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Vibrational Stabilization of Statically Unstable Systems

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One hundred years ago, A. Stephenson [1, 2] showed that the upper statically unstable equilibrium position for the one-, two-, and three-link pendula can be stabilized with the help of a reasonably high-frequency vibration of the suspension point. Among many other investigations devoted to the stabilization of statically unstable systems using vibration, we note [3–10]. All these works are devoted to high-frequency stabilization. In this study, we investigate the possibility of stabilization of statically unstable systems by an arbitrary-frequency vibration. We consider finite-dimensional weakly unstable, i.e., close to critical, systems. The condition of vibrational stabilization with an arbitrary (nonresonant) excitation frequency is obtained. As a special case, we investigate the high-frequency excitation. In examples of systems with one and two degrees of freedom, we show that vibrational stabilization is possible in the entire frequency range—at low, medium, and high excitation frequencies.

1. We consider the linear oscillatory system

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C}(p) + \delta\mathbf{B}(t))\mathbf{q} = 0, \tag{1}$$

where $\mathbf{q} \in \mathbb{R}^n$, $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{B}$ are the real symmetric matrices of the dimension $n \times n$, p and δ are the real parameters, and the dots designate differentiation in time t. It is assumed that the matrix $\mathbf{M}$ is positively definite, and the matrix $\mathbf{B}(t) = \mathbf{B}(t + T)$ is time-periodic with the period $T = \dfrac{2\pi}{\Omega}$ and the frequency Ω. Having put $\delta = 0$, we obtain an autonomous conservative system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0. \tag{2}$$

This system is stable if the matrix $\mathbf{C}(p)$ is positively definite. In this case, the eigenfrequencies $0 < \omega_1 \le \ldots \le \omega_n$ and the vibrational modes $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ satisfy the equations and orthonormalization conditions

$$\mathbf{C}\mathbf{w}_k = \omega_k^2\mathbf{M}\mathbf{w}_k,\quad \mathbf{w}_k^T\mathbf{M}\mathbf{w}_{k'} = \delta_k^{k'},\quad k, k' = 1, 2, \ldots, n, \tag{3}$$

where $\delta_k^{k'}$ is the Kronecker delta. Equations (1)–(3) remain invariant under the transformation

$$t = T\tilde{t},\quad \mathbf{C} = \frac{\tilde{\mathbf{C}}}{T^2},\quad \mathbf{B}(t) = \frac{\tilde{\mathbf{B}}(\tilde{t})}{T^2},\quad \omega_k = \frac{\tilde{\omega}_k}{T}, \tag{4}$$

where $\tilde{\mathbf{B}}(\tilde{t}) = \tilde{\mathbf{B}}(\tilde{t} + \tilde{T})$ has the period $\tilde{T} = 1$. Further, we assume that $T = 1$ omitting tildes.

Let $p = p_0$ be the critical value so that system (2) is stable at $p < p_0$ and unstable at $p > p_0$. We consider the nondegenerate case, when the frequency $\omega_1 = 0$, and $0 < \omega_2 \le \ldots \le \omega_n$ for $p = p_0$. The eigenvector $\mathbf{w}_1$ of the matrix $\mathbf{C}_0 = \mathbf{C}(p_0)$ (the critical mode) corresponds to the zero frequency

$$\mathbf{C}_0\mathbf{w}_1 = 0. \tag{5}$$

Because the system is stable at $p < p_0$, the matrix $\mathbf{C}(p) \approx \mathbf{C}_0 + \dfrac{d\mathbf{C}}{dp}\Delta p$ is positively definite for small negative $\Delta p = p - p_0$. In particular, $\mathbf{w}_1^T\mathbf{C}\mathbf{w}_1 > 0$ which, taking into account Eq. (5), results in the condition

$$\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1 < 0,\quad \mathbf{C}_1 = \left.\frac{d\mathbf{C}}{dp}\right|_{p = p_0}. \tag{6}$$

System (1) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x},\quad \mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix},$$
$$\mathbf{A}(t) = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{C}(p) + \delta\mathbf{B}(t)) & 0 \end{pmatrix}. \tag{7}$$

