

On the observability and asymmetry of adiabatic state flips generated by exceptional points

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Abstract

In open quantum systems where the effective Hamiltonian is not Hermitian, it is known that the adiabatic (or instantaneous) basis can be multivalued: by adiabatically transporting an eigenstate along a closed loop in the parameter space of the Hamiltonian, it is possible to end up in an eigenstate different from the initial eigenstate. This ‘adiabatic flip’ effect is an outcome of the appearance of a degeneracy known as an ‘exceptional point’ inside the loop. We show that contrary to what is expected of the transport properties of the eigenstate basis, the interplay between gain/loss and non-adiabatic couplings imposes fundamental limitations on the observability of this adiabatic flip effect.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Exceptional points (EPs) are an intriguing form of degeneracy that exists only in systems which are described by an effective non-Hermitian Hamiltonian (e.g. open systems). ‘Point’ refers to a point in the parameter space of the system where this degeneracy occurs. Consider a two-level system with two parameters that can be controlled. When these parameters are changed so that a closed loop that encloses the EP forms, the eigenstates gradually transform to each other. That is, after completing the loop, the end eigenstate is different from the initial eigenstate. This interchange of the eigenstates is called a ‘flip’. This is quite different from the transport around a degeneracy in Hermitian systems, where the initial and final eigenstates differ only by the Berry phase. What we mean exactly by ‘transported’ will become clear later on.

This flip property has been beautifully experimentally verified in microwave systems [1]. EPs have also been observed experimentally in electrical circuits [2]. They also appear in

loss/gain waveguide structures [3, 4]. In addition, they have been used to explain phenomena in unstable laser resonators [5], electron scattering from hydrogen molecules [6] and quantum phase transitions [7]. EPs can also arise in diffraction from absorbing gratings [8], in crystal optics [9] and in atomic physics [10, 11]. For more examples and information on EPs, see [12, 13] and also chapter 9 in [14]. Finally, we note that EPs also appear in the context of the PT symmetric system as the branch point singularities that determine the boundary of the real spectrum regime. In our case, however, the Hamiltonian does not generally have a PT symmetry and the energies are complex.

The Berry phase can be observed if the parameters are changed slowly enough so that the non-adiabatic effects are negligible. Is it possible to follow the same procedure with an EP and observe that a system that starts with one eigenstate ends the evolution in the other state (i.e. a flip occurred)? This procedure has been suggested for EPs several times in the literature [10, 11, 15, 16]. Yet, to the best of our knowledge, it has never been studied analytically or numerically.

In this work, we show that as the propagation time increases, the non-adiabatic effects refuse to vanish and impose a severe limitation on the ability of the system to make adiabatic flips for both initial eigenstates (for two-level systems). Our findings reflect a fundamental property of systems with dissipation/gain—the adiabatic theorem does not hold in general for all eigenstates. In [17], it has been shown that in the slow evolution limit of a two-level non-Hermitian system, only the least decaying state is guaranteed (under certain conditions) to follow the adiabatic solution, while the other state which decays faster does not necessarily evolve adiabatically. If so, there are reasons to doubt if the flip expected from adiabatic considerations can be observed in practice for both states. More importantly, for [17] to be applicable, it is necessary that the imaginary part of the instantaneous energy of the gain state is always larger than that of the loss state—a condition that is never met when an EP is encircled. Our strategy, then, is to show that in the cases in which the ‘gain’ state *does* evolve adiabatically (we give an example), the other state does not, and so the adiabatic flip cannot be observed for both states. This is related to the fact that, as the propagation time increases, the final state becomes almost insensitive to initial conditions due to non-adiabatic effects. We show that the flip quality of one state is the inverse of the flip quality of the other state in the slow evolution limit. That is, only one flip can be successfully observed (a condition is given).

In section 2, we introduce formalism and notations. Section 3 presents numerical results for the flip quality of the two eigenstates, and the physical mechanism is discussed in section 4. In section 5, we introduce the ‘periodic modes’ in order to better understand the time dynamics of the system and explain our findings. In section 6, we conclude.

