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Instability of a general rotating system with small axial asymmetry and damping ☆

Alexei A. Mailybaev [a,b], Alexander P. Seyranian [b,*]

[a] Instituto Nacional de Matemática Pura e Aplicada – IMPA, Rio de Janeiro, Brazil
[b] Institute of Mechanics, Lomonosov Moscow State University, Russia

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ABSTRACT

The paper presents new results on instability of general rotating systems with small axial asymmetry and damping. Rotating systems with arbitrary finite degrees of freedom are considered. Instability conditions are derived for nonresonant and resonant rotation speeds characterized by simple and double eigenvalues. As an application, the stability problem of a rotating elastic shaft with small internal and external damping forces and asymmetric imperfections is studied. The numerical example shows validity and accuracy of the derived analytical formulae for the instability regions.

1. Introduction

The vibrations and stability of rotating systems have been the important problem for many years due to increased speed of rotation, stability and safe operation demands. For the dynamist it is extremely important to understand the cause and nature of instabilities associated with internal and external damping as well as imperfections breaking axial symmetry of rotating systems. There is the vast literature on the topic of stability and vibrations of rotating systems. Among essential sources on this subject we should mention the books by Bolotin [1] and Ziegler [2]. In these works the classical results on stability of rotating shafts with symmetric and unsymmetric cross-sections with internal and external damping are presented. It is shown that the stability is sensitive with respect to small factors like internal and external damping coefficients or imperfections related to mass and stiffness distributions. In the book by Huseyin [3] the divergence and flutter instability regions of shafts were examined depending on magnitude of axial force and rotation speed. Among another early source of rotor dynamics we cite [4] containing several interesting papers devoted to practical problems. As an important practical application we indicate the problem to avoid squeal of brakes, see [5–7] and references therein. Some basic knowledge of rotor dynamics with discrete and continuous models is given in the books [8–10].

It is useful to create a formalism for stability analysis of rotating systems, which allows understanding and describing the known instability phenomena on a general basis. Natural approach to this problem, which we follow in this paper, includes describing a general axially symmetric rotating system and then studying its stability under general small perturbation due to damping, geometric imperfections and external forces.

Description of equations governing small vibrations of a general axially symmetric rotating system with finite degrees of freedom is the first problem, which we solve in this paper. It is shown that the structure of mass, stiffness and

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* Corresponding author.
E-mail address: seyran@imec.msu.ru (A.P. Seyranian).

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gyroscopic matrices is essentially restricted by the axial symmetry. In particular, the form of system eigenvectors (or their linear combinations) is prescribed by the axial symmetry, irrespective of a physical nature of the system.

Next we provide a general description of the instability effects in undamped axially symmetric rotating systems. It is shown that instabilities may occur only due to interaction of modes from the same class of rotational symmetry. Interactions of other modes do not lead to instability. This fact is seen on well-known Campbell diagrams, where crossings of eigenvalues do not lead to instability.

As it is stated in [10] important objectives of rotordynamics analysis are in predicting critical speeds of instability and determining design modifications, which change the critical speeds. The central result of our paper is the derivation of general stability conditions for axially symmetric systems under small perturbations, which describe effects of dissipation and geometric imperfections. These conditions are obtained both for nonresonant and resonant values of the rotation speed. The resonant case is related to instabilities distinguished for static (divergence) and dynamic (flutter) forms, which are associated to the double eigenvalues $\lambda=0$ and $\lambda=i\omega\neq 0$, respectively. With the use of perturbation technique [11], the instability regions for all these cases are found in terms of small damping coefficients, small symmetry breaking parameters and deviation of the rotation speed from the critical one as well as eigenvalues and eigenvectors of the undisturbed system.

As the first example, we consider the stability of a rigid disk mounted on a slightly asymmetric shaft. This is the classical problem with two degrees of freedom. We derive the instability conditions and compare them with the known results. As the second example, we consider a simply supported rotating elastic shaft with nonuniform cross-section. We assume small asymmetry of the cross section and small internal and external damping forces described by the classical Kelvin–Voigt and viscous damping models. New formulae for the instability regions are derived in terms of small asymmetry and damping coefficients and deviation of rotation speed from the critical one. It is shown that among possible summed and difference types of resonant rotation speeds only the summed type leads to instability. Instability regions are computed numerically for specific parameters of the shaft demonstrating good agreement with the analytical results.

Attempts to describe a general axially symmetric rotating system disturbed by small stiffness and damping modifications and nonconservative positional forces were made earlier in [12–14]. In these papers, the gyroscopic matrix corresponding to a rotating string model, taken from [15], is treated as the gyroscopic matrix of a general system. We show in Section 3 that this choice of the gyroscopic matrix is incorrect. Besides, considering the rotating shaft with two degrees of freedom in [12,14], the conclusion was made that the system is subjected to instability at zero rotation speed, interpreted as the sub-critical flutter. This statement contradicts to classical results, see Section 8. It is also shown in Section 9 that for a rotating elastic shaft the sub-critical flutter instability is ruled out.

The present paper is organized as follows. In Sections 2 and 3 axially symmetric rotating systems with arbitrary finite degrees of freedom are considered. In Section 4 we introduce small damping and imperfections breaking axial symmetry of rotating systems. In Sections 5–7 the nonresonant and resonant divergence and flutter instability conditions are derived in terms of small perturbation matrices and eigenvectors of the undisturbed system. Section 8 treats instability of a rigid disk mounted on a slightly asymmetric shaft. In Section 9 the stability problem of a rotating elastic shaft with small axial asymmetry and damping is analyzed including comparison with the numerical results. The paper ends up with the conclusion in Section 10.

2. Axial rotation

We consider an axially symmetric mechanical system described by a vector of generalized coordinates $\mathbf{q}\in\mathbb{R}^N$. Rotation by an angle φ about the symmetry axis is characterized by the linear transformation

$$\mathbf{q}\to\mathbf{R}(\varphi)\mathbf{q} \tag{1}$$

with the rotation matrix $\mathbf{R}(\varphi)$. It is convenient to represent this matrix in the exponential form

$$\mathbf{R}(\varphi)=\exp(\varphi\mathbf{A}), \tag{2}$$

which satisfies the relation $\mathbf{R}(\varphi_1+\varphi_2)=\mathbf{R}(\varphi_1)\mathbf{R}(\varphi_2)$, and $\mathbf{R}(0)=\mathbf{I}$ is the identity matrix. The $N\times N$ matrix $\mathbf{A}$ describes the infinitesimal rotation

$$\mathbf{q}\to\mathbf{q}+\delta\varphi\mathbf{A}\mathbf{q}+o(\delta\varphi). \tag{3}$$

The generalized coordinates can always be chosen (and it is usually a natural choice) such that the rotation matrix $\mathbf{R}$ is orthogonal, $\mathbf{R}^T=\mathbf{R}^{-1}$. This implies

$$\exp(\varphi\mathbf{A}^T)=\exp(-\varphi\mathbf{A}) \tag{4}$$

and, hence, $\mathbf{A}^T=-\mathbf{A}$. Therefore, the matrix $\mathbf{A}$ is skew-symmetric.

Since rotation by the angle $\varphi=2\pi$ brings the system to the initial position, we obtain $\mathbf{R}(2\pi)=\exp(2\pi\mathbf{A})=\mathbf{I}$. Eigenvalues of the matrix $\exp(2\pi\mathbf{A})$ have the form $\exp(2\pi\mu)$, where μ is an eigenvalue of $\mathbf{A}$. Since $\exp(2\pi\mu)=1$, the eigenvalues of $\mathbf{A}$ are integer multiples of the imaginary unit. Thus, the eigenvalues μ are

$$\pm\,\mathrm{i}a_1,\ldots,\pm\,\mathrm{i}a_n,\ \underbrace{0,\ldots,0}_{m}, \tag{5}$$

where $a_1,\ldots,a_n$ are positive integer numbers, not necessarily different. The system dimension is, thus, $N=2n+m$.

The eigenvectors of the matrix $\mathbf{A}$ are determined by the equation

$$\mathbf{A}\mathbf{u}=\mu\mathbf{u}. \tag{6}$$

The eigenvectors $\mathbf{u}$ are complex conjugate for $\mu=\pm\,\mathrm{i}a_j$ and real for $\mu=0$. Since $\mathrm{i}\mathbf{A}$ is a Hermitian matrix, there are N eigenvectors, which can be chosen to be orthogonal. We denote the three groups of eigenvectors for $\mu=\mathrm{i}a_1,\ldots,\mathrm{i}a_n$, $\mu=-\mathrm{i}a_1,\ldots,-\mathrm{i}a_n$ and $\mu=0$, respectively, by

$$\mathbf{u}_1,\ldots,\mathbf{u}_n\in\mathbb{C}^N,\quad \overline{\mathbf{u}}_1,\ldots,\overline{\mathbf{u}}_n\in\mathbb{C}^N,\quad \tilde{\mathbf{u}}_1,\ldots,\tilde{\mathbf{u}}_m\in\mathbb{R}^N, \tag{7}$$

where $\overline{\mathbf{u}}_j$ is a complex conjugate vector.

The eigenvector $\mathbf{u}$ for the eigenvalue $\mu=0$ is determined as an arbitrary linear combination of $\tilde{\mathbf{u}}_1,\ldots,\tilde{\mathbf{u}}_m$. Since $\mathbf{A}\mathbf{u}=0$, we have $\mathbf{R}(\varphi)\mathbf{u}=\exp(\varphi\mathbf{A})\mathbf{u}=\mathbf{u}$ for any φ. Similarly, the eigenvectors $\mathbf{u}$ corresponding to $\mu=\mathrm{i}a$ change under rotation as $\mathbf{R}(\varphi)\mathbf{u}=\exp(\varphi\mathbf{A})\mathbf{u}=e^{\mathrm{i}a\varphi}\mathbf{u}$. We see that all eigenvectors in (7) are preserved under rotation up to a scalar factor. This factor characterizes the rotation symmetry of each eigenvector.

For each eigenvalue μ in (5), we associate a set of corresponding eigenvectors (eigenspace) of the matrix $\mathbf{A}$ and call it the *symmetry class*. The vectors of the same symmetry are multiplied by the same factor $e^{\mathrm{i}a\varphi}$ under rotation.

3. Rotating axially symmetric system

Let us consider an axially symmetric system rotating with a constant angular velocity Ω about the symmetry axis, $\varphi=\Omega t$. Let $\mathbf{q}$ be the vector of generalized coordinates describing deviation of the system from the axially symmetric equilibrium corresponding to $\mathbf{q}=0$. Small oscillations of the system in the rotating frame are described by the equation

$$\mathbf{M}(\Omega)\ddot{\mathbf{q}}+\mathbf{G}(\Omega)\dot{\mathbf{q}}+\mathbf{P}(\Omega)\mathbf{q}=0, \tag{8}$$

where $\mathbf{M}$ is a symmetric positive definite mass matrix, $\mathbf{P}$ is a symmetric stiffness matrix, and $\mathbf{G}$ is a skew-symmetric gyroscopic matrix. Dependence of the matrices on Ω is attributed to centrifugal and Coriolis forces. Examples of such systems are rotating rods, disks, etc.