For $\delta = p - p_0 = 0$, the matrix $\mathbf{A}_0 = \mathbf{A}(t)$ is independent of time and has a double zero eigenvalue with a second-

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order Jordan block. The corresponding right-hand and left-hand eigenvectors and associated vectors are determined by the equations and orthonormalization conditions

$$\mathbf{A}_0\mathbf{u}_0 = 0, \quad \mathbf{A}_0\mathbf{u}_1 = \mathbf{u}_0, \quad \mathbf{v}_0^T\mathbf{A}_0 = 0, \quad \mathbf{v}_1^T\mathbf{A}_0 = \mathbf{v}_0^T, \tag{8}$$
$$\mathbf{v}_0^T\mathbf{u}_0 = \mathbf{v}_1^T\mathbf{u}_1 = 0, \quad \mathbf{v}_1^T\mathbf{u}_0 = \mathbf{v}_0^T\mathbf{u}_1 = 1$$

and have the form

$$\mathbf{u}_0 = \begin{pmatrix} \mathbf{w}_1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} 0 \\ \mathbf{w}_1 \end{pmatrix}, \quad \mathbf{v}_0 = \begin{pmatrix} 0 \\ \mathbf{M}\mathbf{w}_1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} \mathbf{M}\mathbf{w}_1 \\ 0 \end{pmatrix}. \tag{9}$$

The fundamental matrix $\mathbf{X}(t)$ of system (7) satisfies the differential equation $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ with the initial condition $\mathbf{X}(0) = \mathbf{I}$. The monodromy matrix is determined as $\mathbf{F} = \mathbf{X}(1)$. Its eigenvalues ρ are called multiplicators. The multiplicators of system (1) have symmetry: if ρ is a multiplicator, $1/\rho$ is also a multiplicator. The system is stable only in the case when all multiplicators lie on a unit circle $|\rho| = 1$ and form no Jordan blocks.

For $\delta = p - p_0 = 0$, the system is autonomous. Hence, $\mathbf{X}_0(t) = \exp(t\mathbf{A}_0)$ and $\mathbf{F}_0 = \exp(\mathbf{A}_0)$, and the multiplicators $\rho = \exp(\pm i\omega_k)$, $k = 1, 2, \ldots, n$. It is easy to understand that the double multiplicator $\rho = 1$ of the matrix $\mathbf{F}_0$ forms the Jordan block with the same eigenvectors and associated vectors (9) for $k = 1$. It is assumed below that other multiplicators $\rho = \exp(\pm i\omega_k)$, $k = 2, 3, \ldots, n$ are simple and complex; i.e., the system is out of resonance. This means that

$$\omega_k \pm \omega_{k'} \neq 2\pi j \tag{10}$$

for arbitrary positive integers k, k', and j. For an arbitrary period $T = \dfrac{2\pi}{\Omega}$, this condition takes the form $\Omega \neq \dfrac{\omega_k \pm \omega_{k'}}{j}$. For small perturbations of parameters p and δ, the simple multiplicators remain on the unit circle $|\rho| = 1$. Hence, for the analysis of stability in the vicinity of $p = p_0$, $\delta = 0$, it is necessary to investigate the bifurcation of the double multiplicator $\rho = 1$ under a change of parameters.

We note that the analysis of stability in the resonant cases of $\omega_k \pm \omega_{k'} \approx 2\pi j$ with $k, k' > 1$ can be carried out using the methods of the theory of parametrical resonance [11]. The resonances at $k' = 1$ (so that $\omega_k \approx 2\pi j$) are degenerate: at the resonance point, the multiplicity of the multiplicator $\rho = 1$ increases to 4. These cases require special study.

