{1122}} \tag{49}$$

At these points, the secondary (postcritical) bifurcations occur. Critical points (48) and (49) exist if the quantities α^2 and β^2 determined by the corresponding expressions are positive.

With a change of parameters $\epsilon_2,\ldots,\epsilon_m$, the bimodal bifurcation splits into a series of unimodal bifurcations. For understanding the structure of the bifurcating equilibria, let us plot solutions (45)–(47) in the $(\epsilon,\alpha^2,\beta^2)$ space. Each of these solutions is represented by a straight line, Fig. 4. The line corresponding to the symmetric equilibria lies in the (ϵ,α^2) plane, the line corresponding to the antisymmetric equilibria lies in the (ϵ,β^2) plane, and the line corresponding to the mixed-type equilibria intersects the two previous lines. Of course, only the $\alpha^2\geqslant 0$, $\beta^2\geqslant 0$ part of the space has physical meaning. Therefore, we can distinguish four qualitatively different situations. If $\alpha_s^2<0$ and $\beta_a^2<0$, then the mixed-type equilibrium line does not intersect the physical domain (equilibria of mixed type do not exist). If $\alpha_s^2>0$ and $\beta_a^2<0$, then the mixed-type equilibrium half-line belongs to the physical domain (equilibria of mixed type exist and appear in the bifurcation of symmetric equilibria); this is the case shown in Fig. 4. If $\alpha_s^2<0$