Vibration modes $\mathbf{q}=\mathbf{u}e^{\lambda t}$ with the eigenvalues λ and eigenvectors $\mathbf{u}$ are determined by the equation

$$(\lambda^2\mathbf{M}+\lambda\mathbf{G}+\mathbf{P})\mathbf{u}=0 \tag{9}$$

following from (8). Due to symmetry of the matrices $\mathbf{M}$ and $\mathbf{P}$, and skew-symmetry of $\mathbf{G}$, we have

$$(\lambda^2\mathbf{M}+\lambda\mathbf{G}+\mathbf{P})^T=\lambda^2\mathbf{M}-\lambda\mathbf{G}+\mathbf{P}. \tag{10}$$

Hence, $-\lambda$ is also an eigenvalue. Since the system matrices are real, the complex conjugate $\overline{\lambda}$ and $-\overline{\lambda}$ are eigenvalues too. This proves the well-known fact that the eigenvalues of a gyroscopic system are symmetric with respect to real and imaginary axes. The system is stable if and only if all eigenvalues λ are purely imaginary and simple or semi-simple, see, e.g., [11]. Real eigenvalues $\lambda=\pm\sigma$ or quadruples of complex eigenvalues $\lambda=\pm\sigma\pm\mathrm{i}\omega$ with nonzero σ determine exponentially growing modes of the unstable system. The change from stability to instability occurs when eigenvalues cross on the imaginary axis. However, an eigenvalue crossing not necessarily leads to instability.

As we showed above, the eigenvalues (5) of the matrix $\mathbf{A}$ are integer numbers and, thus, cannot change with Ω. It is usually easy and practical to choose the coordinates $\mathbf{q}$ such that the rotation matrix $\mathbf{A}$ in (2) does not depend on Ω. We will assume this choice from now on.

Matrices of the axially symmetric system must be invariant under rotation (1), i.e.,

$$\mathbf{M}=\mathbf{R}^{-1}\mathbf{M}\mathbf{R},\quad \mathbf{P}=\mathbf{R}^{-1}\mathbf{P}\mathbf{R},\quad \mathbf{G}=\mathbf{R}^{-1}\mathbf{G}\mathbf{R}, \tag{11}$$

for any angle of rotation φ. These conditions can be written as the commutation rules, $[\mathbf{M},\mathbf{R}]=[\mathbf{P},\mathbf{R}]=[\mathbf{G},\mathbf{R}]=0$, where we used the notation $[\mathbf{M},\mathbf{R}]=\mathbf{M}\mathbf{R}-\mathbf{R}\mathbf{M}$. Using the Taylor expansion

$$\mathbf{R}=\exp(\varphi\mathbf{A})=\mathbf{I}+\varphi\mathbf{A}+o(\varphi), \tag{12}$$

one obtains

$$[\mathbf{M},\mathbf{A}]=[\mathbf{P},\mathbf{A}]=[\mathbf{G},\mathbf{A}]=0. \tag{13}$$

Due to commutation rules (13) and Eq. (6), we have

$$0=(\mathbf{A}\mathbf{M}-\mathbf{M}\mathbf{A})\mathbf{u}=(\mathbf{A}-\mu\mathbf{I})\mathbf{M}\mathbf{u}. \tag{14}$$

This means that $\mathbf{M}\mathbf{u}$ is an eigenvector of the matrix $\mathbf{A}$ for the same eigenvalue μ, i.e., it belongs to the same symmetry class as $\mathbf{u}$. Similarly, one can show that $\mathbf{P}\mathbf{u}$ and $\mathbf{G}\mathbf{u}$ belong to the same symmetry class. In particular, if the symmetry class contains a single vector $\mathbf{u}$, then the vectors $\mathbf{u}$, $\mathbf{Mu}$, $\mathbf{Pu}$, $\mathbf{Gu}$ are equal up to a scalar factor. This means that $\mathbf{u}$ is the eigenvector for each of the four matrices $\mathbf{A}$, $\mathbf{M}$, $\mathbf{P}$, $\mathbf{G}$, see [16] for the theory of commuting matrices.

First, let us consider a particular case when the integers $a_1,\ldots,a_n$ in (5) are all distinct, and there are no zeros ($m=0$). Then the eigenvectors

$$\mathbf{u}_1,\ldots,\mathbf{u}_n, \quad \overline{\mathbf{u}}_1,\ldots,\overline{\mathbf{u}}_n \tag{15}$$

of the matrix $\mathbf{A}$ are determined uniquely up to a scalar factor. As we mentioned in the previous paragraph, the vectors (15) are also the eigenvectors of the matrices $\mathbf{M}$, $\mathbf{P}$ and $\mathbf{G}$. Hence, they are eigenvectors of the problem (9). In the orthogonal complex basis (15), system (9) decouples to $2n$ independent equations. For any eigenvector $\mathbf{u}$ from (15), the corresponding equation has the form

$$m\lambda^2+2\mathrm{i}g\lambda+p=0, \quad m=\mathbf{u}^*\mathbf{M}\mathbf{u}, \quad p=\mathbf{u}^*\mathbf{P}\mathbf{u}, \quad 2\mathrm{i}g=\mathbf{u}^*\mathbf{G}\mathbf{u}, \tag{16}$$

which is obtained from (9) after multiplication by $\mathbf{u}^*$ on the left. Here the asterisk means the conjugate transpose $\mathbf{u}^*=\overline{\mathbf{u}}^T$. Since the matrices $\mathbf{M}$ and $\mathbf{P}$ are symmetric and $\mathbf{G}$ is skew-symmetric, m, p, and g are real numbers. The eigenvalues are found as

$$\lambda=(-\mathrm{i}g\pm\sqrt{-g^2-mp})/m. \tag{17}$$

The system is unstable if $mp+g^2<0$, when one of the eigenvalues has a positive real part. On the stability boundary, one has $mp+g^2=0$ and the two eigenvalues (17) coincide. We see that instability of the system is a result of crossing of two eigenvalues λ having the same eigenvector $\mathbf{u}$. Note that the double eigenvalue with a single eigenvector means origination of a Jordan block [11]. Crossing of eigenvalues with different eigenvectors cannot lead to instability, since the corresponding systems (16) are uncoupled.

In the general case, when (5) contains multiple or zero eigenvalues, the eigenvalue crossings are determined by symmetry classes. As we showed in (14) with similar equations for $\mathbf{P}$ and $\mathbf{G}$, each symmetry class is an invariant subspace for $\mathbf{M}$, $\mathbf{P}$ and $\mathbf{G}$. Hence, (9) reduces to a set of uncoupled subsystems in the basis (7), with one subsystem for each symmetry class. The system can lose stability when eigenvalues corresponding to the same subsystem cross and leave the imaginary axis. Crossings of eigenvalues from different subsystems (and, thus, different symmetry classes) cannot lead to instability, since these systems are uncoupled.

Recall that the rotation by an angle φ leads to multiplication by $e^{\mathrm{i}a\varphi}$ for the eigenvectors corresponding to $\mu=\mathrm{i}a$. Hence, vibrational modes corresponding to the symmetry class of $\mu=0$ are axially symmetric, while modes corresponding to the symmetry classes of $\mu=\mathrm{i}a_j$ and $\mu=-\mathrm{i}a_j$ are symmetric under rotation by an angle $\varphi=2\pi/a_j$.

We conclude that instability of the axially symmetric rotating system is determined by crossings of eigenvalues λ corresponding to the same symmetry class. Crossings of eigenvalues corresponding to different symmetry classes do not lead to instability.

Example. Consider small vibrations of an elastic axially symmetric rod rotating with angular velocity Ω. Small vibrations in two lateral directions are described by the vector $\mathbf{w}(x)=(w_y,w_z)^T$. Axial rotation is characterized by the operator

$$\mathbf{w}\to\mathbf{R}(\varphi)\mathbf{w}, \quad \mathbf{R}(\varphi)=\exp(\varphi\mathbf{J}), \quad \mathbf{J}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{18}$$

It is easy to check that

$$\mathbf{R}(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}. \tag{19}$$

The matrix $\mathbf{J}$ has two eigenvalues $\mu=\pm\mathrm{i}$ determining two symmetry classes.

The dimensionless equation of motion in the rotating frame is

$$\ddot{\mathbf{w}}+2\Omega\mathbf{J}\dot{\mathbf{w}}+\pi^{-4}\mathbf{w}''''-\Omega^2\mathbf{w}=0. \tag{20}$$

The primes and dots denote derivatives with respect to x and t, which are nondimensionalized using the rod length l and natural frequency $\omega_0=\pi^2\sqrt{EI/(ml^4)}$. We will consider a simply supported rod with the boundary conditions

$$x=0: \ \mathbf{w}=\mathbf{w}''=0, \quad x=1: \ \mathbf{w}=\mathbf{w}''=0. \tag{21}$$

Normal modes $\mathbf{w}=\mathbf{u}_j e^{\mathrm{i}\omega_j t}$ of the nonrotating rod ($\Omega=0$) have dimensionless frequencies

$$\omega_j=j^2, \quad j=1,2,\ldots . \tag{22}$$

We determine the eigenvectors

$$\mathbf{u}_j=\sin(j\pi x)\begin{pmatrix}\mathrm{i}\\ 1\end{pmatrix}, \quad j=1,2,\ldots \tag{23}$$

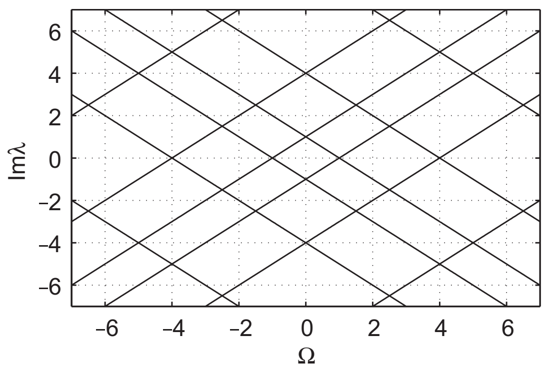

Fig. 1. Campbell diagram for an elastic axially symmetric rod rotating with speed Ω.

corresponding to the symmetry class of $\mu = \mathrm{i}$ (they are eigenvectors of $\mathbf{J}$ for the eigenvalue $\mu = \mathrm{i}$). The complex conjugate vectors $\overline{\mathbf{u}}_j$ correspond to the symmetry class of $\mu = -\mathrm{i}$. Under axial rotation, $\mathbf{u}_j$ is multiplied by $\mathrm{e}^{\mathrm{i}\varphi}$, while $\overline{\mathbf{u}}_j$ is multiplied by $\mathrm{e}^{-\mathrm{i}\varphi}$.