The bifurcation of the double multiplicator is described by the asymptotic formula [11]

$$\rho = 1 \pm \sqrt{g_p\Delta p + g_\delta\delta}, \quad g_\alpha = \mathbf{v}_0^T\frac{\partial \mathbf{F}}{\partial \alpha}\mathbf{u}_0, \quad \alpha \in \{p, \delta\}, \tag{11}$$

where the derivatives of the monodromy matrix at $p = p_0$, $\delta = 0$, are equal to

$$\frac{\partial \mathbf{F}}{\partial \alpha} = \mathbf{F}_0\int_0^1 \mathbf{H}_\alpha(t)dt, \quad \mathbf{H}_\alpha(t) = \mathbf{X}_0^{-1}(t)\frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{X}_0(t). \tag{12}$$

Using Eqs. (7)–(9) and (12) in Eq. (11), we find that

$$g_p = -\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1, \quad g_\delta = -\mathbf{w}_1^T\bar{\mathbf{B}}\mathbf{w}_1, \quad \bar{\mathbf{B}} = \int_0^1 \mathbf{B}(t)dt. \tag{13}$$

The equality $|\rho| = 1$ taking into account Eqs. (11), (13) gives the first-order stability condition

$$\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1(p - p_0) + \mathbf{w}_1^T\bar{\mathbf{B}}\mathbf{w}_1\delta > 0. \tag{14}$$

Below we consider the case $\bar{\mathbf{B}} = 0$. Then we have $p < p_0$ in the first approximation (we recall that $\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1 < 0$) from Eq. (14). Thus, the stabilization effect is described by the second-order approximation. In the general case, the stabilization condition has the form

$$p < p_0 + \frac{a\delta^2}{2} + o(\delta^2) \tag{15}$$

with an unknown coefficient a. The critical value p (the stability limit) is

$$p_{\text{cr}} = p_0 + \frac{a\delta^2}{2} + o(\delta^2). \tag{16}$$

We consider the perturbation along the stability boundary

$$\Delta p = p - p_0 = \frac{a\delta^2}{2} + o(\delta^2).$$

Because $\Delta p \sim \delta^2$ and $g_\delta = 0$, the square root in Eq. (11) is proportional to δ and, hence, exceeds the accuracy of the approximation $o(\delta^{1/2})$. In this degenerate case, the asymptotic expression for ρ begins with the first order term with respect to δ [11]

$$\rho = 1 + \mu\delta + o(\delta), \tag{17}$$

where the values of μ are determined from the quadratic equation $\mu^2 + \alpha_1\mu + \alpha_2 = 0$ with the coefficients

$$\alpha_1 = -\mathbf{v}_0^T\frac{\partial \mathbf{F}}{\partial \delta}\mathbf{u}_1 - \mathbf{v}_1^T\frac{\partial \mathbf{F}}{\partial \delta}\mathbf{u}_0,$$

$$\alpha_2 = \mathbf{v}_0^T\left(\frac{\partial \mathbf{F}}{\partial \delta}\mathbf{G}^{-1}\frac{\partial \mathbf{F}}{\partial \delta} - \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial \delta^2}\right)\mathbf{u}_0 - \frac{1}{2}\mathbf{v}_0^T\frac{\partial \mathbf{F}}{\partial p}\mathbf{u}_0\frac{d^2 p}{d\delta^2}, \quad (18)$$

$$\mathbf{G} = \mathbf{F}_0 - \mathbf{I} + \mathbf{u}_1\mathbf{v}_1^T,$$

$$\frac{\partial^2 \mathbf{F}}{\partial \delta^2} = 2\mathbf{F}_0\int_0^1\int_0^t \mathbf{H}_\delta(t)\mathbf{H}_\delta(\tau)d\tau dt.$$

Using Eqs. (7)–(9) and (12) in Eq. (18), we find $\alpha_1 = \mathbf{w}_0^T\bar{\mathbf{B}}\mathbf{w}_0 = 0$ due to $\bar{\mathbf{B}} = 0$. Then $\mu = \pm\sqrt{-\alpha_2}$. The stability condition $|\rho| = 1$ with expansion (17) gives $\alpha_2 > 0$; the equation $\alpha_2 = 0$ is critical (it gives the stability boundary). According to Eqs. (11), (13), and (16), the last term for α_2 in Eq. (18) is $\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1\frac{a}{2}$. Using Eq. (18) in the condition $\alpha_2 = 0$, we determine the constant

$$a = 2\mathbf{v}_0^T\left(\frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial \delta^2} - \frac{\partial \mathbf{F}}{\partial \delta}\mathbf{G}^{-1}\frac{\partial \mathbf{F}}{\partial \delta}\right)\frac{\mathbf{u}_0}{\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1}. \quad (19)$$

Expression (16) with the coefficient a found in Eq. (19) gives the critical load of the system under parametrically excited. Let the value of parameter p be fixed, and $p > p_0$ (the system is statically unstable). The stabilization of the parametrical-excitation system is possible only for $a > 0$. Then, the system is stabilized according to Eq. (15) at the excitation amplitude $\delta > \sqrt{\frac{2\Delta p}{a}}$.