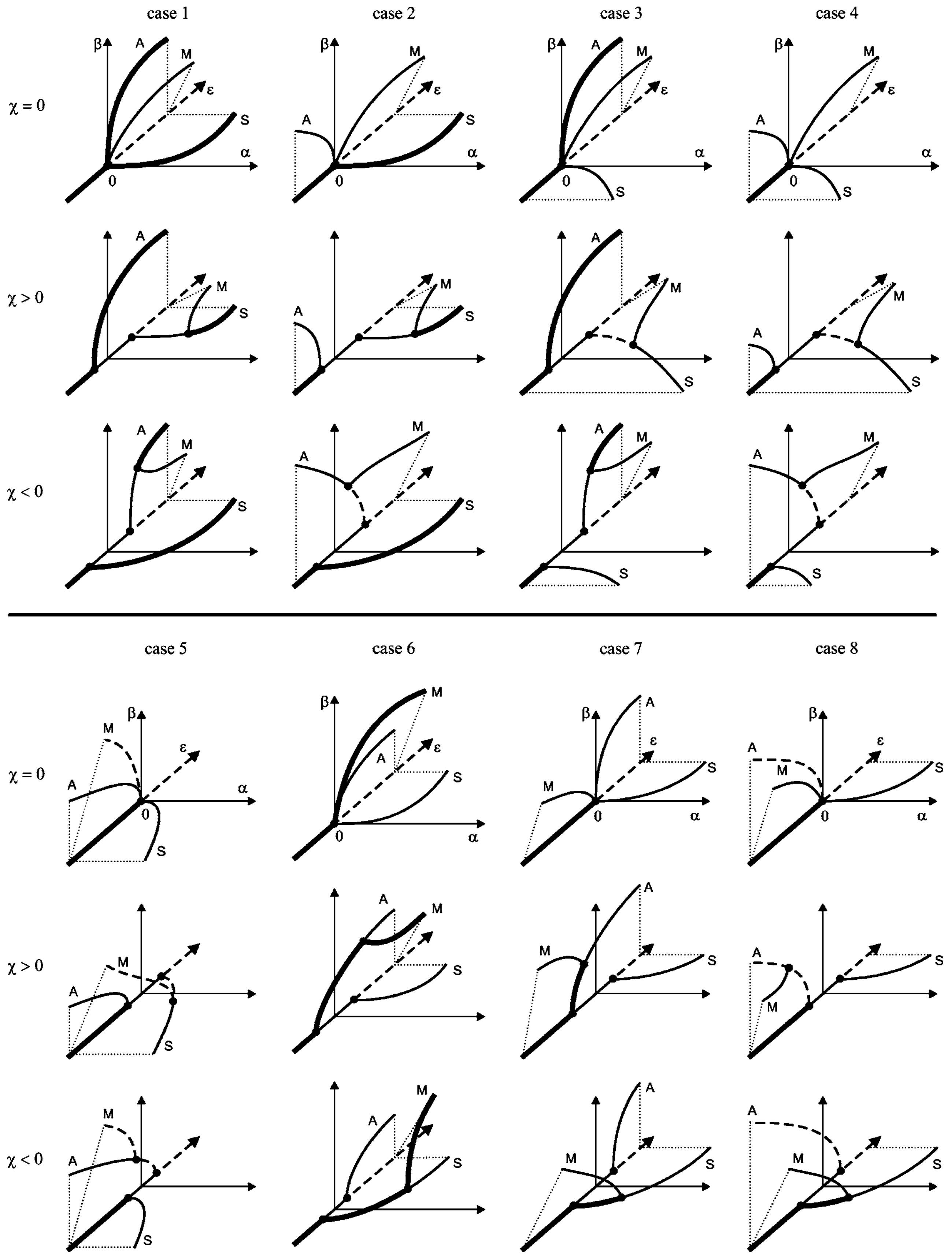

Fig. 2 Unfolding of the bimodal bifurcation: Cases 1–8

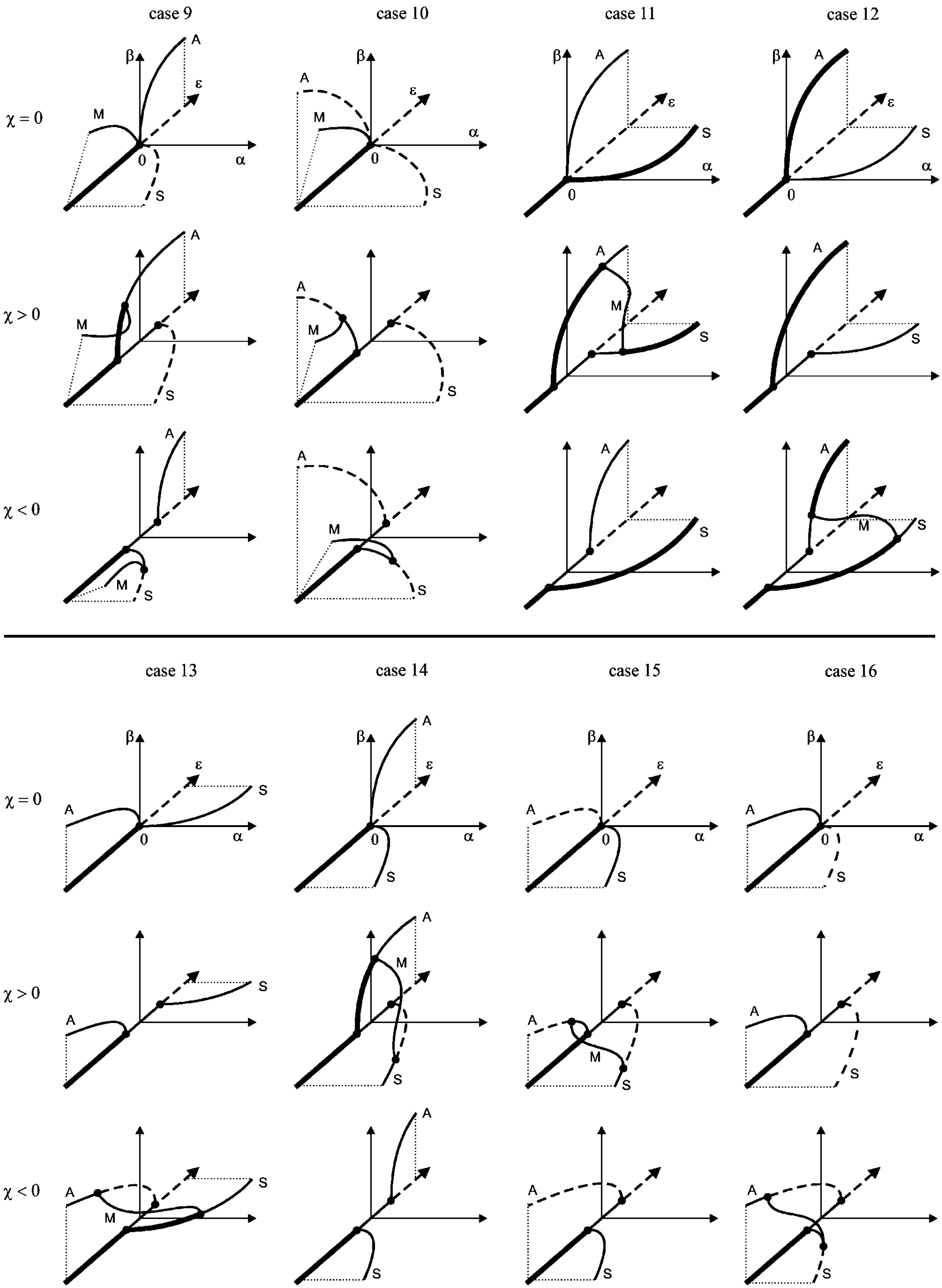

Fig. 3 Unfolding of the bimodal bifurcation: Cases 9–16

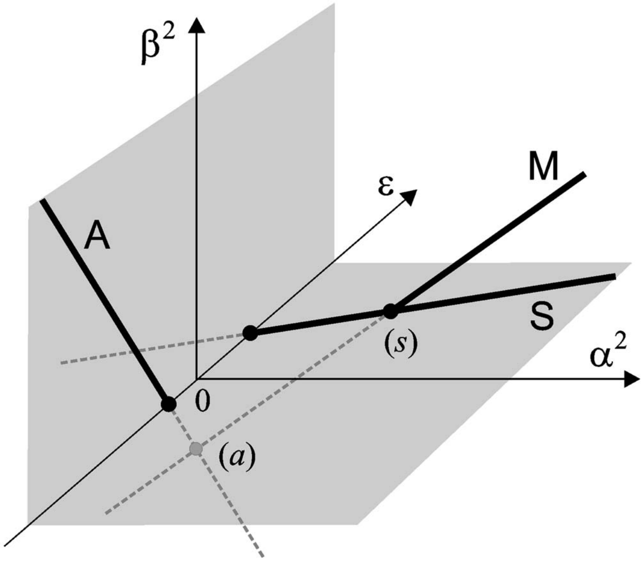

Fig. 4 Structure of nontrivial equilibria for nearly bimodal critical point

and $\beta_a^2>0$, then the mixed-type equilibrium half-line belongs to the physical domain (equilibria of mixed type exist and appear in the bifurcation of antisymmetric equilibria). Finally, if $\alpha_s^2>0$ and $\beta_a^2>0$, then the mixed-type equilibrium segment between the points $(\epsilon_s,\alpha_s^2,\beta_s^2)$ and $(\epsilon_a,\alpha_a^2,\beta_a^2)$ belongs to the physical domain (equilibria of mixed type exist and are bounded by the bifurcations of symmetric and antisymmetric equilibria).

In the (ϵ,α,β) space, the equilibrium lines become curves, which are orthogonal to the planes $\alpha=0$ and $\beta=0$. Eliminating ϵ from Eqs. (47), we obtain

$$(v_{2222}-3v_{1122})\beta^2-(v_{1111}-3v_{1122})\alpha^2=6\chi \quad \chi=\tilde{v}_{11\epsilon}-\tilde{v}_{22\epsilon} \tag{50}$$

On the (α,β) plane, this is a hyperbola, ellipse, or empty set depending on the signs of the coefficients $v_{1111}-3v_{1122}$, $v_{2222}-3v_{1122}$, and χ.

Unfolding of bifurcations for 16 cases of Table 1 is depicted in Figs. 2 and 3. These figures are based on relations (45)–(50). Note that the bifurcation points of the trivial equilibrium correspond to $\epsilon=\tilde{v}_{11\epsilon}$ for the symmetric