Let us consider solution of the form $\mathbf{w} = \mathbf{u}_j \mathrm{e}^{\lambda t}$ corresponding to the symmetry class of $\mu = \mathrm{i}$. Eqs. (20), (22) and (23) yield

$$\lambda^2 + 2\mathrm{i}\Omega\lambda + (\omega_j^2 - \Omega^2) = 0. \tag{24}$$

This equation determines the eigenvalues

$$\lambda = \mathrm{i}(\pm\omega_j - \Omega), \quad j = 1,2,\ldots\ . \tag{25}$$

Vibrational modes corresponding to the symmetry class of $\mu = -\mathrm{i}$ are $\mathbf{w} = \overline{\mathbf{u}}_j \mathrm{e}^{\overline{\lambda} t}$, where

$$\overline{\lambda} = \mathrm{i}(\mp\omega_j + \Omega), \quad j = 1,2,\ldots\ . \tag{26}$$

The eigenvalues from different symmetry classes (25) and (26) cross at

$$\Omega = \frac{\pm\omega_a \pm \omega_b}{2}, \quad \lambda = \mathrm{i}(\pm\omega_a - \Omega) = \mathrm{i}(\mp\omega_b + \Omega) \tag{27}$$

for every pair of eigenfrequencies ω_a and ω_b, see Fig. 1. As expected, these crossings do not lead to instability. Within each symmetry class (25) or (26), the eigenvalues do not cross because they have the same linear dependence on Ω. This yields the well-known result that the axially symmetric rod is stable for all Ω.

Let us represent the solution in the form of the Fourier series

$$\begin{pmatrix} w_y \\ w_z \end{pmatrix} = \sum_{j=1}^{\infty} \sin(j\pi x) \begin{pmatrix} q_{2j-1} \\ q_{2j} \end{pmatrix}, \tag{28}$$

which satisfies boundary conditions (21). The vector of generalized coordinates is given by $\mathbf{q} = (q_1, q_2, q_3, \ldots)^T$. We substitute (28) into (20), multiply the equations by $2\sin(k\pi x)$ and integrate with respect to x from 0 to 1. This yields the system in the matrix form (8) with the following block diagonal matrices:

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{G}(\Omega) = 2\Omega\mathbf{A}, \quad \mathbf{P}(\Omega) = \mathbf{P}_0 - \Omega^2\mathbf{I}, \tag{29}$$

$$\mathbf{P}_0 = \mathrm{diag}(\omega_1^2\mathbf{I}_2, \omega_2^2\mathbf{I}_2, \omega_3^2\mathbf{I}_2, \ldots), \quad \mathbf{A} = \mathrm{diag}(\mathbf{J}, \mathbf{J}, \mathbf{J}, \ldots), \tag{30}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix. Note that the matrix $\mathbf{G} = 2\Omega\,\mathrm{diag}(\mathbf{J}, 2\mathbf{J}, \ldots, n\mathbf{J})$ was suggested in [12–14] as a gyroscopic matrix of a general axially symmetric rotating system. Results of this section show that such a choice corresponds to a particular case, see (29) and (30) as the counterexample.

Fig. 1, known as the Campbell diagram, shows eigenvalue behavior typical for many axially symmetric systems, which are stable for all rotation speeds. However, the instability may occur in ideal axially symmetric systems due to crossings of eigenvalues from the same symmetry class. For example, see [5, Fig. 6], where in-plane vibrations of a rotating disk become unstable due to centrifugal forces.

4. Rotating system with small asymmetry and damping

We assume that the axial symmetry of the system is broken due to geometric imperfections and perturbations of potential and gyroscopic forces. This can be modeled by considering (8) with the perturbed matrices $\mathbf{M} + \mathbf{M}_1$, $\mathbf{G} + \mathbf{G}_1$, and $\mathbf{P} + \mathbf{P}_1$, where the matrices $\mathbf{M}_1(\Omega)$, $\mathbf{G}_1(\Omega)$, and $\mathbf{P}_1(\Omega)$ represent small perturbations and do not commute with $\mathbf{A}$. We also consider small dissipation effects described by the additional term of the form $\mathbf{D}\dot{\mathbf{q}} + \mathbf{N}\mathbf{q}$ in (8), where $\mathbf{D}(\Omega)$ is a symmetric matrix. We assume that the matrix $\mathbf{N}(\Omega)$ is skew-symmetric, which is often the case (in general, its symmetric part can be included into $\mathbf{P}_1$). All the perturbation matrices $\mathbf{M}_1$, $\mathbf{G}_1$, $\mathbf{P}_1$, $\mathbf{D}$, and $\mathbf{N}$ are assumed to be small. In this paper, we assume that these perturbation matrices are time independent, which is the case in many applications.

The perturbed system is nonconservative and has the form

$$(\mathbf{M}+\mathbf{M}_1)\ddot{\mathbf{q}}+(\mathbf{G}+\mathbf{G}_1+\mathbf{D})\dot{\mathbf{q}}+(\mathbf{P}+\mathbf{P}_1+\mathbf{N})\mathbf{q}=0. \tag{31}$$

The corresponding eigenvalue problem becomes

$$(\lambda^2(\mathbf{M}+\mathbf{M}_1)+\lambda(\mathbf{G}+\mathbf{G}_1+\mathbf{D})+\mathbf{P}+\mathbf{P}_1+\mathbf{N})\mathbf{u}=0. \tag{32}$$

The system is asymptotically stable if Re $\lambda<0$ for all λ. If there is an eigenvalue with Re $\lambda>0$, the system is unstable.

When the axially symmetric rotating system is unstable and has the exponentially growing mode with Re $\lambda>0$, then the perturbed system remains unstable for small perturbations. On the contrary, small perturbations can destabilize a stable axially symmetric rotating system, whose eigenvalues lie on the imaginary axis. Due to this reason, we will focus on the case of rotational speeds Ω corresponding to stable axially symmetric systems.

5. Nonresonant destabilization condition

First, let us consider an eigenvalue $\lambda=\mathrm{i}\omega$ of the unperturbed system (9), assuming that it is simple. Due to the symmetry of eigenvalues with respect to real and imaginary axes, simple eigenvalues are nonzero. For the same reason, small imperfections $\mathbf{M}_1$, $\mathbf{G}_1$, $\mathbf{P}_1$ can only shift a simple purely imaginary eigenvalue λ along the imaginary axis. Therefore, imperfections have no effect on stability for simple eigenvalues, and destabilization is possible only due to nonconservative terms $\mathbf{D}\dot{\mathbf{q}}+\mathbf{N}\mathbf{q}$.

Small perturbations of the eigenvalue $\lambda=\mathrm{i}\omega$ and of the corresponding eigenvector $\mathbf{u}$ are denoted as

$$i\omega+\delta\lambda, \quad \mathbf{u}+\delta\mathbf{u}. \tag{33}$$

Since imperfections lead to purely imaginary perturbations $\delta\lambda$, which are unimportant for the stability analysis, we consider the perturbed system (32) with $\mathbf{M}_1=\mathbf{G}_1=\mathbf{P}_1=0$. For small terms of first order, this system yields

$$(-\omega^2\mathbf{M}+\mathrm{i}\omega\mathbf{G}+\mathbf{P})\delta\mathbf{u}+(\delta\lambda(2\mathrm{i}\omega\mathbf{M}+\mathbf{G})+\mathrm{i}\omega\mathbf{D}+\mathbf{N})\mathbf{u}=0. \tag{34}$$

Now we multiply expression (34) by $\mathbf{u}^*$ on the left. The term with $\delta\mathbf{u}$ vanishes due to the equation

$$\mathbf{u}^*(-\omega^2\mathbf{M}+\mathrm{i}\omega\mathbf{G}+\mathbf{P})=0, \tag{35}$$

which is a conjugate transpose of (9). As a result, we obtain

$$\delta\lambda=-\frac{\mathbf{u}^*(\omega\mathbf{D}-\mathrm{i}\mathbf{N})\mathbf{u}}{\mathbf{u}^*(2\omega\mathbf{M}-\mathrm{i}\mathbf{G})\mathbf{u}}. \tag{36}$$

Since the matrices $\omega\mathbf{D}-\mathrm{i}\mathbf{N}$ and $2\omega\mathbf{M}-\mathrm{i}\mathbf{G}$ are Hermitian, $\delta\lambda$ is a real number. The asymptotic stability condition Re $\lambda<0$ is satisfied if

$$\frac{\mathbf{u}^*(\omega\mathbf{D}-\mathrm{i}\mathbf{N})\mathbf{u}}{\mathbf{u}^*(2\omega\mathbf{M}-\mathrm{i}\mathbf{G})\mathbf{u}}>0. \tag{37}$$

When the sign in (37) is opposite, the corresponding vibration mode is destabilized by the nonconservative forces.

The inequality (37) can be modified using the relation

$$\mathrm{i}\omega\mathbf{u}^*\mathbf{G}\mathbf{u}=\omega^2\mathbf{u}^*\mathbf{M}\mathbf{u}-\mathbf{u}^*\mathbf{P}\mathbf{u} \tag{38}$$

obtained from (35) after multiplication by $\mathbf{u}$. Multiplying numerator and denominator of (37) by ω and using (38), we obtain

$$\frac{\omega^2\mathbf{u}^*\mathbf{D}\mathbf{u}-\mathrm{i}\omega\mathbf{u}^*\mathbf{N}\mathbf{u}}{\omega^2\mathbf{u}^*\mathbf{M}\mathbf{u}+\mathbf{u}^*\mathbf{P}\mathbf{u}}>0. \tag{39}$$

It is seen that the denominator and the first term in the numerator are positive, if the matrices $\mathbf{M}$, $\mathbf{P}$ and $\mathbf{D}$ are positive definite. In this case, the system can be destabilized only by nonconservative positional forces described by the matrix $\mathbf{N}$, in agreement with the Thomson–Tait–Chetaev theorem [17].

6. Divergence instability

Crossings of eigenvalues from different symmetry classes (resonances) do not lead to instability in axially symmetric systems. However, near the resonances, the system may be destabilized by symmetry breaking and damping.

In this section, we analyze the crossing at the origin $\lambda=0$, which characterizes divergence instability. We consider the case when the double eigenvalue $\lambda=0$ appears at an isolated value of rotation velocity $\Omega=\Omega_c$ due to crossing of two eigenvalues with the eigenvectors $\mathbf{u}_j$ and $\overline{\mathbf{u}}_j$. These eigenvectors belong to the symmetry classes corresponding to $\mu=\mathrm{i}a_j$ and $\mu=-\mathrm{i}a_j$ for some $a_j>0$, see (5) and (7). The general eigenvector of $\lambda=0$ has the form

$$\mathbf{u}=c_1\mathbf{u}_j+c_2\overline{\mathbf{u}}_j \tag{40}$$

with arbitrary coefficients c_1 and c_2. Note that $\mathbf{u}_j \neq \overline{\mathbf{u}}_j$ because they are eigenvectors of the matrix $\mathbf{A}$ corresponding to different eigenvalues. Eigenvectors (40) satisfy

$$\mathbf{P}_c\mathbf{u} = 0 \tag{41}$$

following from (9) with $\lambda = 0$; here and below we denote matrix values at Ω_c by the subscript c, i.e., $\mathbf{P}_c = \mathbf{P}(\Omega_c)$.