2. Formalism

To demonstrate the effect, it is enough to consider symmetric Hamiltonians. Let $H = H(\vec{\lambda}(t))$ be a two-level non-Hermitian symmetric Hamiltonian where $(\lambda_1, \lambda_2) = \vec{\lambda}$ are the coordinates in parameter space. $\Phi_{a,b}(\vec{\lambda})$ are the *instantaneous* eigenstates of the Hamiltonian that satisfy

$$H(\vec{\lambda})\Phi_{a,b}(\vec{\lambda}) = E_{a,b}(\vec{\lambda})\Phi_{a,b}(\vec{\lambda}). \quad (1)$$

The instantaneous energies $E_{a,b}$ are complex numbers. This basis is sometimes called the adiabatic basis since it is the solution of the Schrödinger equation if the non-adiabatic couplings are neglected.

The instantaneous basis can be used to describe the wavefunction evolution in time:

$$\psi = a(t)\Phi_{\mathbf{a}}(\vec{\lambda}(t)) + b(t)\Phi_{\mathbf{b}}(\vec{\lambda}(t)). \quad (2)$$

We normalize the vectors $\Phi = (\Phi_1, \Phi_2)$ for both states ‘ a ’ and ‘ b ’ by the condition $\Phi_1^2 + \Phi_2^2 = 1$. This condition differs from the Hermitian product by dropping complex conjugation, and it represents the C-product for the case of symmetric non-Hermitian Hamiltonians [14]. This normalization is ill-defined at the EP itself; in this work, however, we encircle the EP and never pass through it.

Since it is always possible to remove the trace of H by a simple gauge transformation, H is taken to be traceless to begin with. This means that the mutual decay of both states is removed and only the *relative* loss and gain remain. In the rest of this paper, we speak only about relative gain and loss. At some point in parameter space, there is an EP. In our coordinates, this point is always at $\vec{\lambda} = 0$. At this special point, the Hamiltonian is not diagonalizable and has a Jordan form rather than a diagonal form. We pick a closed curve around this point and let the system evolve in time as the parameters, $\vec{\lambda}$, change in time to complete the loop. At the cycle time T , $\vec{\lambda}(T) = \vec{\lambda}(0)$.

The adiabatic solutions of the Schrödinger equation are $a_{ad}(t) = a(0)e^{-i\int_0^t E_a dt'}$, $b_{ad}(t) = b(0)e^{+i\int_0^t E_a dt'}$, where we have used $E_b = -E_a$ for traceless system. In this paper, ‘ a ’ will always refer to the state with total adiabatic relative gain: $\text{Im}\{\int_0^T E_a dt'\} > 0$. ‘ b ’ will be referred to as the ‘loss’ state.

For closed loops that encircle a single EP, the instantaneous basis has the following property:

$$\Phi_a(\vec{\lambda}(T)) = e^{i\varphi_a} \Phi_a(\vec{\lambda}(0)), \quad \Phi_b(\vec{\lambda}(T)) = e^{i\varphi_b} \Phi_b(\vec{\lambda}(0)). \quad (3)$$

Since $\vec{\lambda}(T) = \vec{\lambda}(0)$, (3) shows that the instantaneous/adiabatic basis is not single valued in parameter space. If there are no EPs inside the loop, the basis is single valued. The coefficients $e^{i\varphi_{a,b}}$ equal ± 1 for symmetric Hamiltonians but these coefficients are not important for this work. By virtue of (2) and (3), we see that a system that starts with $(a(0) \neq 0, b(0) = 0)$ and ends up with $(a(T) \neq 0, b(T) = 0)$ has performed a perfect flip. That is, the flip has already been taken into account in the instantaneous basis. If $a(t) \gg b(t)$ for every $0 \leq t \leq T$, we call it an ‘adiabatic flip’. Throughout this paper, $(a_1(t), b_1(t))$ will describe the exact time evolution of the initial condition $(1, 0)$ (in the instantaneous basis) in which only the gain state is initially populated, and $(a_2(t), b_2(t))$ will correspond to the initial condition $(0, 1)$ where only the loss state is initially populated. We assume that with a slow enough evolution, the gain state evolves adiabatically:

$$a_1(t) \gg b_1(t). \quad (4)$$

Under this condition, the evolution of a_1 is well approximated by

$$a_1(t) \simeq a_{ad}(t) = e^{-i\int E_a dt}. \quad (5)$$

The Berry phase does not appear in these expressions, since the parallel transport condition is automatically satisfied for C-normalized eigenstates of symmetric Hamiltonians. In our numerical studies, we have seen indications that adiabatic behavior for at least one state is not always observed. For some loops (not close to the EP), none of the states evolve adiabatically as far as we can tell within the limits of our best numerical accuracy. Although interesting, these cases are irrelevant for this work, as neither of the states show an adiabatic flip. We define the quantity $R_1 = \left| \frac{b_1(T)}{a_1(T)} \right|$ as the gain state flip error. In the same way, $R_2 = \left| \frac{a_2(T)}{b_2(T)} \right|$ describes the error when the system starts at the loss state. The key point of this paper is that although it is possible to make the flip error arbitrarily small for the state with relative gain by increasing the cycle time, the other state will have a very large flip error. Specifically we show in section 5 that the error product satisfies $R_1 R_2 \rightarrow 1$ as $T \rightarrow \infty$.