path and $\epsilon=\tilde{v}_{22\epsilon}$ for the antisymmetric path. For the sake of simplicity, in the figures, we took $\tilde{v}_{11\epsilon}>0$ and $\tilde{v}_{22\epsilon}<0$, which does not change the pictures qualitatively.

As an example, let us consider unfolding of the first case in Table 1. The unperturbed situation $(\tilde{v}_{11\epsilon}=\tilde{v}_{22\epsilon}=0)$, is shown in Fig. 2 (Case 1, $\chi=0$). From Table 1, it follows that the denominators in formulas (48) and (49) are positive. Hence, if $\chi=\tilde{v}_{11\epsilon}-\tilde{v}_{22\epsilon}>0$, then there are only intersections (bifurcations) between symmetric and mixed-type equilibrium branches at the point (48), Fig. 2 (Case 1, $\chi>0$). If $\chi<0$, then only antisymmetric and mixed-type equilibrium branches intersect at the point (49), Fig. 2 (Case 1, $\chi<0$). Therefore, in the unfolding picture, the mixed-type equilibria appear due to the secondary bifurcation of symmetric $(\chi>0)$ or antisymmetric $(\chi<0)$ equilibria.

It should be noted that the unfolding of a bimodal critical point is qualitatively different for systems without symmetry property (30). In the latter case, typically, there are no secondary pitchfork bifurcations.

In multiparameter case, stability criterion for the equilibria is the condition of positive definiteness of the matrix (36), where one must substitute $-\epsilon$ by $-\epsilon+\tilde{v}_{11\epsilon}$ and $-\epsilon+\tilde{v}_{22\epsilon}$ in the first and second diagonal elements, respectively.