2. For the applications, it is useful to express the coefficient a in Eq. (19) through the frequencies ω_k and the modes $\mathbf{w}_k$ of the conservative system (2), (3). It is possible to show that this coefficient is represented as the integral

$$a = \frac{4}{\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1}\int_0^1\int_0^t\left[B_1(t)B_1(\tau)(t-1)\tau + \sum_{k=2}^{n}\mathrm{Im}\frac{B_k(t)B_k(\tau)e^{i\omega_k(t-\tau)}}{\omega_k(1-e^{i\omega_k})}\right]d\tau dt, \quad (20)$$

where the real scalar values $B_k(t) = \mathbf{w}_1^T\mathbf{B}(t)\mathbf{w}_k$ describe the interaction between modes. For the systems with an arbitrary period T, the backward substitution of values (4) into Eq. (20) results in the integral

$$a = \frac{4}{\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1}\int_0^T\int_0^t\left[\frac{B_1(t)B_1(\tau)(t-T)\tau}{T^2} + \sum_{k=2}^{n}\mathrm{Im}\frac{B_k(t)B_k(\tau)e^{i\omega_k(t-\tau)}}{\omega_k T(1-e^{i\omega_k T})}\right]d\tau dt. \quad (21)$$

Relation (21) determines the change in critical load (16) in terms of the frequencies and modes of the initial unperturbed system.

In the particular case of the harmonic excitation $\mathbf{B}(t) = \mathbf{B}_0\cos\Omega t$, integration in Eq. (21) gives

$$a = -\frac{1}{\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1}\sum_{k=1}^{n}\frac{(\mathbf{w}_1^T\mathbf{B}_0\mathbf{w}_k)^2}{\Omega^2 - \omega_k^2}. \quad (22)$$

According to Eq. (6), we have $\mathbf{w}_1^T\mathbf{C}_1\mathbf{w}_1 < 0$. Hence, in the case of harmonic excitation, the eigenmodes with the frequencies $\omega_k < \Omega$ give positive terms in the expression for the coefficient a (the stabilizing effect), while the modes with the frequencies $\omega_k > \Omega$ give the negative terms (the destabilizing effect). The term corresponding to the critical mode $\omega_1 = 0$ is always positive (the stabilization).

3. We investigate the effect of a small dissipation for the system of the type

$$\mathbf{M}\ddot{\mathbf{q}} + \gamma\mathbf{D}\dot{\mathbf{q}} + (\mathbf{C}(p) + \delta\mathbf{B}(t))\mathbf{q} = 0, \quad (23)$$

where $\mathbf{D}$ is the real symmetric positively definite matrix of the dimension $n \times n$, and $\gamma > 0$ is the small dissipation parameter.

The critical parameter $p_{cr}(\delta, \gamma)$ can be expanded in a series of powers of δ and γ. If there is no parametrical excitation $\delta = 0$, the dissipative forces have no effect on the critical value of the parameter p [12]. Hence, $p_{cr}(0, \gamma) \equiv p_0$; i.e., there are no terms of the type γ^k, $k = 1, 2, \ldots$, in the expansion of $p_{cr}(\delta, \gamma)$. Investigating system (23) for the fixed dissipation parameter $\gamma > 0$ by the method of the perturbation theory (similar to the case of $\gamma = 0$), it is possible to show that, in the expansion of $p_{cr}(\delta, \gamma)$, there are also no terms of the type $\gamma^k\delta$.

