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Stabilization of statically unstable columns by axial vibration of arbitrary frequency

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Dedicated to the memory of Professor V.V. Bolotin

ABSTRACT

The classical Chelomei's problem of stabilization of a statically unstable elastic column by axial harmonic vibration is reconsidered. The excitation frequency is assumed to be arbitrary. Two types of boundary conditions of the column, simply supported and clamped–hinged ends, are considered. Both cases are studied analytically with check by numerical analysis. For undamped columns with the axial force close to the critical stability value, a finite number of triangular stabilization zones appears. This implies that stabilization is attained at medium frequency range (compared to the second eigenfrequency of the unexcited column). The high-frequency stabilization is impossible, as the stabilization regions vanish with the increasing excitation frequency. With addition of internal damping, a continuous stabilization region appears for small excitation amplitudes, which starts at medium frequencies and extends to high-frequency range. The influence of external damping on stabilization regions is shown to be small.

1. Introduction

The influence of vibration on stability properties of elastic systems is an important topic from the point of view of the theory and applications in mechanical engineering [1,2]. Chelomei [3] predicted that statically unstable elastic systems can be stabilized by high-frequency parametric excitation. In the next paper [4], he showed experimentally that an elastic simply supported column, compressed by an axial force exceeding the critical Euler value, can be stabilized in its straight position by high-frequency axial vibration applied to the end of the column. A heavy mass was mounted on a simply supported column and caused the column to buckle. With the axial vibration applied to the mass, the column was reported to straighten out. However, Chelomei did not report particular values of the stabilization frequencies achieved in the experiment. Bolotin in [5] analyzed numerically the stabilization regions of a column compressed by a periodic axial force exceeding in average Euler's critical value. He arrived at the conclusion that the analogy with the problem of stabilization of the inverted pendulum is not correct due to the presence of intermittent resonance zones caused by higher harmonics, which diminish the stabilization region. Bolotin also warned about the use of averaging and similar high-frequency asymptotic methods for the stability study of multiple degrees-of-freedom systems, since parametric excitation stabilizing some of the modes may destabilize others, and the authors share this point of view.

New attempts to investigate the stabilization of statically unstable columns by means of axial vibration were undertaken in [6–11]. It was stated [7], in particular, that a column subjected to high-frequency excitation has both a

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curved stable configuration and a straight stable equilibrium state for loads almost as low as the original buckling load. The theory and experiments in [6,8,12,13] confirmed the stiffening effect of the longitudinal high-frequency excitation (increase of the natural frequencies of transverse oscillations) for loads below the Euler critical value. Experimental investigation of a buckled column under high-frequency excitation was undertaken in the works [14,15], see also the review paper [16]. The increase of the buckling load with the excitation frequency was confirmed both theoretically and experimentally. However, we note that the stabilization frequency reported in [14,15] is lying between the first and second eigenfrequencies of the column, i.e. it is not high.

In the recent papers [17,18], formulas for higher and lower boundaries of the stabilization frequency of a statically unstable simply supported column were derived and analyzed. It was shown that the column is stabilized by excitation frequencies in the interval near the first frequency of free oscillations of the column, and the high-frequency assumption used in many papers could be invalid. The problem of stabilization of statically unstable finite degrees-of-freedom systems by parametric excitation with arbitrary frequency was studied in our previous paper [19]. It was shown that stabilization may be achieved by low, medium and high excitation frequencies.

In the present paper we study stabilization regions of statically unstable columns excited by axial harmonic vibration. Two types of boundary conditions of the columns are considered: simply supported at both ends and clamped–hinged. In Section 1 for simply supported columns we reduce the stability analysis to Mathieu equations and derive analytical expressions for the stability and instability regions. For undamped columns with the axial force close to the critical stability value, a finite number of triangular stabilization zones appear. This means that, for rather high excitation frequencies, stabilization of the column is impossible. The influence of internal damping, described by the Kelvin–Voigt model, and external damping, described by viscous friction, on the stability bounds is investigated. It is shown that the internal damping suppresses all the instability (parametric resonance) regions for large excitation frequencies when the excitation amplitude is below a certain small value. The effect of the external damping is shown to be small.

In Section 2 the clamped–hinged column is considered. We note that this case cannot be reduced to a system of uncoupled Mathieu equations, which is a general case. Here, using the results of [19], we derive the stabilization condition in terms of eigenfrequencies and eigenmodes of the column for the whole range of excitation frequencies. A high-frequency asymptotic stabilization condition is also derived, which is similar to the formula obtained in Section 1 for the simply supported column. The influence of external and internal damping is studied. Main properties of stabilization regions are shown to be similar to those for the simply supported column. However, some new resonance effects appear due to mode coupling. It is shown that the analytical results for the stability and instability regions are in good agreement with the numerical analysis.

Some necessary derivations are given in the Appendix, and the main results are summarized in the Conclusion.