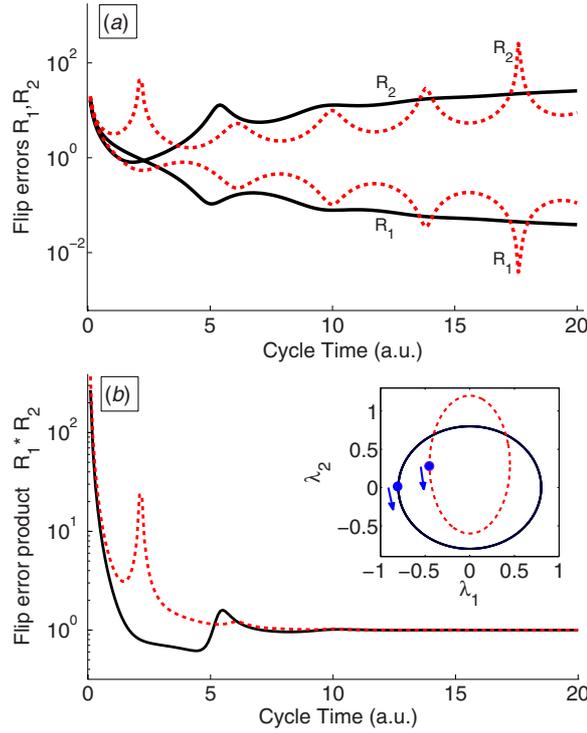


Figure 1. (a) The flip errors R_1 and R_2 (see the text) are plotted as a function of the cycle time, T . Each line style corresponds to a different closed curve in parameter space (inset). (b) For a large T the error product, $R_1 R_2$, converges to unity which imposes a severe limitation of the ability to observe an adiabatic flip for both states.

In our numerical examples, we consider the symmetric Hamiltonian

$$H = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} + \sum_j \lambda_j(t) H_j, \quad (6)$$

where H_j are symmetric coupling matrices that do not commute with the first matrix on the RHS and λ_j is the j th coordinate in parameter space. There is a positive ‘chirality’ [18] EP at $\vec{\lambda} = 0$. The parameter space we are considering is two dimensional: either two real coordinates or one complex coordinate. In the examples that follow, we have used two real parameters and the Pauli matrices σ_x and σ_z for H_1 and H_2 , respectively. For this choice there is another negative chirality EP at $\vec{\lambda} = \{0, -2\}$. The loops in parameter space are of the form $\lambda_1 = \alpha_1 + \beta_1 \cos(\frac{2\pi}{T}t + \varphi)$, $\lambda_2 = \alpha_2 + \beta_2 \sin(\frac{2\pi}{T}t + \varphi)$. $\vec{\alpha}$ is the center of the loop and $\vec{\beta}$ is the radii of the ellipse. φ is the starting point along the curve.

3. Numerical results

Figure 1(a) shows the flip error for two different trajectories in parameter space (inset of (b)) as a function of the cycle time. The first loop (solid black) is a centered circle of radius 0.8 ($\alpha_1 = \alpha_2 = 0, \beta_1 = \beta_2 = 0.8, \varphi = \pi$) and the second loop (dashed-red) is an off-center ellipse ($\alpha_1 = 0, \alpha_2 = 0.3, \beta_1 = 0.9, \beta_2 = 0.45, \varphi = \pi$). For each loop there are two curves corresponding to the two possible initial conditions. As $T \rightarrow 0$ (or $T = 0$), the EP is encircled