Now let us consider the perturbed eigenvalue problem (32). The double zero eigenvalue splits into two small eigenvalues denoted by $\delta\lambda$. Using notation (33) with $\omega = 0$, expression (40), and assuming also a small variation of the rotation frequency $\Omega = \Omega_c + \delta\Omega$ in (32), we obtain the following equation for the first order terms:

$$\mathbf{P}_c\delta\mathbf{u} + (\delta\lambda\mathbf{G}_c + \delta\Omega\mathbf{P}_c' + \mathbf{P}_{1c} + \mathbf{N}_c)(c_1\mathbf{u}_j + c_2\overline{\mathbf{u}}_j) = 0, \tag{42}$$

where $\mathbf{P}_c' = \mathrm{d}\mathbf{P}/\mathrm{d}\Omega$ at Ω_c. Note that the matrices $\mathbf{P}(\Omega)$ and $\mathbf{G}(\Omega)$ corresponding to the axially symmetric system commute with $\mathbf{A}$ for all Ω. Hence, $\mathbf{P}_c$, $\mathbf{P}_c'$, and $\mathbf{G}_c$ commute with $\mathbf{A}$. This property does not hold in general for the matrices $\mathbf{P}_{1c}$ and $\mathbf{N}_c$ describing symmetry breaking and nonconservative forces.

Recall that vectors (7) are orthogonal. This implies that any two vectors $\mathbf{u}_a$ and $\mathbf{u}_b$ from different symmetry classes are orthogonal, $\mathbf{u}_a^*\mathbf{u}_b = 0$. Let $\mathbf{B}$ be an arbitrary matrix commuting with $\mathbf{A}$. As we have shown in (14), the vector $\mathbf{B}\mathbf{u}_b$ belongs to the same symmetry class as $\mathbf{u}_b$ and, hence, it is also orthogonal to $\mathbf{u}_a$. We obtained the relation

$$[\mathbf{B},\mathbf{A}] = 0 \Rightarrow \mathbf{u}_a^*\mathbf{B}\mathbf{u}_b = 0 \tag{43}$$

for any eigenvectors $\mathbf{u}_a$ and $\mathbf{u}_b$ from different symmetry classes. In particular, the matrix $\mathbf{P}_c'$ commutes with $\mathbf{A}$ and the vectors $\mathbf{u}_j$ and $\overline{\mathbf{u}}_j$ belong to different symmetry classes. Therefore,

$$\overline{\mathbf{u}}_j^*\mathbf{P}_c'\mathbf{u}_j = \mathbf{u}_j^*\mathbf{P}_c'\overline{\mathbf{u}}_j = 0. \tag{44}$$

Also,

$$\overline{\mathbf{u}}_j^*\mathbf{G}_c\mathbf{u}_j = \mathbf{u}_j^T\mathbf{G}_c\mathbf{u}_j = 0, \quad \overline{\mathbf{u}}_j^*\mathbf{N}_c\mathbf{u}_j = \mathbf{u}_j^T\mathbf{N}_c\mathbf{u}_j = 0 \tag{45}$$

due to skew-symmetry of the matrices $\mathbf{G}_c$ and $\mathbf{N}_c$.

Multiplying (42) by $\mathbf{u}_j^*$ and $\overline{\mathbf{u}}_j^*$ on the left, we obtain two equations. The term with $\delta\mathbf{u}$ vanishes due to (41). Then using (40), (44), (45), we obtain

$$\begin{pmatrix} 2\mathrm{i}g\delta\lambda + p\delta\Omega + \varepsilon_1 + \mathrm{i}\gamma & \varepsilon_2 - \mathrm{i}\varepsilon_3 \\ \varepsilon_2 + \mathrm{i}\varepsilon_3 & -2\mathrm{i}g\delta\lambda + p\delta\Omega + \varepsilon_1 - \mathrm{i}\gamma \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0, \tag{46}$$

where the real numbers ε_1, ε_2, ε_3, γ, p, and g are determined by the symmetricmatrices P_{1c}, $\mathbf{P}_c{}'$ and skew-symmetric matrices $\mathbf{G}_c$, $\mathbf{N}_c$ as

$$\varepsilon_1 = \mathbf{u}_j^*\mathbf{P}_{1c}\mathbf{u}_j, \quad \varepsilon_2 + \mathrm{i}\varepsilon_3 = \overline{\mathbf{u}}_j^*\mathbf{P}_{1c}\mathbf{u}_j, \quad \mathrm{i}\gamma = \mathbf{u}_j^*\mathbf{N}_c\mathbf{u}_j, \quad p = \mathbf{u}_j^*\mathbf{P}_c'\mathbf{u}_j, \quad 2\mathrm{i}g = \mathbf{u}_j^*\mathbf{G}_c\mathbf{u}_j. \tag{47}$$

Here we used the Greek letters ε and γ for small quantities.

Nontrivial solution (c_1,c_2) of (46) exists when the matrix determinant vanishes. This yields

$$4g^2(\delta\lambda)^2 + 4g\gamma\delta\lambda + \gamma^2 + (p\delta\Omega + \varepsilon_1)^2 - \varepsilon_2^2 - \varepsilon_3^2 = 0. \tag{48}$$

Note that $g \neq 0$ for the double semi-simple eigenvalue $\lambda = 0$, see [11]. The Routh–Hurwitz conditions for asymptotic stability, Re $\delta\lambda < 0$, are

$$g\gamma > 0, \quad \gamma^2 + (p\delta\Omega + \varepsilon_1)^2 - \varepsilon_2^2 - \varepsilon_3^2 > 0. \tag{49}$$

Note that the first condition $g\gamma > 0$ is equivalent to (37) for $\omega = 0$. In the absence of nonconservative forces, we have $\gamma = 0$ and the stability criterion requires $(\delta\lambda)^2 < 0$ (the two eigenvalues are purely imaginary). This yields the inequality

$$(p\delta\Omega + \varepsilon_1)^2 - \varepsilon_2^2 - \varepsilon_3^2 > 0. \tag{50}$$

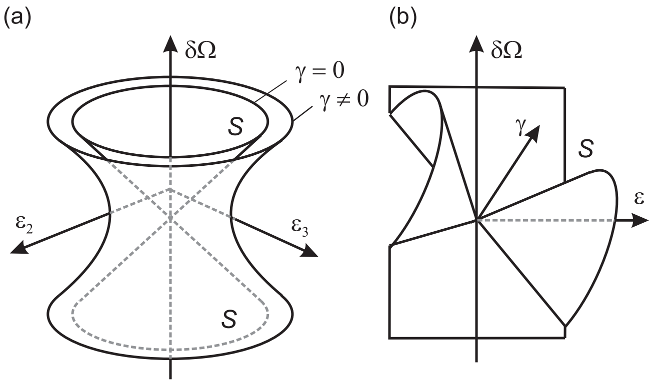

Fig. 2. Divergence instability. (a) Stability regions S in the parameter space $(\delta\Omega,\varepsilon_2,\varepsilon_3)$ for a fixed value of ε_1. (b) Stability region in the parameter space $(\varepsilon,\delta\Omega,\gamma)$ for $g > 0$.

We see that imperfections described by the quantities ε_1, ε_2, ε_3 may destabilize the system with or without nonconservative forces. In the absence of nonconservative forces, the stability condition is given by (50). For fixed ε_1, it defines a conical region in the parameter space $(\delta\Omega, \varepsilon_2, \varepsilon_3)$ with the stability inside the cone, see Fig. 2(a). When nonconservative forces are applied, the system is unstable if $g\gamma<0$. Otherwise, the second inequality in (49) defines a hyperboloid with the inner stability region, see Fig. 2(a). When $\gamma \to 0$, the hyperboloid tends to the cone defined by (50). Selecting a single small parameter ε for imperfections as $\varepsilon_1 = \varepsilon b_1$, $\varepsilon_2 = \varepsilon b_2$, $\varepsilon_3 = \varepsilon b_3$, one can represent the stability domain (49) as the outer part of a cone cut by the plane in the parameter space $(\varepsilon, \delta\Omega, \gamma)$, Fig. 2(b). Note that the cone and cut cone singularities just described are typical singularities of stability boundaries related to double semi-simple eigenvalues [11,18–20].

7. Resonant flutter instability

Now let us consider a crossing of two purely imaginary eigenvalues corresponding to different symmetry classes, i.e., $\lambda_a = \lambda_b = \mathrm{i}\omega$ at $\Omega = \Omega_c$. We denote the corresponding eigenvectors by $\mathbf{u}_a$ and $\mathbf{u}_b$, which satisfy

$$(-\omega^2\mathbf{M}_c + \mathrm{i}\omega\mathbf{G}_c + \mathbf{P}_c)\mathbf{u}_\alpha = 0, \quad \alpha = a, b, \tag{51}$$

and the subscript c denotes the matrix value at Ω_c. The complex transform of (51) yields

$$\mathbf{u}_\alpha^*(-\omega^2\mathbf{M}_c + \mathrm{i}\omega\mathbf{G}_c + \mathbf{P}_c) = 0, \quad \alpha = a, b, \tag{52}$$

where we used symmetry of $\mathbf{M}_c$, $\mathbf{P}_c$ and skew-symmetry of $\mathbf{G}_c$. Note that divergence instability of the previous section corresponds to $\lambda_a = \lambda_b = 0$ and $\mathbf{u}_a = \mathbf{u}_j$, $\mathbf{u}_b = \overline{\mathbf{u}}_j$. Here we will consider the case $\omega \neq 0$ corresponding to flutter instability. The general eigenvector $\mathbf{u}$ corresponding to $\lambda = \mathrm{i}\omega$ is represented similar to (40) as the sum

$$\mathbf{u} = c_1\mathbf{u}_a + c_2\mathbf{u}_b. \tag{53}$$

Now let us consider the perturbed eigenvalue problem (32). The perturbed eigenvalues and eigenvectors are denoted by $\mathrm{i}\omega + \delta\lambda$ and $\mathbf{u} + \delta\mathbf{u}$. Using (53) and assuming a small variation of the rotation velocity $\Omega = \Omega_c + \delta\Omega$ in (32), we obtain the following equation for the first order terms:

$$\begin{aligned}(-\omega^2\mathbf{M}_c + \mathrm{i}\omega\mathbf{G}_c + \mathbf{P}_c)\delta\mathbf{u} + [\delta\lambda(2\mathrm{i}\omega\mathbf{M}_c + \mathbf{G}_c) + \delta\Omega(-\omega^2\mathbf{M}_c{}' + \mathrm{i}\omega\mathbf{G}_c{}' + \mathbf{P}_c{}')\\ -\omega^2\mathbf{M}_{1c} + \mathrm{i}\omega\mathbf{G}_{1c} + \mathbf{P}_{1c} + \mathrm{i}\omega\mathbf{D}_c + \mathbf{N}_c](c_1\mathbf{u}_a + c_2\mathbf{u}_b) = 0,\end{aligned} \tag{54}$$

where primes denote the derivatives $\mathbf{M}_c' = \mathrm{d}\mathbf{M}/\mathrm{d}\Omega$ taken at $\Omega = \Omega_c$. Recall that the matrices $\mathbf{M}(\Omega)$, $\mathbf{G}(\Omega)$ and $\mathbf{P}(\Omega)$ of the axially symmetric system commute with $\mathbf{A}$ for all Ω. Hence, the matrices $\mathbf{M}_c$, $\mathbf{G}_c$, $\mathbf{P}_c$, $\mathbf{M}_c{}'$, $\mathbf{G}_c{}'$, and $\mathbf{P}_c{}'$ commute with $\mathbf{A}$. Multiplying Eq. (54) by $\mathbf{u}_a^*$ and $\mathbf{u}_b^*$ on the left and using (52) and (43), we obtain two equations written as

$$\begin{pmatrix} 2\mathrm{i}h_a\delta\lambda + p_a\delta\Omega + \varepsilon_a + \mathrm{i}\gamma_a & \varepsilon_2 - \mathrm{i}\varepsilon_3 - \gamma_2 + \mathrm{i}\gamma_3 \\ \varepsilon_2 + \mathrm{i}\varepsilon_3 + \gamma_2 + \mathrm{i}\gamma_3 & 2\mathrm{i}h_b\delta\lambda + p_b\delta\Omega + \varepsilon_b + \mathrm{i}\gamma_b \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0, \tag{55}$$

where we introduced the real quantities ε_a, ε_b, γ_a, γ_b, h_a, h_b, p_a, p_b, ε_2, ε_3, γ_2, γ_3 as

$$\begin{aligned}\varepsilon_\alpha = \mathbf{u}_\alpha^*(-\omega^2\mathbf{M}_{1c} + \mathrm{i}\omega\mathbf{G}_{1c} + \mathbf{P}_{1c})\mathbf{u}_\alpha, \quad \mathrm{i}\gamma_\alpha = \mathbf{u}_\alpha^*(\mathrm{i}\omega\mathbf{D}_c + \mathbf{N}_c)\mathbf{u}_\alpha,\\ 2\mathrm{i}h_\alpha = \mathbf{u}_\alpha^*(2\mathrm{i}\omega\mathbf{M}_c + \mathbf{G}_c)\mathbf{u}_\alpha, \quad p_\alpha = \mathbf{u}_\alpha^*(-\omega^2\mathbf{M}_c{}' + \mathrm{i}\omega\mathbf{G}_c{}' + \mathbf{P}_c')\mathbf{u}_\alpha, \quad \alpha = a, b,\\ \varepsilon_2 + \mathrm{i}\varepsilon_3 = \mathbf{u}_b^*(-\omega^2\mathbf{M}_{1c} + \mathrm{i}\omega\mathbf{G}_{1c} + \mathbf{P}_{1c})\mathbf{u}_a, \quad \gamma_2 + \mathrm{i}\gamma_3 = \mathbf{u}_b^*(\mathrm{i}\omega\mathbf{D}_c + \mathbf{N}_c)\mathbf{u}_a.\end{aligned} \tag{56}$$

We used the Greek letters ε and γ to denote small quantities.

Nontrivial solutions (c_1, c_2) of (55) exist when the matrix determinant vanishes. This provides the characteristic equation (taken with the opposite sign)

$$M(\delta\lambda)^2 + (D + \mathrm{i}G)\delta\lambda + P + \mathrm{i}N = 0, \tag{57}$$

where

$$M = 4h_ah_b, \quad D = 2h_b\gamma_a + 2h_a\gamma_b, \quad G = -2h_a(p_b\delta\Omega + \varepsilon_b) - 2h_b(p_a\delta\Omega + \varepsilon_a), \tag{58}$$

$$P = \varepsilon_2^2 + \varepsilon_3^2 - (p_a\delta\Omega + \varepsilon_a)(p_b\delta\Omega + \varepsilon_b) - \gamma_2^2 - \gamma_3^2 + \gamma_a\gamma_b, \tag{59}$$

$$N = 2\varepsilon_2\gamma_3 - 2\varepsilon_3\gamma_2 - \gamma_a(p_b\delta\Omega + \varepsilon_b) - \gamma_b(p_a\delta\Omega + \varepsilon_a). \tag{60}$$

Note that $M \neq 0$ for a semi-simple double eigenvalue $\lambda = \mathrm{i}\omega$, see [11].

In the absence of nonconservative forces, $\gamma_a = \gamma_b = \gamma_2 = \gamma_3 = 0$, we have $D = N = 0$ and the system is gyroscopic. The eigenvalues are found from (57) as

$$\delta\lambda = \frac{-\mathrm{i}G \pm \sqrt{-G^2 - 4MP}}{2M}. \tag{61}$$

The system is stable for $G^2 + 4MP > 0$, when both $\delta\lambda$ are purely imaginary. If $G^2 + 4MP < 0$, then one of the eigenvalues gets a positive real part, and the system becomes unstable. Using (58) and (59) in the stability condition written as

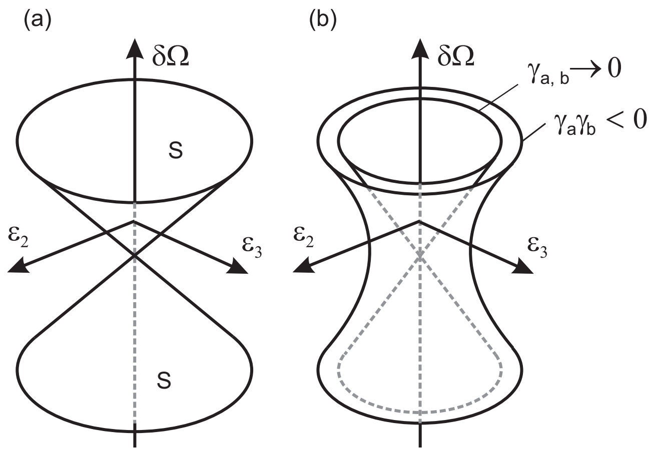

Fig. 3. Resonant flutter instability. (a) Stability regions S in the effective parameter space $(\delta\Omega,\varepsilon_2,\varepsilon_3)$ for a system with no damping in the case $h_1h_2<0$. (b) Stability boundaries of the damped system for the case $\gamma_a/h_a+\gamma_b/h_b>0$ and $\gamma_a\gamma_b<0$. The stability region is inside the hyperboloid if $h_1h_2<0$ and outside if $h_1h_2>0$.

$(G/M)^2+4P/M>0$, we obtain

$$\left(\frac{p_a\delta\Omega+\varepsilon_a}{2h_a}-\frac{p_b\delta\Omega+\varepsilon_b}{2h_b}\right)^2+\frac{\varepsilon_2^2+\varepsilon_3^2}{h_ah_b}>0. \tag{62}$$

If $h_ah_b<0$, the stability boundary represents the cone surface in the space of effective parameters $(\delta\Omega,\varepsilon_2,\varepsilon_3)$. The stability region is inside the cone, Fig. 3(a). If $h_ah_b>0$, the system remains stable for any small imperfections.

The asymptotic stability of the nonconservative system is determined by the condition Re $\delta\lambda<0$ for both roots of (57). This condition can be written in the form of the Bilharz inequalities (see, e.g., [11])

$$\frac{D}{M}>0,\quad \frac{DGN}{M}-N^2+\frac{D^2P}{M}>0. \tag{63}$$

Using (58), we write the first condition in the form

$$\frac{\gamma_a}{h_a}+\frac{\gamma_b}{h_b}>0. \tag{64}$$

The second inequality in (63) can be multiplied by $4/D^2$ and written in the equivalent form $(G/M)^2-(2N/D-G/M)^2+4P/M>0$. We substitute (58)–(60) into this condition. Long but straightforward computation yields

$$\frac{4(h_a\gamma_b\Delta+\varepsilon_3\gamma_2-\varepsilon_2\gamma_3)(h_b\gamma_a\Delta-\varepsilon_3\gamma_2+\varepsilon_2\gamma_3)}{(h_b\gamma_a+h_a\gamma_b)^2}+\frac{\varepsilon_2^2+\varepsilon_3^2}{h_ah_b}+\frac{\gamma_a\gamma_b-\gamma_2^2-\gamma_3^2}{h_ah_b}>0, \tag{65}$$

$$\Delta=\frac{p_a\delta\Omega+\varepsilon_a}{2h_a}-\frac{p_b\delta\Omega+\varepsilon_b}{2h_b}. \tag{66}$$

Inequalities (64) and (65) are the asymptotic stability conditions for small imperfections and dissipation. Let us consider in detail the case $\gamma_2=\gamma_3=0$. This case corresponds to axially symmetric dissipative forces, when the matrices $\mathbf{D}_c$ and $\mathbf{N}_c$ commute with $\mathbf{A}$, see (43) and (56). Then, condition (65) takes the form

$$\frac{4h_ah_b\gamma_a\gamma_b}{(h_b\gamma_a+h_a\gamma_b)^2}\left(\frac{p_a\delta\Omega+\varepsilon_a}{2h_a}-\frac{p_b\delta\Omega+\varepsilon_b}{2h_b}\right)^2+\frac{\varepsilon_2^2+\varepsilon_3^2}{h_ah_b}+\frac{\gamma_a\gamma_b}{h_ah_b}>0. \tag{67}$$

Let us analyze the stability domain given by (64) and (67) in the space $(\delta\Omega,\varepsilon_2,\varepsilon_3)$ for fixed values of dissipation coefficients γ_a and γ_b. The system is unstable for all small perturbations if $\gamma_a/h_a+\gamma_b/h_b<0$. Let us assume that $\gamma_a/h_a+\gamma_b/h_b>0$. Then, due to (67), we have stability if $\gamma_a\gamma_b>0$, $h_ah_b>0$ and instability if $\gamma_a\gamma_b>0$, $h_ah_b<0$. When $\gamma_a\gamma_b<0$, the stability domain given by (67) is bounded by a hyperboloid in the parameter space $(\delta\Omega,\varepsilon_2,\varepsilon_3)$, Fig. 3(b). The stability region is inside or outside the hyperboloid for negative or positive values of h_1h_2, respectively.