Thus, the effect of small dissipative forces is reduced mainly to changing the coefficient a in the relation for critical load (16)

$$p_{cr} = p_0 + \frac{a(\gamma)\delta^2}{2} + o(\delta^2), \quad (24)$$

where $a(0)$ is given by Eq. (21). It can be seen that the effect of dissipative forces is usually very weak. If the excitation matrix is reversible in time, i.e., $\mathbf{B}(t) = \mathbf{B}(t_0 - t)$ for certain t_0, system (23) is invariant with respect to the transformation $t \to t_0 - t$ and $\gamma \to -\gamma$. Hence, the odd powers of γ altering the sign under this transformation cannot appear in the expansion of $p_{cr}(\delta, \gamma)$. In this case, the first order correction when taking into account the dissipation has the order of $\gamma^2\delta^2$.

4. Consider the high-frequency excitation case when the frequency Ω considerably exceeds all eigenfrequencies $\omega_2, \ldots, \omega_n$ of the system. Then $\omega_k T \ll 1$ so that it is possible to expand the exponential functions in Eq. (21) in terms of powers of small values $\omega_k T$ and

$\omega_k(t-\tau)$. As a result of a number of transformations, we obtain

$$a = \frac{4}{T^2 \mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1} \times \sum_{k=1}^{n} \int_0^T \int_0^t B_k(t) B_k(\tau)(t-T)\tau d\tau dt + O(T^4). \qquad (25)$$

This formula is the high-frequency asymptotic form of Eq. (21). Because the integral in Eq. (25) has the order of T^4, $a \sim T^2 \sim \Omega^{-2}$.

We present the matrix $\mathbf{B}(t)$ as the Fourier expansion

$$\mathbf{B}(t) = \sum_{m=1}^{\infty} \mathbf{B}'_m \cos(m\Omega t) + \mathbf{B}''_m \sin(m\Omega t), \qquad \Omega = \frac{2\pi}{T} \qquad (26)$$

(there is no constant term due to $\bar{\mathbf{B}} = 0$). Substituting Eq. (26) into Eq. (25) and carrying out integration and summation over k, we can reduce this expression to the form

$$a = -\frac{1}{\mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1} \sum_{m=1}^{\infty} \frac{\mathbf{w}_1^T (\mathbf{B}'_m \mathbf{M}^{-1} \mathbf{B}'_m + \mathbf{B}''_m \mathbf{M}^{-1} \mathbf{B}''_m) \mathbf{w}_1}{(m\Omega)^2} + O(\Omega^{-4}). \qquad (27)$$

Formula (27) expresses the coefficient a in the high-frequency limit through the coefficients of the Fourier coefficients of the matrix $\mathbf{B}(t)$. We note that Eq. (27) can be obtained also by the averaging method.

In the particular case of $\mathbf{B}(t) = \mathbf{B}_0 \cos\Omega t$, Eq. (27) becomes

$$a = -\frac{\mathbf{w}_1^T \mathbf{B}_0 \mathbf{M}^{-1} \mathbf{B}_0 \mathbf{w}_1}{\mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1} \Omega^{-2} + O(\Omega^{-4}). \qquad (28)$$

This expression can be presented also as the series

$$a = -\frac{\Omega^{-2}}{\mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1} \sum_{k=1}^{n} (\mathbf{w}_1^T \mathbf{B}_0 \mathbf{w}_k)^2 + O(\Omega^{-4}), \qquad (29)$$

which agrees with Eq. (22) for the high excitation frequency Ω.

5. As the first example, we consider the system with one degree of freedom for which Eq. (1) is the Hill equation

$$\ddot{q} + (-p + \delta b(t))q = 0, \qquad (30)$$

where $b(t)$ is a periodic function with the period T and a zero average value. Then from Eqs. (15) and (21), we obtain the stabilization region in the first approximation as

$$p < \frac{a\delta^2}{2}, \quad a = \frac{4}{T} \int_0^T b(t) \int_0^t b(\tau)\tau d\tau dt - \frac{2}{T^2} \left(\int_0^T b(t) t dt \right)^2. \qquad (31)$$

For $T = 2\pi$, this formula coincides with that obtained previously in [10]. If $b(t) = \cos t$, $a = 1$, which is well-known.