2. Stabilization of a simply supported column

Let us consider vibrations of a straight elastic column of length l, compressed by the axial force P, Fig. 1. In this section, we assume that the column is simply supported at both ends. The actuator is attached to the upper end of the column moving freely in the axial direction x. The actuator possesses mass M that is periodically excited in the x direction according to the harmonic law $x_a = -d\cos\Omega t$, and generates the axial force $\Omega^2 Md\cos\Omega t$. The small transverse vibrations $w(x,t)$ of the

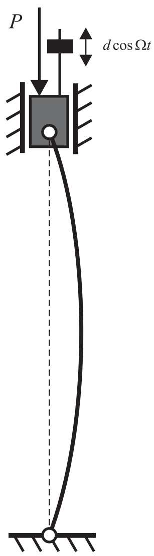

Fig. 1. Buckled elastic column under parametric excitation.

column are governed by the linearized equation

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} + (P + \Omega^2 Md \cos \Omega t) \frac{\partial^2 w}{\partial x^2} = 0, \tag{1}$$

where the weight of the column is neglected. The boundary conditions are

$$x = 0, l: \quad w = \frac{\partial^2 w}{\partial x^2} = 0. \tag{2}$$

We introduce the dimensionless variables and parameters

$$\tilde{x} = \frac{x}{l}, \quad \tilde{t} = \Omega_0 t, \quad \tilde{w} = \frac{w}{l}, \quad p = \frac{P}{P_{cr}}, \quad \delta = \frac{\pi^2 Md}{\rho A l^2}, \quad \omega = \frac{\Omega}{\Omega_0}, \tag{3}$$

where $P_{cr} = \pi^2 EI/l^2$ is the Euler critical load of the simply supported column, and $\Omega_0 = (\pi/l)^2 \sqrt{EI/\rho A}$ is its first natural frequency. The parameter ω describes the excitation frequency and δ is the excitation amplitude parameter. We also introduce the parameter $\mu = p - p_0$ for the deviation of the compressive force from the dimensionless critical value $p_0 = 1$. Omitting the tildes, we reduce (1) and (2) to the form

$$\ddot{w} + \pi^{-4} w'''' + \pi^{-2}(p + \omega^2 \delta \cos \omega t) w'' = 0, \tag{4}$$

$$x = 0, 1: \quad w = w'' = 0, \tag{5}$$

where the dots and primes denote derivatives with respect to dimensionless t and x, respectively.

Looking for the solution in the form $w_k = \varphi(\omega t/2) \sin(k\pi x)$ with $k = 1, 2, \ldots$, we obtain for $\varphi(\tau)$, $\tau = \omega t/2$, the Mathieu equation

$$\frac{d^2 \varphi}{d\tau^2} + (a - 2q \cos 2\tau)\varphi = 0, \tag{6}$$

with the parameters

$$a = 4k^2(k^2 - 1 - \mu)/\omega^2, \quad q = 2k^2 \delta. \tag{7}$$

A linear combination of solutions w_k with all positive integers k provides the general solution of (4) and (5). Therefore, the trivial state of the column $w = 0$ is stable if the solution $\varphi = 0$ of (6) is stable for all $k = 1, 2 \ldots$.

We consider the case when the non-excited system is unstable, which means that $\mu > 0$ (in dimensional form, $P > P_{cr}$). Taking $k = 1$ in (7), we obtain $a = -4\mu/\omega^2 < 0$ and $q = 2\delta$. The approximate stability condition for the Mathieu equation with negative a is $-a < q^2/2$ (accurate up to 3% for $q \lesssim 0.5$), see, e.g. [20]. In terms of problem parameters, it gives the stabilization condition due to periodic excitation as

$$\omega > \frac{\sqrt{2\mu}}{\delta}. \tag{8}$$

We must check the condition $-a < q^2/2$ for all modes k giving negative a. For example, if $\mu > 3$, then $a < 0$ for $k = 2$, and the corresponding stability condition becomes $\omega > \sqrt{(\mu - 3)/2}/\delta$. Since $(\mu - 3)/2 < 2\mu$, this condition is always weaker than (8). A similar argument shows that the same is true for larger k. Therefore, analysis of all modes with $a < 0$ gives the approximate stabilization condition (8).

For higher modes, we have $a > 0$, and the instability is related to parametric resonance zones. We will focus on the primary resonance zones located at $a \approx 1$, which are the widest and therefore the important. For small q, the resonance zone is given approximately by $|a - 1| < q$ (accurate up to 6% for $q \lesssim 0.5$). Using (7), the instability condition takes the form

$$|4(k^2 - 1 - \mu)/\omega^2 - 1/k^2| < 2\delta. \tag{9}$$

For different k, this condition determines a set of intersecting instability regions.

Fig. 2(a and b) shows stability charts for $\mu = 0.05$ and 0.5. The stability boundaries are computed numerically for the Mathieu equation by solving a boundary value problem with MATLAB. The stability boundaries are close to those given by asymptotic formulae (8) and (9). One can see that the main stabilization region lies in the medium-frequency interval $\omega < 6$. For higher excitation frequencies ω, triangular stabilization regions exist when μ is small (systems close to the critical stability bound). There is a single triangular zone for $0.18 < \mu < 0.40$, two zones for $0.10 < \mu < 0.18$, etc. The number of these zones is finite for any $\mu > 0$. Indeed, the largest δ in such a triangular zone corresponds to the intersection point of the adjacent boundaries corresponding to the resonant zones of modes k and $k + 1$. Condition (9) gives for this point

$$2\delta = -4(k^2 - 1 - \mu)/\omega^2 + 1/k^2 = 4((k+1)^2 - 1 - \mu)/\omega^2 - 1/(k+1)^2. \tag{10}$$