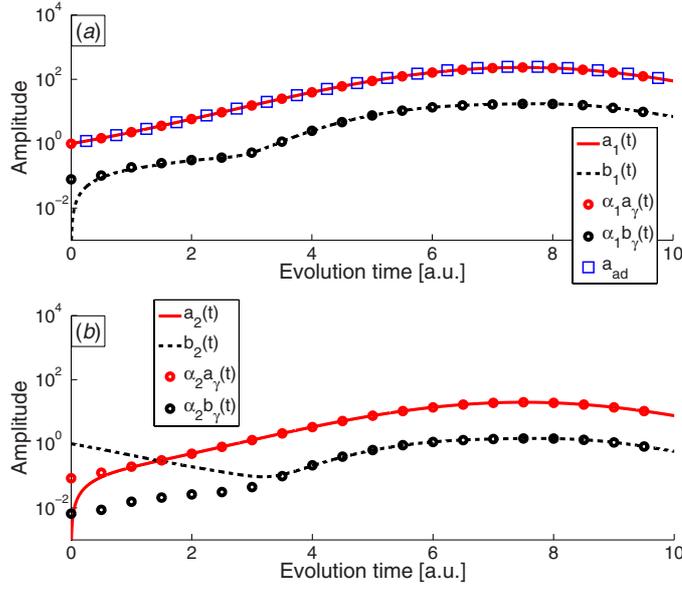


Figure 2. The coefficients a and b in the instantaneous basis as functions of time. The cycle period is $T = 10$. (a) When only the gain eigenstate is initially populated, the population in the instantaneous basis (' a ' and ' b ') evolves adiabatically ($a_1(t) \gg b_1(t)$). The blue squares show how well the adiabatic approximation (5) describes a_1 in this case. (b) If the loss state is initially populated, then after a short adiabatic decay the gain state becomes populated through non-adiabatic coupling and rapidly takes over the state evolution. The circles show how well the periodic mode ($a_\gamma(t)$, $b_\gamma(t)$) describes the dynamics of both possible initial conditions starting from a certain point in time.

so fast that the states do not change at all. If the states do not change, there is no flip and the error goes to infinity for both states. Note that in the adiabatic basis there are sharp changes in the coefficient and in the basis functions as $T \rightarrow 0$ since (3) holds for any T . For larger T values, it is apparent from the graphs that the flip error is nowhere small for both states simultaneously. This fact is seen more clearly in figure 1(b) where the flip error product, $R_1 R_2$, is plotted for the same loops. Hermitian reasoning suggests that the time scale expected for adiabatic behavior can be estimated by $1/\min(|\Delta|)$ (the inverse of the minimal level spacing). For the circular loop, this time scale is roughly 0.51 au and for the oval loop it is about 0.57 au. Yet figure 1 shows that the system is not adiabatic even when the propagation is much longer than these time scales ($T = 20$ is at least 35 times longer).

4. The physical mechanism

In order to gain some insight into the physical mechanism responsible for this effect, we have plotted in figure 2 the evolution of the gain and loss states in the instantaneous basis. In the plot, we used the circular curve from the previous section (inset of figure 1(b)) with a cycle time of $T = 10$. Figure 2(a) shows the evolution in time when only the gain state is initially populated. Throughout the evolution, the occupancy, a_1 (solid-red), of the instantaneous eigenstate Φ_a remains larger than the occupancy, b_1 (dashed-black), of the instantaneous eigenstate Φ_b . The blue squares show clearly how well the adiabatic approximation (5) describes the coefficient $a_1(t)$. This shows that assumptions (4) and (5) are fairly justified in this case (and they

become more justified if T is increased). While the amplitude $a_1(t)$ follows the adiabatic expression, $a_{ad}(t)$, $b_1(t)$ does not follow at all its adiabatic estimate $b_{ad}(t)$. The considerably large amplitudes of $a_1(t)$ overcome the smallness of the non-adiabatic coupling and excite the other state Φ_b . Thus, b_1 does not remain zero as expected from adiabatic reasoning. Still, the total evolution can be considered to be adiabatic in the sense that almost all the population remains in the same instantaneous eigenstate. *If only the other ('loss') eigenstate is initially populated, the situation is completely different* (see figure 2(b)). In the beginning, the state decays practically adiabatically. Yet during this stage, Φ_a become just slightly populated through the small non-adiabatic coupling. Once the gain state is populated, it rapidly outgrows the population of Φ_b and the system becomes completely dominated by the Φ_a eigenstate. If so, up to a multiplicative constant, the final state is the same for both initial conditions. Simple algebra shows that this proportionality immediately implies $R_1 R_2 = 1$. While this picture offers some rough intuitive understanding of the dynamics, in the next section we introduce more appropriate solutions (circles in figure 2), in order to better quantify this effect.