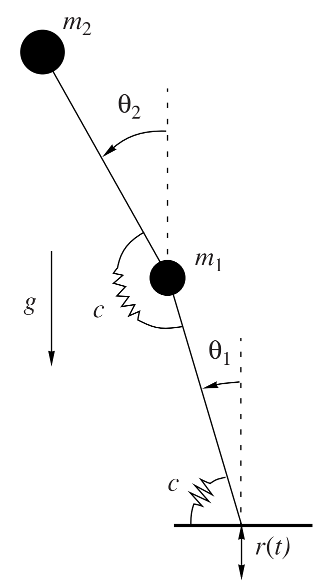

Fig. 1. Inverted pendulum with the periodic excitation of the base.

As the second example, we consider the inverted double pendulum consisting of two concentrated masses m_1 and m_2 connected by rigid massless rods of identical length l and rigidity c in the gravity field (Fig. 1). The Lagrange function $L = K - V$ of the system is determined by the expressions

$$K = \frac{1}{2}(m_1 + m_2) l^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l^2 \dot{\theta}_2^2 + m_2 l^2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2),$$
$$V = \frac{c\theta_1^2}{2} + \frac{c(\theta_2 - \theta_1)^2}{2} + (m_1 + m_2) g l \cos\theta_1 + m_2 g l \cos\theta_2, \qquad (32)$$

where g is the acceleration of gravity. We consider the periodic excitation of the base $z = a\cos\Omega t$. Then, according to the d'Alembert principle, it is necessary to substitute $g + \ddot{z}$ instead of g. The equations of motion of the system are found with the help of the Lagrange function and linearized in the vicinity of the vertical position $\theta_1 = \theta_2 = 0$.

We introduce the dimensionless time and parameters

$$\tilde{t} = \Omega^* t, \quad \delta = \frac{a}{l}, \quad p = -\frac{c}{m_1 g l}, \quad \tilde{\Omega} = \frac{\Omega}{\Omega^*}, \quad \eta = \frac{m_2}{m_1}, \tag{33}$$

where $\Omega^* = \sqrt{\frac{g}{l}}$. In the dimensionless variables, the linearized equations of motion of the system take the form of Eq. (1) with $\mathbf{B}(t) = \mathbf{B}_0 \cos\Omega t$ and the matrices

$$\mathbf{M} = \begin{pmatrix} \eta+1 & \eta \\ \eta & \eta \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -2p-\eta-1 & p \\ p & -p-\eta \end{pmatrix}, \quad \mathbf{B}_0 = \Omega^2 \begin{pmatrix} 1+\eta & 0 \\ 0 & \eta \end{pmatrix}. \tag{34}$$

Here and below, we omitted tildes. The negative parameter p is introduced, in order to determine the instability condition as $p > p_0$. Taking into account the forces of viscous friction in the hinges determined by the dissipative function $F = \frac{\gamma\dot{\theta}_1^2}{2} + \frac{\gamma(\dot{\theta}_2 - \dot{\theta}_1)^2}{2}$, we come to system (23) with the dimensionless dissipation parameter $\tilde{\gamma} = \frac{\gamma}{\Omega^* m_1 l^2}$ and the matrix

$$\mathbf{D} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}. \tag{35}$$

In the case of $\delta = 0$, the system is stable if and only if the matrix $\mathbf{C}$ is positively definite. This condition leads to the inequality

$$p < p_0, \quad p_0 = -\frac{3\eta + 1 + \sqrt{5\eta^2 + 2\eta + 1}}{2}. \tag{36}$$

For $p = p_0$, we find the frequencies and the corresponding modes from Eqs. (3) and (34) as

$$\omega_1 = 0, \quad \mathbf{w}_1 = \alpha_1 \begin{pmatrix} p_0 \\ 2p_0 + \eta + 1 \end{pmatrix}, \tag{37}$$

$$\omega_2 = \sqrt{-\left(5 + \frac{1}{\eta}\right)p_0 - 2(\eta+1)}, \quad \mathbf{w}_2 = \alpha_2 \begin{pmatrix} \eta + \omega_2^2\eta + p_0 \\ -\omega_2^2\eta + p_0 \end{pmatrix}, \tag{38}$$

where the multipliers α_1 and α_2 are determined from orthonormalization conditions (3).