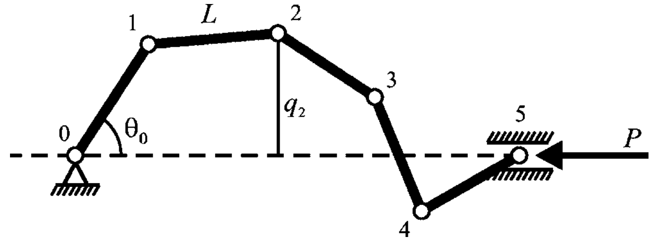

Fig. 5 Elastic articulated column loaded by an axial force

First, consider the trivial equilibrium $\alpha=\beta=0$. In this case, the eigenvalues of the matrix (36) become

$$\lambda_1=-\epsilon+\tilde{v}_{11\epsilon} \quad \lambda_2=-\epsilon+\tilde{v}_{22\epsilon} \tag{51}$$

The bimodal critical point is defined by the conditions $\lambda_1=\lambda_2=0$. With the use of Eq. (44), these conditions define a plane of codimension 2 in parameter space $(\epsilon_1,\ldots,\epsilon_m)$. Hence, the codimension of a bimodal critical point equals 2 (this critical point can be typically found by adjusting values of two parameters). Here, the symmetry (30) is very important: Due to this symmetry, the off-diagonal elements $\tilde{v}_{12\epsilon}$ vanish. These elements are nonzero in systems without symmetry (30) or if both unstable modes are symmetric (or antisymmetric). In that case, the codimension of a bimodal critical point equals 3, which agrees with general results of the singularity theory [15].

Stability of nontrivial equilibria can be studied similarly by computing eigenvalues of the 2×2 second variation matrix. However, in the perturbed case, we can avoid these computations by using known properties of unimodal bifurcations (Sec. 3), and the properties of postcritical paths for large $\epsilon_1 \gg \epsilon_2,\ldots,\epsilon_m$ (at these values of ϵ_1, the stability type of a postcritical path is the same as for $\epsilon_2=\cdots=\epsilon_m=0$). The results of stability analysis are shown in Figs. 2 and 3. Recall that stable equilibria are shown by thick lines, while thin solid and dashed lines correspond to unstable equilibria with one and two negative eigenvalues of the matrix (36), respectively. For example, in Fig. 2 (Case 1, $\chi<0$), the first bifurcation is supercritical (symmetric equilibria are stable), and antisymmetric equilibria appear when the unstable trivial equilibrium bifurcates (antisymmetric equilibria are unstable). After the secondary bifurcation, antisymmetric equilibria become stable as in the bimodal picture $(\chi=0)$, and unstable mixed-type equilibria appear. We can see that for higher values of ϵ, the stability properties of all the equilibria are the same as for the bimodal bifurcation for $\chi=0$.

We remark that the unfoldings in Cases 5 and 10, as well as 15 and 16, are similar from the physical point of view since the unstable paths differ only by degrees of instability.

Note that for symmetric two degrees-of-freedom systems, classification of four cases with respect to the parameters $v_{1111}-3v_{1122}$ and $v_{2222}-3v_{1122}$ was given in [3], and Cases 1, 7, 11, and 12 $(\chi<0)$ of Figs. 2 and 3 were drawn in [4], while we have recognized 16 different cases, each of them corresponding to different pictures in 3D space.

8 Mechanical Example

As a mechanical example, we consider an elastic articulated column with elastically clamped ends loaded by an axial force P, Fig. 5. The column consists of five segments of length L connected by six elastic hinges with the bending stiffnesses $b_0,b_1,\ldots,b_5$. Linear stability problem for the straight equilibrium of the column has been treated in [10]. We consider a symmetric structure with symmetric boundary conditions, so that $b_0=b_5$, $b_1=b_4$, and $b_2=b_3$. Deflection of the column is determined by the vector of coordinates $\mathbf{q}=(q_1,q_2,q_3,q_4)$, which are related to the angles between the segments and the horizontal axis as

$$q_{i+1} - q_i = L \sin\theta_i \quad i = 0, \ldots, 4 \quad q_0 = q_5 = 0 \tag{52}$$

The potential function of the column is

$$V = \sum_{i=0}^{5} \left(\frac{b_i}{2}(\theta_i - \theta_{i-1})^2 - PL(1 - \cos\theta_i) \right) \quad \theta_{-1} = 0 \quad \theta_5 = 0 \tag{53}$$

For the sake of simplicity, we introduce nondimensional quantities

$$\tilde{q}_i = \frac{q_i}{L} \quad \tilde{P} = \frac{PL}{b*} \quad \tilde{b}_i = \frac{b_i}{b*} \quad \tilde{V} = \frac{V}{b*} \tag{54}$$

where $b*$ is a reference stiffness.