For large k, the intersection point is found from (10) by expansion with respect to $1/k$ as

$$\omega \approx 2k^2, \quad \delta \approx k^{-3} \approx (\omega/2)^{-3/2}. \tag{11}$$

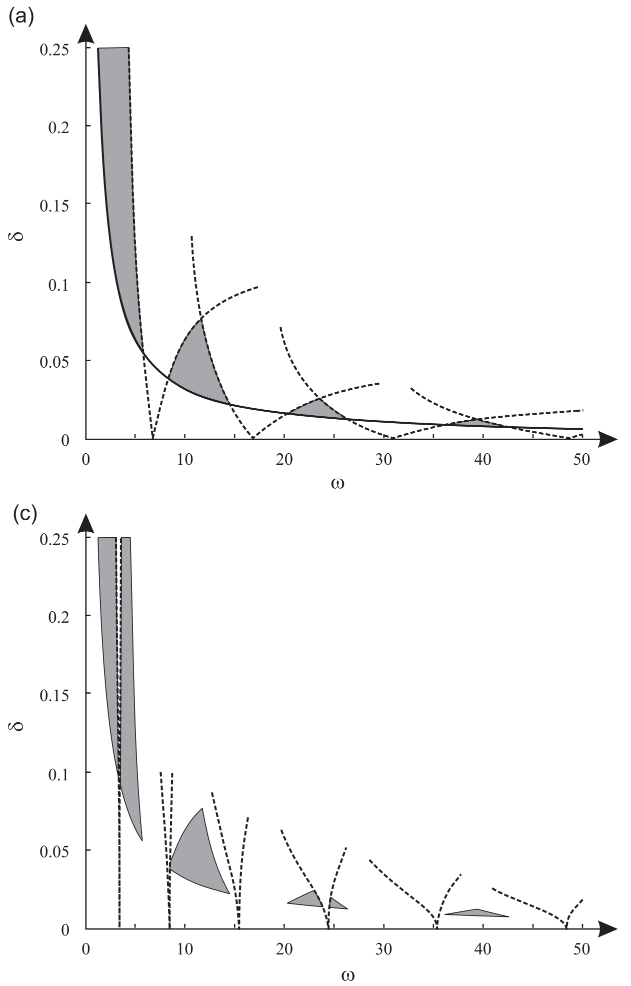

Fig. 2. Stability regions (shaded) for (a) $\mu = 0.05$, (b) $\mu = 0.5$, (c) effect of secondary parametric resonance zones for $\mu = 0.05$, (d) effect of internal damping for $\mu = 0.05$, $\gamma_i = 0.1$. The solid lines are the stability boundaries of the first mode, and the dashed lines show parametric resonance boundaries for higher modes.

But, for large k, these points do not satisfy the stabilization condition (8). Note that, at these points, $q = 2k^2\delta \approx 2/k$ and the condition for smallness of q is satisfied.

The effect of secondary resonances on the stabilization region is shown in Fig. 2(c) for $\mu = 0.05$; here only the second parametric resonance zones of the Mathieu equation are shown. Due to these zones, the stabilization region slightly diminishes. For larger μ, this effect is stronger: the stabilization region in Fig. 2(b) becomes almost one-half as large.

Let us study the effect of external (viscous friction) and internal (Kelvin–Voigt model) damping described, respectively, by the extra terms $c\dot{w}$ and $\eta I\dot{w}''''$ in Eq. (1). In the dimensionless equation (4), these terms become $(\gamma_e\dot{w} + \pi^{-4}\gamma_i\dot{w}'''')$. The dimensionless parameters $\gamma_e = c/(\rho A\Omega_0)$ and $\gamma_i = \eta\Omega_0/E$ are assumed to be small. Then Eq. (6) for each mode takes the form

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\tau^2} + 2\varepsilon\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} + (a - 2q\cos 2\tau)\varphi = 0, \quad \varepsilon = (\gamma_e + \gamma_i k^4)/\omega. \tag{12}$$

The asymptotic stabilization condition for Eq. (12) with negative values of a and small $\varepsilon > 0$ is [21]

$$a > -\frac{q^2(1-\varepsilon^2)}{2}. \tag{13}$$

In terms of problem parameters this inequality yields

$$\omega^2 > \frac{2\mu}{\delta^2} + (\gamma_e + \gamma_i)^2. \tag{14}$$

The effect of damping appears to be very small.

For primary parametric resonance, the first-order condition for the instability region is (see, e.g. [20])

$$q^2 > (a-1)^2 + 4\varepsilon^2. \tag{15}$$

Resonance is not possible if $q < q_{cr}$ with the critical value $q_{cr} = 2\varepsilon$ corresponding to $a = 1$. Taking q and ε from (7) and (12), we write this condition in terms of problem parameters as

$$k^2\delta < \frac{\gamma_e + \gamma_i k^4}{\omega}. \tag{16}$$

For high modes (large k), the resonance condition $a \approx 1$ yields $\omega \approx 2k^2$, so the inequality (16) becomes asymptotically

$$\delta < \gamma_i/2. \tag{17}$$

Therefore, if the excitation amplitude is small enough the internal damping completely suppresses parametric resonances. We also note that, for $k \gtrsim \sqrt{2/\gamma_i}$, the damping coefficient becomes $\varepsilon \gtrsim 1$, so that free oscillations of the k-th mode get completely suppressed. The stability chart with resonance zones (15) is shown in Fig. 2(d) for $\mu = 0.05$, $\gamma_e = 0$ and $\gamma_i = 0.1$. One can see that parametric resonance does not occur in the region (17).