5. The periodic modes, the preferred final state and the flip quality

Using (2), the Schrödinger equation can be written as a set of two coupled equations for the coefficients a and b only. Let U_t^{ab} be the evolution operator of the coefficients: $(a(t), b(t)) = U_t^{ab}(a(0), b(0))$. The one cycle evolution operator, U_T^{ab} , has eigenstates that describe periodic evolution of the coefficients up to a constant factor:

$$\begin{aligned} \begin{pmatrix} a_\gamma(T) \\ b_\gamma(T) \end{pmatrix} &= e^{-iE_\gamma T} \begin{pmatrix} a_\gamma(0) \\ b_\gamma(0) \end{pmatrix} \\ \begin{pmatrix} a_\delta(T) \\ b_\delta(T) \end{pmatrix} &= e^{-iE_\delta T} \begin{pmatrix} a_\delta(0) \\ b_\delta(0) \end{pmatrix}. \end{aligned} \quad (7)$$

It is straightforward to show that for traceless symmetric Hamiltonians, $E_\gamma = -E_\delta$. In addition, we assign γ to the solution with the gain ($\text{Im}\{E_\gamma\} > 0$). Despite the apparent similarity to Floquet solutions, these solutions are not the Floquet states of the system. The coefficients are periodic in T but the basis is not periodic in T (see (3)). Similarly $E_{\gamma,\delta}$ are not the Floquet quasi-energies of the original Hamiltonian. This distinction between periodic modes of the coefficients and the Floquet states is unique to loops that encircle an EP and to the best of our knowledge it is presented here for the first time. If there is no EP inside the loops, then $E_{\gamma,\delta}$ become equal to the Floquet quasi-energies. To prevent confusion we will call these solutions 'periodic modes' and not states. Here after the indices γ and δ will refer to the periodic modes of the system. Unlike the Floquet states in this case, these modes are directly related to adiabatic evolution. This fact makes the periodic modes solutions quite useful as we shall demonstrate shortly.

A general state at $t = 0$ can be spanned using the periodic modes:

$$\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \alpha \begin{pmatrix} a_\gamma(0) \\ b_\gamma(0) \end{pmatrix} + \beta \begin{pmatrix} a_\delta(0) \\ b_\delta(0) \end{pmatrix}. \quad (8)$$

At the end of the evolution, the final state is

$$\begin{pmatrix} a(T) \\ b(T) \end{pmatrix} = e^{-iE_\gamma T} \alpha \begin{pmatrix} a_\gamma(0) \\ b_\gamma(0) \end{pmatrix} + e^{-iE_\delta T} \beta \begin{pmatrix} a_\delta(0) \\ b_\delta(0) \end{pmatrix}. \quad (9)$$

Since the mode γ is the one with the gain and the mode δ is the one with the loss, we are going to argue that the second term becomes negligible and the final state is $(a(T), b(T)) \simeq e^{-iE_\gamma T} \alpha (a_\gamma(0), b_\gamma(0))$. This means that the system has a preferred final state which is insensitive to the initial conditions (except the factor α which *does* depend on

the initial conditions). In particular, both instantaneous initial conditions lead to this same final state $(a_\gamma(0), b_\gamma(0))$. From this, once again, the result $R_1 R_2 \rightarrow 1$ immediately follows. The circles in figures 2(a) and (b) show the occupancies of $\alpha_{1,2}(a_\gamma(t), b_\gamma(t))$ for the two possible initial conditions. For the gain state (a), the periodic gain mode provides a good description throughout the propagation. For the loss state (b), the periodic gain mode also properly describes the evolution once the other periodic mode has decayed.

Since the coefficients α and β are not fixed yet, we can choose the norm of the vectors γ and δ as we wish. In order to simplify the comparison of the two terms, we set the Hermitian norm of the two vectors to be the same, i.e. $|a_\gamma(0)|^2 + |b_\gamma(0)|^2 = |a_\delta(0)|^2 + |b_\delta(0)|^2$. Now that the two vectors are comparable in magnitude, the condition for the second term in (9) to become negligible is given by $|e^{-iE_\gamma T} \alpha| \gg |e^{-iE_\delta T} \beta|$ which can be written as

$$|\alpha/\beta| \gg e^{-2\text{Im}\{E_\gamma\}T} \quad (10)$$

($\text{Im}\{E_\gamma\} > 0$). The exponent in (10) takes a more useful form when our adiabatic gain state assumptions (4) and (5) are valid. Then the evolution of the gain state is described by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{t=0} \rightarrow \begin{pmatrix} a_1(T) \\ b_1(T) \end{pmatrix}_{t=T} \simeq a_1(T) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \simeq e^{-i \int_0^T E_a dt} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (11)$$