Substituting Eq. (52) into Eq. (53) with the use of Eq. (54) and omitting tildes, we obtain the nondimensional potential function as

$$\begin{aligned} V = {} & \frac{b_0}{2}(\arcsin q_1)^2 + \frac{b_1}{2}(\arcsin(q_2 - q_1) - \arcsin q_1)^2 \\ & + \frac{b_2}{2}(\arcsin(q_3 - q_2) - \arcsin(q_2 - q_1))^2 + \frac{b_2}{2}(\arcsin(q_4 - q_3) \\ & - \arcsin(q_3 - q_2))^2 + \frac{b_1}{2}(\arcsin q_4 + \arcsin(q_4 - q_3))^2 \\ & + \frac{b_0}{2}(\arcsin q_4)^2 - P(5 - \sqrt{1 - q_1^2} - \sqrt{1 - (q_2 - q_1)^2} \\ & - \sqrt{1 - (q_3 - q_2)^2} - \sqrt{1 - (q_4 - q_3)^2} - \sqrt{1 - (q_4)^2}) \end{aligned} \tag{55}$$

For small values of the coordinates q_i, the potential can be expanded in Taylor series

$$\begin{aligned} V = {} & \frac{b_0}{2}q_1^2 + \frac{b_1}{2}(q_2 - 2q_1)^2 + \frac{b_2}{2}(q_3 - 2q_2 + q_1)^2 + \frac{b_2}{2}(q_4 - 2q_3 + q_2)^2 \\ & + \frac{b_1}{2}(-2q_4 + q_3)^2 + \frac{b_0}{2}q_4^2 - \frac{P}{2}(q_1^2 + (q_2 - q_1)^2 + (q_3 - q_2)^2 \\ & + (q_4 - q_3)^2 + q_4^2) + \cdots \end{aligned} \tag{56}$$

The second order terms given in Eq. (56) define the stiffness matrix **C**. Equation (6) for the linear buckling problem takes the form

$$\begin{aligned} & (b_0 + 4b_1 + b_2 - 2P)u_1 + (-2b_1 - 2b_2 + P)u_2 + b_2u_3 = 0 \\ & (-2b_1 - 2b_2 + P)u_1 + (b_1 + 5b_2 - 2P)u_2 + (-4b_2 + P)u_3 + b_2u_4 = 0 \\ & b_2u_1 + (-4b_2 + P)u_2 + (b_1 + 5b_2 - 2P)u_3 + (-2b_1 - 2b_2 + P)u_4 = 0 \\ & b_2u_2 + (-2b_1 - 2b_2 + P)u_3 + (b_0 + 4b_1 + b_2 - 2P)u_4 = 0 \end{aligned} \tag{57}$$

Due to symmetry of the column, Eq. (57) possesses symmetric and antisymmetric solutions. For the symmetric solution, we take $u_4 = u_1$, $u_3 = u_2$. Then, from the first two (or the last two) equations (57), we get the quadratic equation for buckling loads

$$P_s^2 - P_s(b_0 + 2b_1 + b_2) + b_0b_1 + b_0b_2 + b_1b_2 = 0 \tag{58}$$

Both roots of this equation are positive, and the smaller root gives the critical buckling load if buckling is symmetric.

For the antisymmetric solution, we take $u_4 = -u_1$, $u_3 = -u_2$ and similarly obtain the quadratic equation

$$P_a^2 - P_a\left(\frac{3}{5}b_0 + 2b_1 + 3b_2\right) + \frac{1}{5}b_0b_1 + \frac{9}{5}b_0b_2 + 5b_1b_2 = 0 \tag{59}$$

The smaller root of this equation yields the critical buckling load if buckling is antisymmetric.

The condition of bimodality is that the smaller P_s is equal to the smaller P_a. So, we have

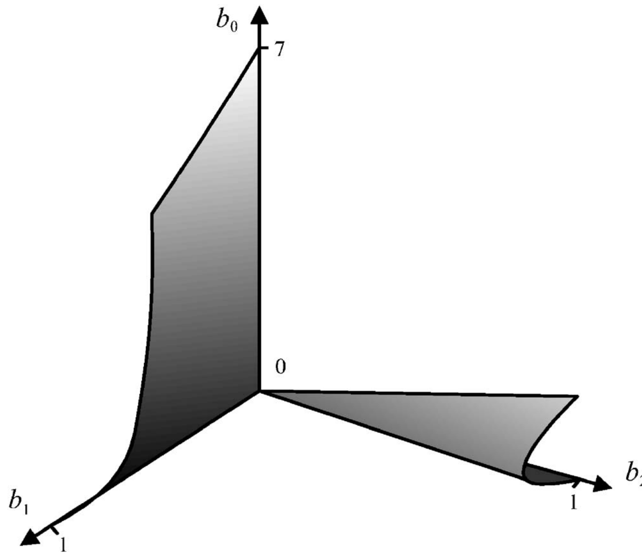

Fig. 6 Stiffness parameters for columns undergoing bimodal buckling

$$2b_2 - \frac{2}{5}b_0 + \sqrt{b_0^2 - 2b_0b_2 + 4b_1^2 + b_2^2} = \frac{1}{5}\sqrt{9b_0^2 + 40b_0b_1 - 90b_0b_2 + 100b_1^2 - 200b_1b_2 + 225b_2^2} \tag{60}$$

This equation defines a surface in three-dimensional space of the column stiffnesses (b_0, b_1, b_2) shown in Fig. 6. Each point on this surface corresponds to a column with the bimodal critical buckling load.