On the contrary, the effect of external damping is small. When $\gamma_i = 0$, near the resonance $\omega \approx 2k^3$ condition (16) yields the inequality $\delta < 2\gamma_e/\omega^2$. This condition is not compatible with the stabilization condition (8) for large ω with fixed value of μ.

3. Stabilization of a clamped–hinged column

General properties of the stabilization region studied in the previous section should be valid for columns with different boundary conditions. At least, this should hold for high frequencies, as influence of boundary conditions is small for high modes. On the other hand, the simply supported column is specific, since this case reduces explicitly to a system of uncoupled Mathieu equations. This implies, in particular, that no combination resonances appear. Below we consider the case when the lower end of the column is clamped and the upper end is simply supported (Fig. 1), and study the stabilization conditions.

Let us consider Eq. (4) with the new boundary conditions

$$x = 0: \quad w = w'' = 0, \quad x = 1: \quad w = w' = 0. \tag{18}$$

The dimensionless Euler critical force is $p_0 = 2.0457$, see the Appendix.

In our previous paper [19], we studied systems with finite degrees of freedom of the form

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C}_0 + p\mathbf{C}_1 + \delta\mathbf{B}_0\cos\omega t)\mathbf{q} = 0, \quad \mathbf{q} \in \mathbb{R}^n, \tag{19}$$

where $\mathbf{M}$, $\mathbf{C}_0$, $\mathbf{C}_1$ and $\mathbf{B}_0$ are real symmetric matrices. The stability condition for the equilibrium $\mathbf{q} = 0$ is found to be

$$p < p_0 + a(\omega)\delta^2/2 + o(\delta^2), \tag{20}$$

with the coefficient depending on the excitation frequency

$$a(\omega) = -\frac{1}{\mathbf{w}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{w}_1}\sum_{k=1}^{n}\frac{(\mathbf{w}_1^{\mathrm{T}}\mathbf{B}_0\mathbf{w}_k)^2}{\omega^2 - \omega_k^2}. \tag{21}$$

Here ω_k and $\mathbf{w}_k$ are the eigenfrequencies and eigenmodes of system (19) at $p = p_0$ and $\delta = 0$, normalized as $\mathbf{w}_k^{\mathrm{T}}\mathbf{M}\mathbf{w}_k = 1$. Since p_0 is the critical force, $\omega_1 = 0$ and $\omega_k > 0$ for $k > 1$. Expression (21) was obtained under the non-resonance condition $\omega_k \pm \omega_{k'} \neq \omega j$ for any positive integers k, k', j.

If $a(\omega) > 0$, the excitation increases the critical force $p_{cr} \approx p_0 + a(\omega)\delta^2/2$. For fixed $p > p_0$, the approximate lower bound on the stabilizing excitation amplitude is found as $\delta > \sqrt{2\mu/a(\omega)}$ with $\mu = p - p_0$. Note that the resonance effects (studied below) provide an upper stability bound on δ.

Eq. (4) represents the infinite degree-of-freedom analog of (19), where instead of matrices we have the linear differential operators

$$\mathbf{M} \to 1, \quad \mathbf{C}_0 \to \pi^{-4}\frac{\partial^4}{\partial x^4}, \quad \mathbf{C}_1 \to \pi^{-2}\frac{\partial^2}{\partial x^2}, \quad \mathbf{B}_0 \to \pi^{-2}\omega^2\frac{\partial^2}{\partial x^2}. \tag{22}$$

Eigenfrequencies ω_k and eigenmodes $w_k(x)$ for system (4), (18) with $p = p_0$, $\delta = 0$ are given in the Appendix; the first several eigenfrequencies are listed in Table 1. The eigenmodes are normalized as

$$\int_0^1 w_k^2(x)\,\mathrm{d}x = 1, \tag{23}$$

Table 1
Values of eigenfrequencies and integrals (24) for the clamped–hinged column.

k	1	2	3	4	5	6	7	8
ω_k	0	4.091	9.595	17.090	26.584	38.079	51.575	67.072
J_{1k}	12.114	−5.6879	−4.0413	−3.1312	−2.5569	−2.1614	−1.8722	−1.6516

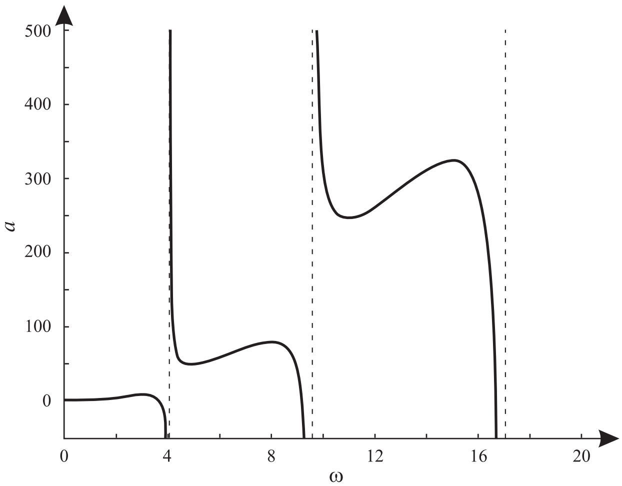

Fig. 3. Dependence of the coefficient a in (20) and (25) on the excitation frequency ω.