This relation shows that in this case the gain state takes the periodic form that appears in (7) as $T \rightarrow \infty$. From this we conclude that for slow evolution, the *gain state* must be identified with the periodic *gain mode* and therefore $|e^{-iE_\gamma T}| \cong |e^{-i \int_0^T E_a dt}|$. Finally for slow evolution, condition (10) for the validity of the result $R_1 R_2 \rightarrow 1$ can be written as

$$|\alpha/\beta| \gg e^{-2\text{Im} \int_0^T E_a dt}, \quad (12)$$

where $\text{Im} \int_0^T E_a dt > 0$, so the RHS is a small number. If the angle along the loop in parameter space is swept at a constant rate, $2\pi/T$, we can write $\int_0^T E_a dt = \frac{T}{2\pi} \int_0^{2\pi} E_a d\theta$. Thus, the ratio of the weights α and β has to be larger than a number which is exponentially small in T . The domain of initial conditions that do not satisfy this condition is reduced as T is increased. To be more specific, in this work we consider the instantaneous eigenstate as initial conditions. So we are interested to know whether or not these initial conditions populate both periodic modes. The Hamiltonian at $t = 0$ which determines the instantaneous eigenstates is symmetric and different from the identity matrix. The propagator U_T^{ab} is typically asymmetric. Thus, $H(\lambda(t=0))$ and U_T^{ab} do not commute in general and their eigenvectors are not the same. From this we see that typically the instantaneous eigenstate will populate both periodic modes. Let us give some numbers to demonstrate this. In the example given in figure 2, the ratio of coefficients is $|\alpha_1/\beta_1| \simeq 12.736$ for the gain eigenstate ‘1’ and $|\alpha_2/\beta_2| \simeq 0.0846$ for the loss state ‘2’. The exponential term in (12) is found to be $e^{-2|\text{Im} \int_0^T E_a dt|} \simeq 0.000126$. As both initial states satisfy condition (12), we indeed obtain that $R_1 R_2 \simeq 1$ in this case. For $T = 20$, the ratio of coefficients for the loss state decreases to $|\alpha_2/\beta_2| \simeq 0.0388$ but the exponential term drops down to $e^{-2|\text{Im} \int_0^T E_a dt|} \simeq 1.44 \times 10^{-8}$, so the condition is even more easily satisfied. α_1/β_1 is not important since $|\alpha_1/\beta_1| \gg |\alpha_2/\beta_2|$.

It is interesting to note that if we propagate the exact loss mode, we observe that it is not adiabatic at all. In contrast, the gain mode evolves adiabatically as expected (figure 2(a)). Therefore, not only is the flip associated with the loss periodic mode very hard to observe in slow evolution, but also it is generally not an adiabatic flip.

In the light of these results, we find that schemes for observing the adiabatic flip such as those proposed in the literature in the past [11, 15, 10, 16] may be successful only for one of the states. If the laser pulse used to move in parameter space is too short, it is likely that the non-adiabatic effects will be strong for both states (as can be seen in figure 1 for small T).

Longer pulses, on the other hand, will give rise to a very large flip error for one of the states. Either way, the result presented here clearly indicates that non-adiabatic terms should not be ignored when circling an EP.

6. Conclusion

We have shown that there are fundamental limits on the observability of the flip effect associated with exceptional points. For slow evolution, the inherent relative gain in the system magnifies even the most minute non-adiabatic effect. Consequently, undesired states get populated, and the perfect flip effect expected from the instantaneous eigenstate basis is ruined. The flip errors for the two different cases have been shown to be connected and exactly inversely proportional to each other in the limit of slow evolution. To explain this, we have introduced the notion of ‘periodic modes’. These modes differ from the Floquet state whenever an EP is encircled in parameter space. Finally, we comment that this work has motivated several other studies of time evolution when an EP is cycled. These studies complement this work and reveal other new and interesting aspects of this problem. In [19], an analytically solvable model has been introduced, and another intriguing manifestation of the asymmetry associated with EP cycling is explained in terms of the Stokes phenomenon. A physical realization of this model in crystal optics is explored in [20]. Moiseyev and Gilary have studied adiabatic flips in the context of atom/molecules in chirped laser pulses [21].

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