Note that for rigid clamping of the column (as b_0 tends to infinity), Eqs. (58) and (59) furnish the buckling loads

$$P_s = b_1 + b_2 \quad P_a = \frac{b_1}{3} + 3b_2 \tag{61}$$

Thus, for rigid clamping, the bimodality condition is $b_1 = 3b_2$. This means that the bimodal surface tends to the plane $b_1 = 3b_2$ for the stiffness b_0 tending to infinity, see Fig. 6.

Let us study postbuckling behavior of the symmetric column for the parameters $b_0 = 1$, $b_1 = 0.25$, $b_2 = 1$, satisfying the bimodality condition (60). According to Eqs. (57)–(59), we compute the bimodal critical buckling load $P = 1$ and the corresponding eigenmodes (eigenvectors) $\mathbf{u}_1 = (1, 2, 2, 1)$ and $\mathbf{u}_2 = (1, 0.4, -0.4, -1)$. Expanding the potential function (55) up to fourth order terms and using Eq. (20), we compute the coefficients $v_{11\epsilon} = -4.0$ and $v_{22\epsilon} = -3.36$. Then, we normalize the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ dividing them by $\sqrt{-v_{11\epsilon}}$ and $\sqrt{-v_{22\epsilon}}$, respectively, so that the condition (32) is satisfied. Using normalized eigenvectors in Eq. (20), we calculate the coefficients

$$v_{1111} = 0.25 \quad v_{2222} = 0.38605 \quad v_{1122} = 0.25 \tag{62}$$

The bifurcation belongs to Type 1 in Table 1. It means that both symmetric and antisymmetric solutions are supercritical and stable while the mixed-type solution is supercritical and unstable, see Fig. 2 (Case 1, $\chi = 0$). The nontrivial equilibria according to Eqs. (33)–(35) are given asymptotically as

$$\begin{aligned} \mathbf{q}_s &= \pm\sqrt{\epsilon}(0, 2.4494, 4.8989, 4.8989, 2.4494, 0) \\ \mathbf{q}_a &= \pm\sqrt{\epsilon}(0, 2.1507, 0.8602, -0.8602, -2.1507, 0) \\ \mathbf{q}_{m1} &= \pm\sqrt{\epsilon}(0, 2.4665, 2.7184, 1.6110, -0.3018, 0) \end{aligned} \tag{63}$$

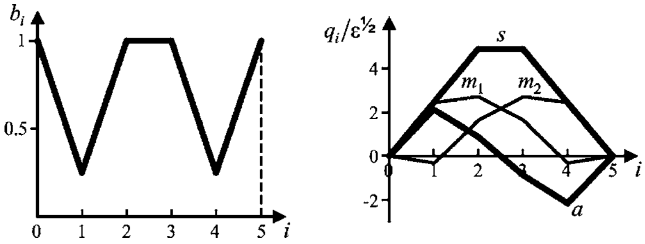

Fig. 7 Stiffnesses and buckling modes of the elastic column ($b_0=1, b_1=0.25, b_2=1$)

$$\mathbf{q}_{m2} = \pm\sqrt{\epsilon}(0,-0.3018,1.6110,2.7184,2.4665,0)$$

The stiffnesses of the bimodal column and corresponding new equilibrium states, divided by $\pm\sqrt{\epsilon}$, are presented in Fig. 7.

Let us study unfolding of this bifurcation due to change of the stiffness δb_0. According to Eqs. (44) and (56), we find

$$\chi = \tilde{v}_{11\epsilon} - \tilde{v}_{22\epsilon} = 2\delta b_0(u_{11}^2 - u_{21}^2) = -0.0952\delta b_0 \qquad (64)$$

Hence, if we decrease the stiffness $\delta b_0 < 0$, then the antisymmetric form of instability becomes critical, and the corresponding unfolding is shown in Fig. 2, Case 1 ($\chi > 0$). If the stiffness is increased $\delta b_0 > 0$, then the symmetric form of instability becomes critical with the unfolding shown in Fig. 2, Case 1 ($\chi < 0$).

For the stiffnesses $b_0=1$, $b_1=0.15$, $b_2=0.76465$, we compute the bimodal critical buckling load $P=0.84167$, the corresponding eigenmodes $\mathbf{u}_1=(1,3.0554,3.0554,1)$ and $\mathbf{u}_2=(1,0.3888,-0.3888,-1)$, and the coefficients $v_{11\epsilon}=-10.449$ and $v_{22\epsilon}=-3.3517$. Then, we normalize the eigenvectors and calculate the coefficients

$$v_{1111} = 0.29058 \quad v_{2222} = 0.35097 \quad v_{1122} = 0.05163 \qquad (65)$$

The bifurcation belongs to Type 6 in Table 1. This means that symmetric and antisymmetric solutions are supercritical and unstable while the mixed-type solution is supercritical and stable, see Fig. 2 (Case 6, $\chi=0$). Thus, we have recognized a surprising effect that a symmetric bimodal column loaded by an axial force can buckle with a stable asymmetric mode!