which corresponds to the discrete condition $\mathbf{w}_k^T \mathbf{M} \mathbf{w}_k = 1$. Finally, using the operators (22), we must substitute in formula (21)

$$\mathbf{w}_1^T \mathbf{B}_0 \mathbf{w}_k \to -\pi^{-2}\omega^2 J_{1k}, \quad \mathbf{w}_1^T \mathbf{C}_1 \mathbf{w}_1 \to -\pi^{-2} J_{11}, \quad J_{jk} = \int_0^1 w_j'(x) w_k'(x)\,dx, \tag{24}$$

where J_{1k} is obtained using integration by parts with boundary conditions (18). This yields

$$a(\omega) = \frac{\omega^2}{\pi^2}\left(J_{11} + \sum_{k=2}^{\infty} \frac{J_{1k}^2 \omega^2}{J_{11}(\omega^2 - \omega_k^2)}\right). \tag{25}$$

Numerical values of the integrals J_{1k} are given in Table 1. The function $a(\omega)$ is shown in Fig. 3. The vertical asymptotes in the figure correspond to the resonant frequencies ω_k. For large k, it can be shown that $J_{1k} \sim k^{-1}$ and $\omega_k \sim k^2$, so the sum in (25) converges as $\sum k^{-6}$.

Note that, up to the small factor $\delta^2/2$, Fig. 3 represents dependence of the critical parameter $\mu_{cr} = p_{cr} - p_0$ on the excitation frequency ω away from resonances.

The stabilization condition (20) and (25) takes the form

$$\frac{2\mu}{\delta^2} < \frac{\omega^2}{\pi^2}\left(J_{11} + \sum_{k=2}^{\infty} \frac{J_{1k}^2 \omega^2}{J_{11}(\omega^2 - \omega_k^2)}\right), \tag{26}$$

with $\mu = p - p_0$. Note that viscous damping has a very small effect on this condition, as shown in [19].

In order to obtain the stabilization regions of the column, we must also take into account parametric resonances. The resonance excitation frequencies are $\omega \approx (\omega_k + \omega_{k'})/\mathrm{j}$ with integer numbers k, k' and j. The instability regions for simple resonances $\omega \approx 2\omega_k$ and combination resonances $\omega \approx \omega_k + \omega_k'$ $(k, k' > 1)$ are given for small δ by the inequality [22, Section 11.8]

$$(\omega - \omega_k - \omega_{k'})^2 < \frac{J_{kk'}^2 \omega^4 \delta^2}{4\pi^4 \omega_k \omega_{k'}}. \tag{27}$$

Like in the previous section, we can take into account the external and internal damping with the dissipative terms $\gamma_e \dot{w} + \pi^{-4}\gamma_i \dot{w}''''$ in (4). Then the resonance condition (27) becomes

$$\gamma_k \gamma_{k'} - \frac{J_{kk'}^2 \omega^4 \delta^2}{4\pi^4 \omega_k \omega_{k'}} + \frac{4\gamma_k \gamma_{k'}}{(\gamma_k + \gamma_{k'})^2}(\omega - \omega_k - \omega_{k'})^2 < 0, \tag{28}$$

with the constants

$$\gamma_k = \gamma_e + \gamma_i(\omega_k^2 + \pi^{-4}p_0 J_{kk}), \quad k = 1, 2, \ldots . \tag{29}$$

Note that the frequencies ω_k and eigenmodes $w_k(x)$ in (27)–(29) correspond to $p = p_0 + \mu$. However, for small μ, one can use ω_k and $w_k(x)$ for $p = p_0$ computed in the Appendix.