Let us consider the vanishing damping, $\gamma_a\to 0$ and $\gamma_b\to 0$, when the ratio $\chi=(h_b/h_a)(\gamma_a/\gamma_b)$ is fixed, Fig. 3(b). The last term in (67) vanishes, and the factor of the first term tends to

$$\frac{4h_ah_b\gamma_a\gamma_b}{(h_b\gamma_a+h_a\gamma_b)^2}\longrightarrow 4\chi(1+\chi)^{-2}\le 1,\quad \chi=\frac{h_b\gamma_a}{h_a\gamma_b}. \tag{68}$$

The equality $4\chi(1+\chi)^{-2}=1$ is attained for $\chi=1$. Thus, inequality (67) takes the limiting form

$$\frac{4\chi}{(1+\chi)^2}\left(\frac{p_a\delta\Omega+\varepsilon_a}{2h_a}-\frac{p_b\delta\Omega+\varepsilon_b}{2h_b}\right)^2+\frac{\varepsilon_2^2+\varepsilon_3^2}{h_ah_b}>0. \tag{69}$$

Comparing (69) with (62), we conclude that the stability region for the vanishing damping is smaller than the stability region of the undamped system when $\chi\neq 1$. But these regions coincide for $\chi=1$. This represents the effect of destabilization by small damping, which is similar to the destabilization phenomenon for combination resonance in parametrically exited systems, see [11].

8. Instability of a rigid disk mounted on a slightly asymmetric shaft

As the first example, we consider a rigid disk mounted on a massless asymmetric flexible shaft in the presence of internal and external damping forces. It is assumed that the shaft is rotating with constant angular velocity Ω. Equations of motion of the disk in the rotating frame take the form [1,2]

$$\begin{aligned}&\ddot{\xi}+(\varepsilon_i+\varepsilon_e)\dot{\xi}-2\Omega\dot{\eta}+(\omega_1^2-\Omega^2)\xi-\varepsilon_e\Omega\eta=0,\\ &\ddot{\eta}+(\varepsilon_i+\varepsilon_e)\dot{\eta}+2\Omega\dot{\xi}+(\omega_2^2-\Omega^2)\eta+\varepsilon_e\Omega\xi=0,\end{aligned} \tag{70}$$

where ω_1 and ω_2 are the eigenfrequencies of the undamped nonrotating shaft, and $\varepsilon_i\geq 0$ and $\varepsilon_e\geq 0$ are the internal and external damping coefficients. Note that ε_i and ε_e are also referred to as rotating and non-rotating damping coefficients, respectively. We assume that the difference $\delta\omega=\omega_2-\omega_1>0$ is small which implies slightly asymmetric cross-section of the shaft.

Let us study the influence of small asymmetry and small internal and external damping on stability of the rotating shaft. For this purpose we can use the results of Sections 5–7. In this example, Eqs. (70) have the form of (31) with the matrices

$$\mathbf{M}=\mathbf{I},\quad \mathbf{M}_1=0,\quad \mathbf{P}=(\omega_1^2-\Omega^2)\mathbf{I},\quad \mathbf{D}=(\varepsilon_i+\varepsilon_e)\mathbf{I},\quad \mathbf{G}_1=0, \tag{71}$$

$$\mathbf{P}_1=\begin{pmatrix}0 & 0\\ 0 & 2\omega_1\delta\omega\end{pmatrix},\quad \mathbf{G}=2\Omega\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \mathbf{N}=\varepsilon_e\Omega\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{72}$$

where we took $\omega_2^2-\omega_1^2\approx 2\omega_1\delta\omega$ in the matrix $\mathbf{P}_1$ in the first approximation. We note that the nonconservative positional (circulatory) forces with the matrix $\mathbf{N}$ appear due to the external damping introduced in the static frame.

Rotation about the symmetry axis is described by (18), where $\mathbf{w}=(\xi,\eta)^T$ and the matrix $\mathbf{J}$ has the eigenvalues $\mu=\mathrm{i}$ and $\overline{\mu}=-\mathrm{i}$ determining two symmetry classes. The corresponding eigenvectors are

$$\mathbf{u}=\begin{pmatrix}\mathrm{i}\\ 1\end{pmatrix},\quad \overline{\mathbf{u}}=\begin{pmatrix}-\mathrm{i}\\ 1\end{pmatrix}. \tag{73}$$

The eigenvalues of the undamped axially symmetric system ($\omega_1=\omega_2$ and $\varepsilon_i=\varepsilon_e=0$) are

$$\lambda=\mathrm{i}\omega,\quad \overline{\lambda}=-\mathrm{i}\omega,\quad \text{where } \omega=\pm\omega_1-\Omega, \tag{74}$$

with the eigenvectors $\mathbf{u}$ and $\overline{\mathbf{u}}$ from (73).

First, we consider simple eigenvalues λ. Then, according to (37) and (71)–(74) we find the stability condition

$$\frac{(\pm\omega_1-\Omega)(\varepsilon_i+\varepsilon_e)+\varepsilon_e\Omega}{2(\pm\omega_1-\Omega)+2\Omega}>0. \tag{75}$$

From this inequality we obtain

$$|\Omega|<\Omega_*=\omega_1\left(1+\frac{\varepsilon_e}{\varepsilon_i}\right). \tag{76}$$

Here Ω_* is the critical angular velocity known in the literature [1]. We note that the small difference $\delta\omega$ does not influence the critical velocity in the first approximation.

Consider now the divergence instability. The eigenvalues (74) of the unperturbed system become double $\lambda=0$ at $\Omega_c=\omega_1$ with the two eigenvectors $\mathbf{u}$ and $\overline{\mathbf{u}}$ given by (73). According to expressions (47) with $\mathbf{u}_j=\mathbf{u}$ we find

$$\varepsilon_1=\varepsilon_2=2\omega_1\delta\omega,\quad \varepsilon_3=0,\quad \gamma=2\varepsilon_e\omega_1,\quad g=2\omega_1,\quad p=-4\omega_1. \tag{77}$$

Substituting these quantities in (49) with $\delta\Omega=\Omega-\omega_1$, we see that the first condition is satisfied. The second condition takes the form

$$\frac{\varepsilon_e^2}{4}+(\delta\Omega)^2-\delta\omega\delta\Omega>0. \tag{78}$$

This inequality describes a conical stability domain in three-dimensional space $(\delta\Omega,\varepsilon_e,\delta\omega)$. Thus, due to small asymmetry and external damping the instability region appears.

From inequality (78) we obtain the conditions for instability as

$$\frac{\delta\omega-\sqrt{(\delta\omega)^2-\varepsilon_e^2}}{2}<\delta\Omega<\frac{\delta\omega+\sqrt{(\delta\omega)^2-\varepsilon_e^2}}{2}. \tag{79}$$

If we assume $\varepsilon_e \ll \delta\omega$ then from conditions (79) by expanding the square roots we get the instability domain as

$$\omega_1+\frac{\varepsilon_e^2}{4(\omega_2-\omega_1)}<\Omega<\omega_2-\frac{\varepsilon_e^2}{4(\omega_2-\omega_1)}, \tag{80}$$

since $\omega_2=\omega_1+\delta\omega$. This formula shows the stabilizing effect of small external damping. In the first approximation it agrees with the results presented in [1].

Now we study the case of small rotation velocity Ω. We note that at $\Omega_c=0$ the unperturbed system possesses the double eigenvalue $\lambda=\mathrm{i}\omega_1$ with the corresponding eigenvectors $\mathbf{u}_a=\mathbf{u}$ and $\mathbf{u}_b=\overline{\mathbf{u}}$. We can use the results of Section 7 to verify the absence of instability for small imperfections and damping. Indeed, using (56) with $\omega=\omega_1$ and (71)–(73), we calculate the coefficients $\gamma_a=\gamma_b=2\omega_1(\varepsilon_i+\varepsilon_e)$, $h_a=h_b=2\omega_1$. Then, according to inequalities (64) and (67) the system is stable. Hence, in spite of the double eigenvalue $\lambda=\mathrm{i}\omega_1$ at $\Omega_c=0$ the flutter instability at small rotation velocities is eliminated. The same is true for the double eigenvalue $\overline{\lambda}=-\mathrm{i}\omega_1$ at $\Omega_c=0$.

For small rotation velocity Ω it is also easy to verify the stability condition directly. Using characteristic equation (57) with the coefficients (58)–(60) we find the eigenvalues for the damped, slightly asymmetric system. Solving this equation we obtain in the first approximation

$$\delta\lambda=-\frac{\varepsilon_i+\varepsilon_e}{2}+\frac{\mathrm{i}(\omega_2-\omega_1\pm\sqrt{(\omega_2-\omega_1)^2+4\Omega^2})}{2}. \tag{81}$$

This means that under perturbations the double eigenvalue $\lambda=\mathrm{i}\omega_1$ splits into two simple eigenvalues with the negative real part implying asymptotic stability of the system.

Note that in [12,14] instability of the rotating shaft was found at rotation speeds close to zero. This result was obtained by considering constant circulatory forces $\mathbf{Nq}$ in (70) while these forces vanish, $\mathbf{N}=0$ at $\Omega=0$.

9. Stability of rotating elastic shaft with small axial asymmetry and damping

Small oscillations of a rotating shaft with a circular cross-section and simply supported boundary conditions are governed by Eqs. (20) and (21) for the deflection vector-function $\mathbf{w}(t,x)$ defined in the rotating reference frame. As we showed in Section 3, this system is stable for all rotation speeds Ω. Double frequencies appear at resonance points (27), but do not cause instability. In this section, we analyze stability of this system when small asymmetry and damping are introduced.

When external and internal dissipative forces are taken into account, the system (20) becomes

$$\ddot{\mathbf{w}}+2\Omega\mathbf{J}\dot{\mathbf{w}}+\pi^{-4}\mathbf{w}''''-\Omega^2\mathbf{w}+\eta_i\pi^{-4}\dot{\mathbf{w}}''''+\eta_e(\dot{\mathbf{w}}+\Omega\mathbf{J}\mathbf{w})=0, \tag{82}$$

where the coefficient of internal damping $\eta_i>0$ (Kelvin–Voigt model) and the coefficient of external viscous damping $\eta_e>0$ are assumed to be small. The sum $\dot{\mathbf{w}}+\Omega\mathbf{J}\mathbf{w}$ in the last term describes the velocity in the static frame. Note that the dissipative forces considered in this model are axially symmetric.