According to Eqs. (33)–(35), the bifurcating equilibria are given asymptotically as

$$\begin{aligned}
\mathbf{q}_s &= \pm\sqrt{\epsilon}(0,1.4056,4.2949,4.2949,1.4056,0)\\
\mathbf{q}_a &= \pm\sqrt{\epsilon}(0,2.2583,0.8780,-0.8780,-2.2583,0)\\
\mathbf{q}_{m1} &= \pm\sqrt{\epsilon}(0,2.9661,4.3570,2.9848,-0.5632,0)\\
\mathbf{q}_{m2} &= \pm\sqrt{\epsilon}(0,-0.5632,2.9848,4.3570,2.9661,0)
\end{aligned} \qquad (66)$$

The stiffnesses of the bimodal column and corresponding nontrivial equilibrium states, divided by $\pm\sqrt{\epsilon}$, are presented in Fig. 8.

If we study unfolding of this bifurcation due to change of the stiffness δb_1, then according to Eqs. (44) and (56), we get

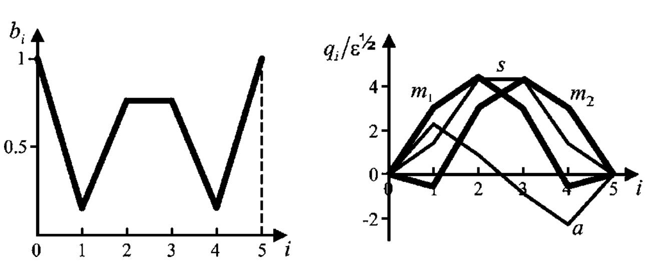

Fig. 8 Stiffnesses and buckling modes of the elastic column ($b_0=1, b_1=0.15, b_2=0.76465$)

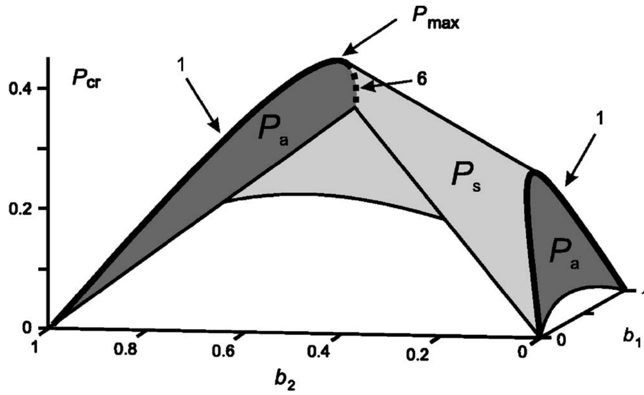

Fig. 9 Critical load depending on stiffness parameters

$$\chi = \tilde{v}_{11\epsilon} - \tilde{v}_{22\epsilon} = 2\delta b_1((u_{12} - 2u_{11})^2 - (u_{22} - 2u_{21})^2) = 1.3358\delta b_1 \qquad (67)$$

Thus, if we decrease the stiffness $\delta b_1 < 0$, then the antisymmetric form of instability becomes critical, and the corresponding unfolding is shown in Fig. 2, Case 6 ($\chi > 0$). If the stiffness is increased $\delta b_1 > 0$, then the symmetric form of instability becomes critical with the unfolding is shown in Fig. 2, Case 6 ($\chi < 0$).

8.1 Bimodal Optimal Column. Let us consider columns under the condition

$$b_0 + b_1 + b_2 = \text{const} \qquad (68)$$

This equality resembles the fixed total volume constraint for a continuous column. Figure 9 shows dependence of the critical load on b_1 and b_2 with b_0 given by Eq. (68) with const=1; due to homogeneity of Eqs. (58) and (59), the plot for any const can be obtained from Fig. 9 by scaling. Columns with bimodal critical loads correspond to edges, where the surfaces P_a and P_s intersect. The analysis (similar to the one given above) shows that the left bimodal arch contains two big parts corresponding to Bifurcations 1 and 6 according to the classification in Fig. 2; between these two parts, there is a tiny part corresponding to Bifurcation 11 (not shown in the figure). The right arch corresponds to the bifurcation of Type 1. The maximal critical load $P_{\max}=0.4465$ is attained at the bimodal point $b_0=0.4717$, $b_1=0.1021$, $b_2=0.4263$ with the bifurcation of Type 1. We note that the postbuckling behavior of the articulated optimal column is similar to that of the continuous optimal column [12]. Clearly, a bimodal optimal solution is the generic phenomenon. In different optimization problems, the bimodal solutions were found [6,10–14].