Stability regions on the (ω, δ) plane were determined numerically for $\mu = 0.1$; see gray regions in Figs. 4 and 5. For numerical computations we used Galerkin method with five eigenmodes expansion (taking more modes does not lead to significant change of the results). Thus, substituting $w(x) = q_1 w_1(x) + \cdots + q_5 w_5(x)$ into Eq. (4), multiplying by $w_k(x)$ and integrating in the interval $0 \le x \le 1$ yields the five degree-of-freedom system (19). In this system, $\mathbf{M} = \mathbf{I}$ is the identity matrix, the stiffness matrix $\mathbf{C}_0 + p\mathbf{C}_1 = \mathrm{diag}(0, \omega_2^2, \omega_3^2, \omega_4^2, \omega_5^2) - \pi^{-2}\mu\mathbf{J}$, and the excitation matrix $\mathbf{B}_0 = -\pi^{-2}\omega^2\mathbf{J}$, where $\mathbf{J}$ is the matrix with elements J_{jk} given in (24). Taking into account the dissipative terms $\gamma_e \dot{w} + \pi^{-4}\gamma_i \dot{w}''''$ in (4) leads to the additional term $\mathbf{D}\dot{\mathbf{q}}$ in (19) with $\mathbf{D} = \gamma_e \mathbf{I} + \gamma_i(\mathrm{diag}(0, \omega_2^2, \omega_3^2, \omega_4^2, \omega_5^2) + \pi^{-2}p_0\mathbf{J})$. The stability was checked using the Floquet method. For this purpose, system (19) is reduced to the form $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. Asymptotic stability is determined by the inequality $|\rho| < 1$ for all the eigenvalues of the matrix $\mathbf{F} = \mathbf{X}(2\pi/\omega)$, where $\mathbf{X}(t)$ is found numerically from the matrix differential equation $\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}$ with the initial condition $\mathbf{X}(0) = \mathbf{I}$, see, e.g. [22]. Results presented in Fig. 4 correspond to $\gamma_e = 0.1$, $\gamma_i = 0$ (external damping). Fig. 5 shows the stabilization region for $\gamma_e = 0$ and $\gamma_i = 0.1$ (internal damping).

For comparison with analytical results, the lower stability bounds (26) are shown by solid lines in Figs. 4 and 5. The boundaries of parametric resonance zones (28) are plotted by dashed lines for the resonances $\omega \approx 2\omega_2$, $\omega_2 + \omega_3$, $2\omega_3$ in Fig. 4 and for the resonances $\omega \approx 2\omega_2$, $2\omega_3$ in Fig. 5. We see that the lower stability bounds (26) are in a very good agreement with the stability regions found numerically. Approximations of resonance regions (28) are less accurate, but still give adequate estimates. A number of thin secondary parametric resonance zones is observed in Fig. 4, giving broken structure of the stability boundary.

We see that the main properties of the stabilization regions, described for a simply supported column in the previous section, are also valid for the clamped–hinged column. In Fig. 4, the stabilization region is formed by several zones that get smaller for larger frequencies. Numerical computations also show that these zones decrease and disappear with the increase of μ. The introduction of internal damping completely suppresses parametric resonances for a certain interval of excitation amplitudes $\delta \lesssim \gamma_e/2$, so the stabilization becomes possible for high excitation frequencies, see Fig. 5.

Some new effects are related to degenerate combination resonances. Since $\omega_1 = 0$, the excitation frequencies close to $\omega \approx \omega_k$ correspond at the same time to the simple resonance ω_k and to the combination resonance $\omega_1 + \omega_k$. Hence, the stabilization bound (26) has singularities for $\omega \approx \omega_k$, $k = 2, 3, \ldots$. These singularities have complicated structure as shown by numerical computations in Fig. 4 for $\omega \approx \omega_2 \approx 4$. Recall that formula (26) is not valid near the points $\omega \approx \omega_k$.

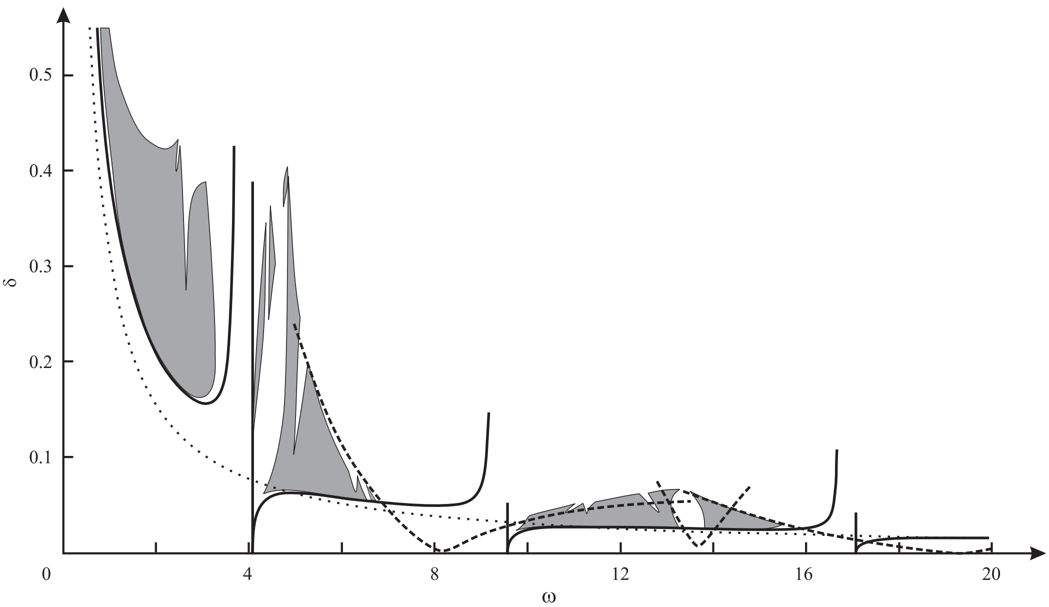

Fig. 4. Stabilization regions (shaded) obtained numerically for $\mu = 0.1$, $\gamma_e = 0.1$, $\gamma_i = 0$ (external damping). The solid lines are the stability boundaries of the first mode, and the dashed lines show parametric resonance boundaries for higher modes. The dotted line shows the stability boundary for the high-frequency asymptotics.