The coefficient a is found from Eq. (22) in the form

$$a = a_1\Omega^2 + \frac{a_2\Omega^4}{\Omega^2 - \omega_2^2}, \quad a_k = -\frac{(\mathbf{w}_1^T \mathbf{B}_0 \mathbf{w}_k)^2}{\Omega^4 \mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1}, \quad k = 1, 2. \tag{39}$$

According to Eq. (34), the coefficients a_1 and a_2 depend only on the mass-ratio parameter η. Both values of a_1 and a_2 are not negative; $a_2 = 0$ for $\eta = 0$ and 1. The stability condition is given by inequality (15), which can be written by taking into account Eq. (39) as

$$\left(a_1 + \frac{a_2\Omega^2}{\Omega^2 - \omega_2^2}\right)\frac{\delta^2\Omega^2}{2} > \Delta p, \quad \Delta p = p - p_0. \tag{40}$$

The second term in the brackets describes the effect of the second mode ω_2. The effect of this term is negligible for $\eta \lesssim 2$, when $a_2 \ll a_1$, but its role is significant for large η.

Inequality (40) is the stabilization condition of the first mode, which is unstable if there is no excitation ($\delta = 0$). Hence, condition (40) is the necessary stabilization condition determining the lower limit of the stabilization amplitude. The instability can be also associated with the second mode due to the resonances giving the upper limit of the stabilization amplitude.

For $\Omega \approx \frac{2\omega_2}{j}$ with integer j, the system is subjected to parametrical resonance. It is known that the resonances corresponding to $j > 1$ are efficiently suppressed by introducing the damping term $\gamma\mathbf{D}$ in Eq. (23) for the harmonic excitation. The basic resonance $\Omega \approx 2\omega_2$ is the most important region of instability. In the first approximation, this region is determined by the inequality [11]

$$\zeta^2\gamma^2 + (\Omega - 2\omega_2)^2 < \xi\delta^2, \quad \zeta = \mathbf{w}_2^T \mathbf{D} \mathbf{w}_2, \quad \xi = \frac{(\mathbf{w}_2^T \mathbf{B}_0 \mathbf{w}_2)^2}{4\omega_2^2}. \tag{41}$$

The case $\Omega \approx \frac{\omega_2}{j}$ corresponds to the special case of instability related to the multiplicator $\rho = 1$ of multiplicity 4 and requires special investigation.

We consider two particular values of η. For $\eta = 1$ (the equality of masses $m_1 = m_2$) $p_0 = -3.414$, $\omega_2 = 4.040$, and the coefficients $a_1 = 2$ and $a_2 = 0$. Then stabilization condition (40) is simplified:

$$\delta^2\Omega^2 > \Delta p. \quad (42)$$

It is the degenerate case because the effect of the second mode disappears and the condition of the high-frequency stabilization becomes applicable over the whole range of excitation frequencies.

In Fig. 2, we show the stability diagram obtained numerically for $\eta = 1$, $\Delta p = 0.5$, and the damping coefficient $\gamma = 0.01$ by the Floquet method (the calculation of the system multiplicators and the check of the asymptotic-stability condition $|\rho| < 1$). The stability (stabilization) region is painted over with gray. The lower boundary (42) of the stability region, which was obtained analytically, agrees perfectly with the numerical results of calculation of the boundary. In Fig. 2, the resonances can be seen at $\Omega \approx \omega_2$ and $2\omega_2$. Basic-resonance region (41) is shown by a bold V-shaped line; certainly, this approximation is inexact at large amplitudes.

Putting $\eta = 10$, we find $p_0 = -26.91$, $\omega_2 = 10.74$, and condition (40) becomes

$$\left(14.34 + \frac{22.30\Omega^2}{\Omega^2 - 10.74^2}\right)\frac{\delta^2\Omega^2}{2} > \Delta p. \quad (43)$$

For high frequencies of excitation $\Omega \gg \omega_2$, this condition is reduced to the following:

$$18.32\delta^2\Omega^2 > \Delta p. \quad (44)$$

In Fig. 3, we show the stability diagram found numerically for $\eta = 10$, $\Delta p = 0.1$ and $\gamma = 0.01$ by the Floquet method. Lower stability-region boundary (43) is shown by solid lines. This theoretical result agrees perfectly with the numerical calculations. High-frequency asymptotic behavior (44) is shown by the dashed line. For the frequencies $\Omega \gtrsim 20$, the boundaries determined by Eqs. (43) and (44) are very close.