9 Conclusion

For general potential systems with symmetry having multiple degrees of freedom, we studied bifurcations at bimodal branching points. Formulas describing postbuckling paths and conditions for their stability are derived. We presented the full list of possible cases for postbuckling paths and their stability depending on three system coefficients v_{1111}, v_{2222}, and v_{1122}. In order to calculate these coefficients, we need to know the derivatives of the potential energy and eigenvectors of the linearized problem taken at the bifurcation point. Then, we studied unfolding of bimodal branching points due to change of system parameters. Classification and analysis of all possible cases given in Table 1 with Figs. 2 and 3 constitute the central result of the paper. It is remarkable that all the formulas derived in this paper are given in terms of the original potential energy.

The presented theory is illustrated by a mechanical example on stability and postbuckling behavior of a bimodal articulated elastic column having four degrees of freedom and depending on three stiffnesses at the hinges (problem parameters). It is shown that bimodal critical points are described by smooth surfaces in parameter space. Numerical results are presented illustrating influence of problem parameters on postbuckling paths, their stability and unfolding. Two different kinds of postbuckling behavior are demonstrated. One is associated with stable symmetric and antisymmetric modes, and unstable mixed-type modes, while the second one is associated with stable mixed-type modes and unstable symmetric and antisymmetric modes. Thus, a surprising phenomenon that a symmetric bimodal column loaded by an axial force can buckle with a stable asymmetric mode is recognized.

A considered example with the constrained sum of the stiffnesses of the articulated column shows that the maximum critical load (optimal design) is attained at the bimodal point with the postbuckling behavior similar to that of the continuous optimal column.

We remark that we have studied bimodal bifurcations of the stable stability path of the potential system with increasing load parameter. Certainly, the case when the trivial equilibrium is unstable on both sides of the bifurcation point could also be useful. It would be interesting to recognize more physical systems and phenomena related to bimodal bifurcations.

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References

[1] Thompson, J. M. T., and Hunt, G. W., 1973, *A General Theory of Elastic Stability*, Wiley, London.

[2] Thompson, J. M. T., and Hunt, G. W., 1984, *Elastic Instability Phenomena*, Wiley, Chichester, UK.

[3] Supple, W. J., 1967, "Coupled Branching Configurations in the Elastic Buckling of Symmetric Structural Systems," Int. J. Mech. Sci., **9**, pp. 97–112.

[4] Supple, W. J., 1973, "Coupled Buckling Modes of Structures," *Structural Instability*, W. J. Supple, ed., IPC Science and Technology Press, Guildford, UK, pp. 28–53.

[5] Golubitsky, M., and Schaeffer, D., 1985, *Singularities and Groups in Bifurcation Theory*, Springer, New York.

[6] Augusti, G., 1964, "Stabilita' di Strutture Elastiche Elementari in Presenza di Grandi Spostamenti," Atti Accad. Sci. Fis. Mat., Napoli, Serie 3a, 4(5).

[7] Koiter, W. T., 1969, "The Nonlinear Buckling Problem of a Complete Spherical Shell Under Uniform External Pressure," Proc. K. Ned. Akad. Wet., Ser. B: Phys. Sci., **72**, pp. 40–123.

[8] Bauer, L., Keller, H. B., and Reiss, E. L., 1975, "Multiple Eigenvalues Lead to Secondary Bifurcation," SIAM Rev., **17**(1), pp. 101–122.

[9] Olhoff, N., and Rasmussen, S. H., 1977, "On Single and Bimodal Optimum Buckling Loads of Clamped Columns," Int. J. Solids Struct., **13**, pp. 605–614.

[10] Prager, S., and Prager, W., 1979, "A Note on Optimal Design of Columns," Int. J. Mech. Sci., **21**, pp. 249–251.

[11] Seyranian, A. P., Lund, E., and Olhoff, N., 1994, "Multiple Eigenvalues in Structural Optimization Problems," Struct. Optim., **8**, pp. 207–227.

[12] Seyranian, A. P., and Privalova, O. G., 2003, "The Lagrange Problem on an Optimal Column: Old and New Results," Struct. Multidiscip. Optim., **25**, pp. 393–410.

[13] Seyranian, A. P., and Mailybaev, A. A., 2003, *Multiparameter Stability Theory With Mechanical Applications*, World Scientific, River Edge, NJ.

[14] Atanackovic, T. M., and Novakovic, B. N., 2006, "Optimal Shape of an Elastic Column on Elastic Foundation," Eur. J. Mech. A/Solids, **25**, pp. 154–165.

[15] Arnold, V. I., 1978, *Mathematical Methods of Classical Mechanics*, Springer, New York.