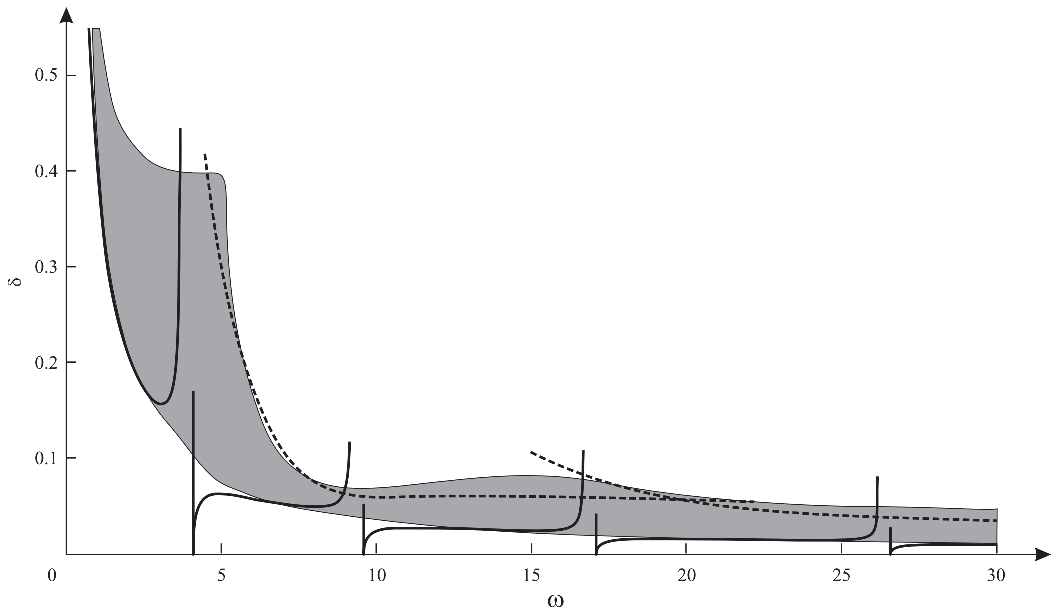

Fig. 5. Stabilization regions (shaded) obtained numerically for $\mu = 0.1$, $\gamma_e = 0$, $\gamma_i = 0.1$ (internal damping). The solid lines are the stability boundaries of the first mode, and the dashed lines show parametric resonance boundaries for higher modes.

Finally, let us consider the well-known high-frequency asymptotic stabilization condition. For large excitation frequencies $\omega \gtrsim \omega_K$ with $K \gg 1$, the asymptotic form of expression (26) is

$$\frac{2\mu}{\delta^2} < \frac{\omega^2}{\pi^2} \sum_{k=1}^{\infty} \frac{J_{1k}^2}{J_{11}}. \tag{30}$$

Here we neglected ω_k^2 in the denominator for $k \ll K$; the terms with $k \gtrsim K$ are small and, thus, unimportant (the system must be far from the resonances $\omega \approx \omega_k$, so that none of the denominators is small). Using the summation formula (38) from the Appendix, we write (30) as

$$\omega > \frac{\sqrt{2\mu/p_0}}{\delta}. \tag{31}$$

This formula is the same as the formula (8) for the simply supported case with $p_0 = 1$.

We remark that (31) can be easily obtained using the averaging method for high-frequency excitation, see [23]. The averaged equation is obtained from (4) by substituting the excitation term $\pi^{-2}\omega^2\delta\cos(\omega t)w''$ with the term $\pi^{-4}\omega^2\delta^2 w''''/2$ corresponding to the effective potential energy $U_{eff} = \frac{1}{4}\pi^{-4}\omega^2\delta^2 \int_0^l w''^2\,dx$. After time scaling, the averaged equation is reduced to the form (4) with $\delta = 0$ and $p/(1+\omega^2\delta^2/2)$ instead of p. This yields the critical force $p_{cr} = p_0(1+\omega^2\delta^2/2)$. The stability condition $p < p_{cr}$ is the same as (31). From the derivation, it is clear that the formula (31) is valid also for other boundary conditions, like clamped–clamped.

The stability boundary determined by (31) is shown in Fig. 4 by the dotted line. We see that this stability boundary is adequate, but it is much less accurate than the original formula (26) in the frequency range $\omega < 10$. The fact that the high-frequency asymptotic (31) gives reasonable results in the whole frequency range is related *not to the high frequency* ω, but to the smallness of the correction terms in the sum in (26). Indeed, already the second coefficient $J_{12}^2 \approx 32$ is about 4.5 times smaller than the coefficient $J_{11}^2 \approx 147$, see Table 1. In other words, the use of the high-frequency formula for stabilization of statically unstable columns is justified by the small effect of non-critical modes. The formula (26) is free from this restriction and gives more accurate results. However, it is valid only for small μ.

4. Conclusion

We reconsidered a classical problem of stabilization of a statically unstable elastic column by axial harmonic excitation with arbitrary frequency for two types of boundary conditions. In the first case corresponding to a column with simply supported ends, the system is reduced to uncoupled Mathieu equations. Therefore, this case allows complete stability study based on the known properties of the Ince–Strutt diagram. However, the simply supported column does not possess all typical resonances since its modes are not coupled through the parametric excitation. The second case of a column with clamped and simply supported ends, we believe, shows the most typical structure of stability zones. This case was studied using the asymptotic stability formulae.