The upper boundary of the stability (stabilization) region corresponds to the resonances when the second mode becomes unstable. In Fig. 3, the resonant regions can be seen for $\Omega \approx \frac{2\omega_2}{k}$, $k = 1$, 2, and 4. The approximation of basic-resonance region (41) is shown by the solid V-shaped line. The upper boundary depends on the small parameter Δp only weakly. In Fig. 3, we show the case of a reasonably small value of $\Delta p = 0.1$. With increasing Δp, the lower boundary of the stability region grows, and the medium stabilization region disappears entirely approximately at $\Delta p \sim 10$, while the low-frequency stabilization region is shifted to higher amplitudes δ and becomes thinner because of resonances. This fact is demonstrated in Fig. 4 corresponding to the case $\Delta p = 2$.

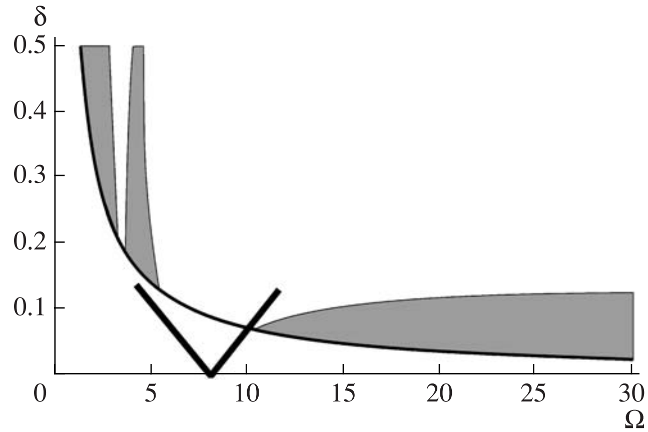

Fig. 2. Stability diagram for $\eta = 1$ and $\Delta p = 0.5$.

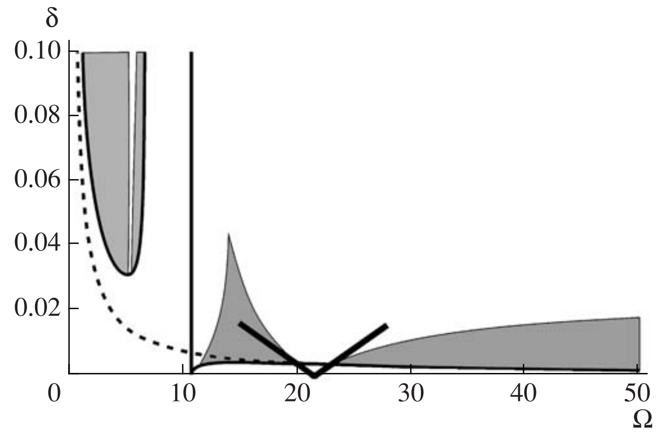

Fig. 3. Stability diagram for $\eta = 10$ and $\Delta p = 0.1$.

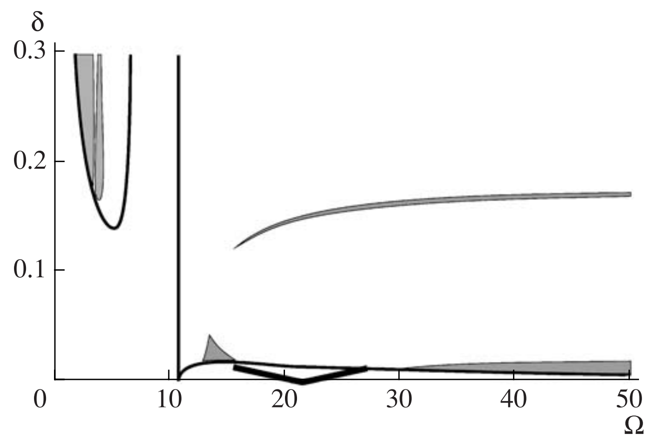

Fig. 4. Stability diagram for $\eta = 10$ and $\Delta p = 2$.

Thus, it is shown that the stabilization by a periodic excitation outside of the resonance regions is possible over the entire frequency range—at low, medium, and high excitation frequencies.

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