Additionally, we consider the breaking of axial symmetry due to variation of the shaft cross-section. We study the case of elliptic cross-section with the axes parallel to y- and z-axes of the rotating frame, Fig. 4. The cross-section is assumed to be nonuniform, with the semi-axes of the ellipses given by

$$a_y(x)=a_0(1-\varepsilon s(x)), \quad a_z(x)=a_0(1+\varepsilon s(x)), \tag{83}$$

where a_0 is the radius of the axially symmetric shaft, ε is a small imperfection parameter, and the function $s(x)$ describes distribution of imperfections. The moments of inertia of the elliptic cross-section are

$$I_z=\pi a_y^3 a_z/4=(1-2\varepsilon s)I_0+o(\varepsilon), \quad I_y=\pi a_z^3 a_y/4=(1+2\varepsilon s)I_0+o(\varepsilon), \tag{84}$$

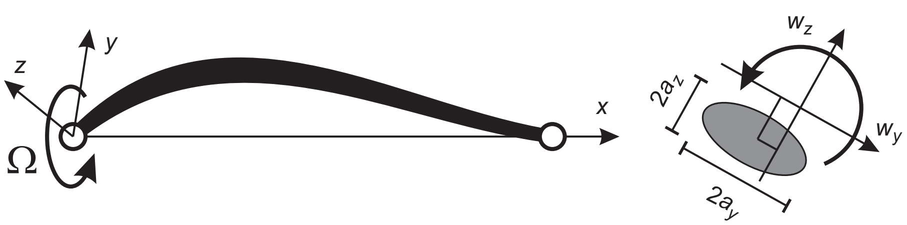

Fig. 4. Elastic shaft of variable cross-section rotating with speed Ω.

where $I_0 = \pi a_0^4/4$ is the moment of inertia for the circular cross-section. Such symmetry breaking leads to the change of the stiffness term in (82) as

$$\pi^{-4}\mathbf{w}'''' \mapsto \pi^{-4}\begin{pmatrix} ((1-2\varepsilon s)w_y'')'' \\ ((1+2\varepsilon s)w_z'')'' \end{pmatrix}. \tag{85}$$

Perturbation of the mass density has higher order of magnitude, since $a_y a_z = a_0^2 - \varepsilon^2 s^2$. Therefore, we neglect small perturbations of the mass and gyroscopic terms in (82). The simply supported boundary conditions (21) remain unchanged.

Note that the change of direction for one of the axes, $w_y \mapsto -w_y$, is equivalent to the change of sign of rotation speed, $\Omega \mapsto -\Omega$. Hence, we can take $\Omega \geq 0$.

Using (28), we write Eq. (82) with (85) in the matrix form (31). As in the example of Section 3, we substitute (28) into (82), multiply the equations by 2 $\sin(k\pi x)$ and integrate with respect to x from 0 to 1. The resulting system matrices are given by (29), (30) and

$$\mathbf{D} = \eta_i \mathbf{P}_0 + \eta_e \mathbf{I}, \quad \mathbf{N}(\Omega) = \eta_e \Omega \mathbf{A}, \quad \mathbf{M}_1 = \mathbf{G}_1 = 0. \tag{86}$$

Due to (28), elements of the matrix $\mathbf{P}_1 = [P_{kj}]$ vanish when the indices have different parity, i.e., $P_{2k,2j-1} = P_{2k-1,2j} = 0$. The other elements are found as

$$P_{2k,2j} = -P_{2k-1,2j-1} = \varepsilon s_{kj}, \quad s_{kj} = 4\omega_j \omega_k \int_0^1 s(x)\sin(j\pi x)\sin(k\pi x)\,\mathrm{d}x, \quad j,k = 1,2,\ldots, \tag{87}$$

where we performed two integrations by parts.

The eigenvalues for the axially symmetric undamped system (9) are found in Section 3 and given in (25) and (26). Comparing (23) with the expansion (28) we write the eigenvalues λ with the corresponding eigenvectors $\mathbf{u}_j$ as

$$\mathbf{u}_j = \mathrm{i}\mathbf{e}_{2j-1} + \mathbf{e}_{2j}, \quad \lambda = \mathrm{i}\omega, \quad \omega = \pm\omega_j - \Omega,\ j = 1,2,\ldots, \tag{88}$$

where $\mathbf{e}_j$ is the vector with all zero components except for the unit j-th component. For the complex conjugate eigenvectors and eigenvalues, we have

$$\overline{\mathbf{u}}_j = -\mathrm{i}\mathbf{e}_{2j-1} + \mathbf{e}_{2j}, \quad \overline{\lambda} = -\mathrm{i}\omega, \quad \omega = \pm\omega_j - \Omega,\ j = 1,2,\ldots\ . \tag{89}$$

In the following we will use the expressions derived from (30), (87)–(89) as

$$\begin{gathered} \mathbf{u}_{j_2}^* \mathbf{u}_{j_1} = 2\delta_{j_1 j_2}, \quad \mathbf{u}_{j_2}^* \mathbf{A}\mathbf{u}_{j_1} = 2\mathrm{i}\delta_{j_1 j_2}, \quad \mathbf{u}_{j_2}^* \mathbf{P}_0 \mathbf{u}_{j_1} = 2\omega_{j_1}^2 \delta_{j_1 j_2}, \quad \overline{\mathbf{u}}_{j_2}^* \mathbf{P}_1 \mathbf{u}_{j_1} = 2\varepsilon s_{j_1 j_2}, \\ \overline{\mathbf{u}}_{j_2}^* \mathbf{u}_{j_1} = \overline{\mathbf{u}}_{j_2}^* \mathbf{A}\mathbf{u}_{j_1} = \overline{\mathbf{u}}_{j_2}^* \mathbf{P}_0 \mathbf{u}_{j_1} = \mathbf{u}_{j_2}^* \mathbf{P}_1 \mathbf{u}_{j_1} = 0, \end{gathered} \tag{90}$$

where $\delta_{j_1 j_2}$ is the Kronecker delta.

First, we consider the nonresonant stability condition (37). Using (29), (86) and (90) in (37), we obtain the asymptotic stability condition for the eigenvector $\mathbf{u} = \mathbf{u}_j$ and frequency ω from (88) as

$$\frac{\mathbf{u}^*(\omega\mathbf{D} - \mathrm{i}\mathbf{N})\mathbf{u}}{\mathbf{u}^*(2\omega\mathbf{M} - \mathrm{i}\mathbf{G})\mathbf{u}} = \frac{2\omega(\eta_i\omega_j^2 + \eta_e) + 2\eta_e\Omega}{4(\omega + \Omega)} = \frac{\eta_i\omega_j}{2}\left(\frac{\eta_e}{\eta_i\omega_j} + \omega_j \mp \Omega\right) > 0. \tag{91}$$

The complex conjugate eigenvector $\mathbf{u} = \overline{\mathbf{u}}_j$ leads to the same result. Condition (91) taken for all vibrational modes yields

$$\Omega < \Omega_* = \min_{j \geq 1}\left(\frac{\eta_e}{\eta_i\omega_j} + \omega_j\right). \tag{92}$$

This condition determines the critical flutter speed Ω_* corresponding to the instability caused by the damping forces (far from resonances). Note that this formula is known in the literature, see e.g. [1]. Since $\omega_j = j^2$, one can show that, for $\eta_e < 4\eta_i$, the mode $\mathbf{u}_1$ is the first to violate condition (92) with increasing rotation speed Ω. When $\eta_e > 4\eta_i$, such instability is caused by higher modes.

Now let us consider the resonance points. First, we consider the divergence instability, i.e., the resonance associated with the modes $\mathbf{u}_j$ and $\overline{\mathbf{u}}_j$. According to (88) and (89), the resonance occurs at the values of the rotation speed

$$\Omega_c = \omega_j, \quad j = 1,2,\ldots, \tag{93}$$

corresponding to the double eigenvalue $\lambda = 0$ with the two eigenvectors $\mathbf{u}_j$ and $\overline{\mathbf{u}}_j$. Using the expressions (29), (86), (90) in (47), we find

$$\varepsilon_1 = \varepsilon_3 = 0, \quad \varepsilon_2 = 2\varepsilon s_{jj}, \quad \gamma = 2\eta_e\Omega_c, \quad p = -4\Omega_c, \quad g = 2\Omega_c. \tag{94}$$

The asymptotic stability conditions given by (49) take the form

$$\eta_e\Omega_c^2 > 0, \quad \eta_e^2\Omega_c^2 + 4\Omega_c^2(\delta\Omega)^2 - \varepsilon^2 s_{jj}^2 > 0. \tag{95}$$

The last condition defines the region bounded by a cone in the space of parameters $(\varepsilon, \eta_e, \delta\Omega)$. Note that the internal damping coefficient η_i does not enter the stability condition in the first approximation. The divergence instability region appears due to symmetry breaking, $\varepsilon \neq 0$.

Recall that the stability condition (95) is obtained for the two eigenvalues, which appear after splitting of the double eigenvalue $\lambda=0$. For stability of the system, one has to verify that all the other (simple) eigenvalues of the system have negative real parts. This means that the resonant rotation speed Ω_c must satisfy inequality (92).

All possible resonant cases are listed in (27). We will distinguish summed and difference types of resonances following the terminology of combination resonances in periodic systems, see, e.g., [11,19]. The summed resonance is given by

$$\Omega_c=\frac{\omega_{j_1}+\omega_{j_2}}{2},\quad \lambda=\mathrm{i}\omega,\quad \omega=\frac{\omega_{j_1}-\omega_{j_2}}{2}. \tag{96}$$

It corresponds to the crossing of the eigenvalue from (88) with $j=j_1$ and the eigenvalue from (89) with $j=j_2$. Thus, the eigenvalue $\lambda=\mathrm{i}\omega$ is double with the two eigenvectors $\mathbf{u}_a=\mathbf{u}_{j_1}$ and $\mathbf{u}_b=\overline{\mathbf{u}}_{j_2}$. Using the matrices (29), (86) and relations (90), (96) in (56), we find

$$\varepsilon_a=\varepsilon_b=0,\quad \gamma_a=2\omega_{j_1}(\omega\eta_i\omega_{j_1}+\eta_e),\quad \gamma_b=2\omega_{j_2}(\omega\eta_i\omega_{j_2}-\eta_e),$$

$$h_a=2\omega_{j_1},\quad h_b=-2\omega_{j_2},\quad p_a=-4\omega_{j_1},\quad p_b=-4\omega_{j_2},$$

$$\varepsilon_2=2\varepsilon s_{j_1j_2},\quad \varepsilon_3=0,\quad \gamma_2=\gamma_3=0. \tag{97}$$

Using (97), we write the first asymptotic stability condition (64) as

$$2\omega^2\eta_i+2\eta_e>0, \tag{98}$$

where we used the expression $\omega_{j_1}-\omega_{j_2}=2\omega$ from (96). Inequality (98) is satisfied for $\eta_i>0$ and $\eta_e>0$. Similarly, we write the second asymptotic stability condition (67) in the form

$$\frac{4z(\delta\Omega)^2}{(\omega^2\eta_i+\eta_e)^2}-\frac{s_{j_1j_2}^2\varepsilon^2}{\omega_{j_1}\omega_{j_2}}+z>0,\quad z=(\eta_e+\omega\eta_i\omega_{j_1})(\eta_e-\omega\eta_i\omega_{j_2}). \tag{99}$$

If $z>0$, inequality (99) defines the asymptotic stability domain bounded by hyperbolas on the parameter plane $(\varepsilon,\delta\Omega)$. If $z<0$, inequality (99) cannot be satisfied and the system is unstable for all small perturbations.