We conclude that for stabilization problems of statically unstable systems, there is an essential difference between finite degrees-of-freedom and continuous column systems, which is caused mainly by the structure of parametric resonance zones. It is known that systems with finite degrees-of-freedom can be stabilized by high frequency excitation in the range where parametric resonances do not occur. For continuous columns, this is not true. Parametric resonance zones get wider for higher modes, eventually filling up the whole space. Therefore, stabilization of columns is possible only by medium excitation frequencies, of order of the first few natural frequencies of the column. In that frequency range, one must be careful when using averaging methods, conventional for high-frequency excitation in finite degree-of-freedom systems.

It turns out that addition of internal damping of the column provides, at rather small excitation amplitudes, the stabilization region extending to high frequency range. This effect is also of continuous system spirit, as it is determined by the specific nonlinear dependence of a damping force on a frequency. The internal damping force grows rapidly for higher modes, and thus, for higher excitation frequencies. On the contrary, external damping forces, which grow much slower with the excitation frequency, cannot provide high-frequency stabilization of the column.

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Appendix

Consider vibrations of a column governed by Eq. (4) with boundary conditions (18) and $\delta = 0$. Its eigenfrequencies and eigenmodes are found by taking $w(x,t) = w_k(x)\,\mathrm{e}^{\mathrm{i}\omega_k t}$, which yields

$$\pi^{-4}w_k'''' + \pi^{-2}pw_k'' = \omega_k^2 w_k. \tag{32}$$

Using $w_k(x) = \mathrm{e}^{\pi\beta x}$ in (32), we obtain $\beta^4 + p\beta^2 = \omega_k^2$. For $\omega_k^2 > 0$, this gives four solutions with two purely imaginary $\beta = \pm\mathrm{i}\mu_k$ and two real $\beta = \pm\nu_k$, where

$$\mu_k^2 = \frac{\sqrt{p^2 + 4\omega_k^2} + p}{2}, \quad \nu_k^2 = \frac{\sqrt{p^2 + 4\omega_k^2} - p}{2}. \tag{33}$$

A linear combination of these solutions satisfying the boundary conditions $w_k(0) = w_k''(0) = w_k(1) = 0$ is

$$w_k(x) = c_k\left(\frac{\sin \pi\mu_k x}{\sin \pi\mu_k} - \frac{\sinh \pi\nu_k x}{\sinh \pi\nu_k}\right), \tag{34}$$

with a constant c_k determined by the normalization condition $\int_0^1 w_k^2(x)\,\mathrm{d}x = 1$. Using (34) in the remaining boundary condition $w_k'(1) = 0$, we obtain the characteristic equation

$$\frac{\tan \pi\mu_k}{\mu_k} = \frac{\tanh \pi\nu_k}{\nu_k}. \tag{35}$$

Together with (33), it determines the eigenfrequencies ω_k of the prestressed column.

The Euler critical force p_0 is obtained by solving (32) with the boundary conditions (18) and $\omega_1 = 0$. The same result can be obtained by taking the limit $\omega_k \to 0$ in (33)–(35). This gives the critical mode

$$w_1(x) = c_1\left(\frac{\sin \pi\sqrt{p_0}x}{\sin \pi\sqrt{p_0}} - x\right) \tag{36}$$

and the equation for p_0:

$$\tan \pi\sqrt{p_0} = \pi\sqrt{p_0}. \tag{37}$$

The critical force, corresponding to the lowest solution of (37), is known to be $p_0 = 2.0457$ (the corresponding dimensional value is $2.0457\pi^2 EI/l^2$). Solving numerically Eqs. (33) and (35) with $p = p_0$, we find the eigenfrequencies ω_k, $k > 1$, see Table 1.

The following summation formula is valid

$$\begin{aligned}\sum_{k=1}^{\infty} J_{1k}^2 &= \sum_{k=1}^{\infty}\left(\int_0^1 w_1'(x)w_k'(x)\,\mathrm{d}x\right)^2 = \sum_{k=1}^{\infty}\left(\int_0^1 w_1''(x)w_k(x)\,\mathrm{d}x\right)^2 \\ &= \int_0^1\int_0^1 w_1''(x)w_1''(y)\left(\sum_{k=1}^{\infty} w_k(x)w_k(y)\right)\mathrm{d}x\,\mathrm{d}y = \int_0^1 w_1''^2(x)\,\mathrm{d}x \\ &= \int_0^1 w_1(x)w_1''''(x)\,\mathrm{d}x = -\pi^2 p_0\int_0^1 w_1(x)w_1''(x)\,\mathrm{d}x = \pi^2 p_0 J_{11}.\end{aligned} \tag{38}$$

Here we used integration by parts with boundary conditions (18), the well-known summation formula $\sum_{k=1}^{\infty} w_k(x)w_k(y) = \delta(x-y)$ (Dirac delta function) valid for eigenfunctions of self-adjoint operators (see, e.g. [24]), and $w_1''''(x)$ expressed from Eq. (32) with $\omega_1 = 0$.

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