Since $\omega=\omega_{j_1}-\Omega_c=\Omega_c-\omega_{j_2}$, we can write z from (99) as

$$z=\eta_i^2\omega_{j_1}\omega_{j_2}\left(\frac{\eta_e}{\eta_i\omega_{j_1}}+\omega_{j_1}-\Omega_c\right)\left(\frac{\eta_e}{\eta_i\omega_{j_2}}+\omega_{j_2}-\Omega_c\right). \tag{100}$$

In the region $\Omega_c<\Omega_*$, we have $z>0$ due to (92) and the instability zones appear. Recall that, for $\Omega>\Omega_*$, the instability is caused by simple eigenvalues, which violate condition (91). If ω_{j_1} or ω_{j_2} in (100) violate condition (91), then one can use $\Omega_c=(\omega_{j_1}+\omega_{j_2})/2$ in (100) and see that $z<0$ (instability). This means that the system is unstable for all rotation speeds $\Omega>\Omega_*$ including the resonance points.

In the case of the difference type resonance, we have

$$\Omega_c=\frac{\omega_{j_1}-\omega_{j_2}}{2},\quad \lambda=\mathrm{i}\omega,\quad \omega=\frac{\omega_{j_1}+\omega_{j_2}}{2}. \tag{101}$$

The eigenvalue $\lambda=\mathrm{i}\omega$ is double with the two eigenvectors $\mathbf{u}_a=\mathbf{u}_{j_1}$ and $\mathbf{u}_b=\overline{\mathbf{u}}_{j_2}$. Instead of (97), one obtains

$$\gamma_a=2\omega_{j_1}(\omega\eta_i\omega_{j_1}+\eta_e),\quad \gamma_b=2\omega_{j_1}(\omega\eta_i\omega_{j_2}+\eta_e),\quad h_a=2\omega_{j_1},\quad h_b=2\omega_{j_2}. \tag{102}$$

The stability conditions (64) and (67) are satisfied since γ_a, γ_b and h_a, h_b are all positive. Thus, flutter instability for difference resonances is ruled out. This effect is analogous to the combination resonances in some periodic systems. For example, the straight beam under axial periodic excitation undergoes only summed combination resonances, see, e.g., [11].

We conclude that the rotating elastic shaft with small asymmetry and damping is unstable for $\Omega>\Omega_*$ and in the resonance regions. The resonance regions appear near the critical values $\Omega_c=(\omega_{j1}+\omega_{j_2})/2$ corresponding to divergence $(j_1=j_2)$ and flutter $(j_1\neq j_2)$ instabilities. Note that the first divergence resonance corresponds to $\Omega_c=\omega_1=1$. This resonance always appears since $\Omega_*>1$. The first flutter resonance corresponds to $\Omega_c=(\omega_1+\omega_2)/2=5/2$ since $\omega_2=4$. Comparing this value with (92), we conclude that the flutter resonance $5/2<\Omega_*$ appears only for $\eta_e/\eta_i>3/2$.

As a numerical example let us consider the imperfections function $s(x)=x$ and small damping coefficients $\eta_i=0.01$, $\eta_e=0.03$. According to (92) with $\omega_j=j^2$, the system with small asymmetry and damping is unstable for

$$\Omega>\Omega_*=4. \tag{103}$$

In the region $\Omega\le 4$, there are two resonances. The first resonance corresponds to the divergence instability with $\Omega_c=\omega_1=1$. Using (95), we find the instability region for small ε and $\delta\Omega=\Omega-1$ as

$$4(\delta\Omega)^2-\varepsilon^2+0.0009<0, \tag{104}$$

where $s_{11}=1$ was computed by formula (87). The second resonance corresponds to the flutter instability with $\Omega_c=(\omega_2+\omega_1)/2=5/2$ and $\omega=(\omega_2-\omega_1)/2=3/2$. Using (99), we find the instability region for small ε and $\delta\Omega=\Omega-5/2$ as

$$1.9592(\delta\Omega)^2-0.5191\varepsilon^2+0.0013<0, \tag{105}$$

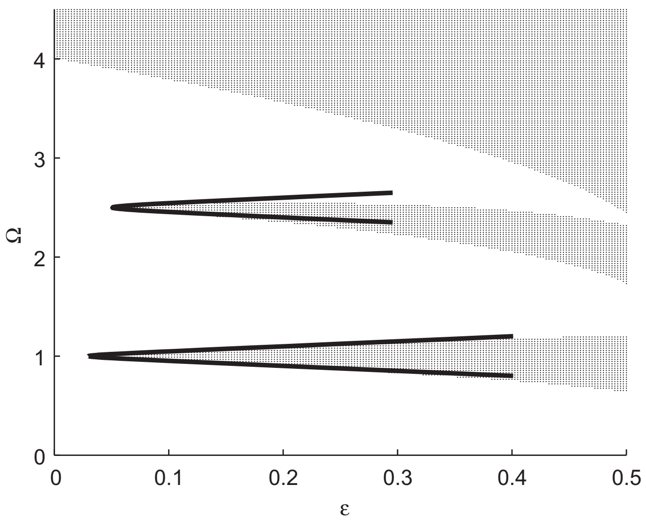

Fig. 5. Instability regions for rotating elastic shaft with asymmetry and damping (gray regions) and approximation of the stability boundary (bold curves).

where $s_{12} = -128\pi^{-2}/9$ was computed by (87). Boundaries of instability regions (104) and (105) are shown in Fig. 5 by bold lines. Gray regions are the instability domains computed numerically by checking condition Re $\lambda < 0$ for eigenvalues of system (32) with the matrices (29), (30), (86), (87), where we kept 10 terms in the sum (28). One can see a good agreement of theoretical results (103)–(105) with numerical computations.

10. Conclusion

Vibrations and stability of rotating systems is a topic of theoretical interest due to a wide range of technical applications. These applications include loss of stability of rods and rotating parts of mechanisms, the brake squeal phenomenon and many others. A large class of systems studied in this area is characterized by axial symmetry. In this paper, we develop a general mathematical theory that treats stability of axially symmetric systems and the systems close to axially symmetric ones in a unified way.

Summarizing specific results of the paper we emphasize the main achievements. We described stability properties of a general axially symmetric rotating system using eigenvectors classified by the symmetry. Then we formulated the stability problem for rotating systems with small damping and symmetry-breaking perturbations. We gave a classification of all possible instability and resonance phenomena and provided new general ready-to-use formulae for nonresonant and resonant divergence and flutter instability regions. As the application, the stability problem for a rotating elastic shaft of variable cross-section and small internal and external damping is solved. In particular it is shown that only the summed type of resonance leads to instability, while the difference type of resonance instability is ruled out. This is a new physical result, which is similar to the combination resonances of parametrically excited beams. Instability regions are computed numerically with good agreement between analytical and numerical results.

Though various methods are available for stability analysis of rotating systems along with a large number of case studies, the presented approach is advantageous due to its universality and constructive form. It yields qualitative conclusions on possible instability effects directly from eigenvalues and eigenvectors of the unperturbed system, and provides quantitative stability conditions derived explicitly in terms of the perturbed system matrices.

References

[1] V.V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability*, Pergamon, New York, 1963 (Russian original: Fizmatlit, Moscow, 1961).
[2] H. Ziegler, *Principles of Structural Stability*, Blaisdell, Waltham, MA, 1968.
[3] K. Huseyin, *Vibrations and Stability of Multiple Parameter Systems*, Sijthoff and Nordhoff, Alphen aan den Rijn, 1978.
[4] F.I. Niordson (Ed.), Dynamics of Rotors. *Proceedings of IUTAM Symposium*, Lyngby, Denmark, Springer, Berlin, 1975.
[5] G. Spelsberg-Korspeter, D. Hochlenert, O.N. Kirillov, P. Hagedorn, In-and out-of-plane vibrations of a rotating plate with frictional contact: *investigations on squeal phenomena*, *ASME Journal of Applied Mechanics* 76 (2009) 041006.
[6] G. Spelsberg-Korspeter, Breaking of symmetries for stabilization of rotating continua in frictional contact, *Journal of Sound and Vibration* 322 (4–5) (2009) 798–807.
[7] G. Spelsberg-Korspeter, Eigenvalue optimization against brake squeal: *symmetry, mathematical background and experiments*, *Journal of Sound and Vibration* 331 (19) (2012) 4259–4268.
[8] G. Genta, *Dynamics of Rotating Systems*, Springer, New York, 2005.
[9] A. Muszynska, *Rotordynamics*, CRC Press, Taylor and Francis Group, Boca Raton, 2005.
[10] J. Vance, F. Zeiden, B. Murthy, *Machinery Vibration and Rotordynamics*, John Wiley and Sons, Hoboken, NJ, 2010.
[11] A.P. Seyranian, A.A. Mailybaev, *Multiparameter Stability Theory with Mechanical Applications*, World Scientific, New Jersey, 2003.
[12] O.N. Kirillov, Campbell diagrams of weakly anisotropic flexible rotors, *Proceedings of the Royal Society A* 465 (2009) 2703–2723.

[13] O.N. Kirillov, Unfolding the conical zones of the dissipation induced subcritical flutter for the rotationally symmetrical gyroscopic systems, *Physics Letters A* 373 (2009) 940–945.
[14] O.N. Kirillov, Sensitivity of sub-critical mode-coupling instabilities in non-conservative rotating continua to stiffness and damping modifications, *International Journal of Vehicle Structures and Systems* 3 (1) (2011) 1–13.
[15] L. Yang, S.G. Hutton, Interactions between an idealized rotating string and stationary constraints, *Journal of Sound and Vibration* 185 (1995) 139–154.
[16] F.R. Gantmacher, *The Theory of Matrices*, AMS Chelsea Publishing, Providence, RI, 1998 (Russian original: Moscow, 1953).
[17] D.R. Merkin, *Introduction to the Theory of Stability*, Springer, New York, 1997.
[18] A.A. Mailybaev, A.P. Seyranian, On stability domains of Hamiltonian systems, *Journal of Applied Mathematics and Mechanics* 63 (4) (1999) 545–555.
[19] A.A. Mailybaev, A.P. Seyranian, Parametric resonance in systems with small dissipation, *Journal of Applied Mathematics and Mechanics* 65 (5) (2001) 755–767.
[20] A.A. Mailybaev, On stability domains of nonconservative systems under small parametric excitation, *Acta Mechanica* 154 (1–4) (2002) 11–33.