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RENORMALIZATION AND UNIVERSAL STRUCTURE OF BLOWUP IN 1D CONSERVATION LAWS

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Abstract. We discuss universality properties of blowup of classical (smooth) solutions of conservation laws in one space dimension. It is shown that the renormalized wave profile tends to a universal function, which is independent both of initial conditions and of the form of a conservation law. This property is explained in terms of the renormalization group theory. A solitary wave appears in logarithmic coordinates of the Fourier space as a counterpart of this universality. As a numerical example, blowup in ideal polytropic gas is considered.

1. **Introduction.** It is well-known that, in inviscid conservation laws, smooth initial conditions typically give rise to a blowup (singularity) in finite time followed by formation of a shock wave in a weak solution. Development of such finite-time singularities is a classical subject described in every book on nonlinear waves, e.g., [11, 7, 1, 3]. Scalar conservation laws in one space dimension capture a lot of what happens in general systems. In this respect, the inviscid Burgers (also called Hopf) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1}$$

plays a special role. Solution of this equation is easily constructed by the method of characteristics. Each characteristic carries a constant value of u, and the blowup occurs when characteristics cross.

In this paper we show that the wave profile $u(t,x)$ creates a universal "core" just before the shock formation in a generic 1D conservation law. This core is described, up to scaling symmetry, by a function, which is independent of initial conditions as well as of the flux function. We explain the universality using the renormalization group approach, and discuss its relation to more sophisticated universal phenomena described by the theory of renormalization group [12, 6, 5]. Finally, we show that in logarithmic coordinates of the Fourier transformed function $u(t,k)$, the blowup is mapped to a stable solitary wave moving with constant speed. As an example, we describe universal structure of shock formation in ideal polytropic gas.

2. **Universal structure of blowup.** First, let us consider the inviscid Burgers equation (1) with smooth initial condition $u(t_0,x) = u_0(x)$. Its solution $u(t,x)$ constructed by the method of characteristics has the implicit form

$$x = x_0 + u_0(x_0)(t-t_0), \quad u = u_0(x_0), \tag{2}$$

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where x_0 is an auxiliary variable. For spatial derivative of $u(t,x)$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u/\partial x_0}{\partial x/\partial x_0} = \frac{u_0'(x_0)}{1+u_0'(x_0)(t-t_0)}. \tag{3}$$

The denominator vanishes at $t = t_0 - 1/u_0'(x_0)$. This yields the well-known result that the classical solution blows up along the characteristic with the minimum negative value of $u_0'(x_0)$.

We choose the origin of time and space so that the blowup singularity appears at $t = x = 0$, and consider the classical solution in the interval $t_0 \le t < 0$. Also, we can take $u = 0$ at the singularity, which can be achieved by the transformation $x \mapsto x - u_0(0)(t-t_0)$ and $u \mapsto u + u_0(0)$, which leaves (1) invariant. In this case, the initial condition $u_0(x)$ satisfies

$$u_0(0) = 0, \quad t_0 = 1/u_0'(0) < 0, \quad u_0''(0) = 0, \quad u_0'''(0) > 0, \tag{4}$$

which are the blowup conditions at $t = x = u = 0$. Using (4) in (2), for small x_0, we obtain

$$x = ut - \frac{u_0'''(0)}{6u'(0)}x_0^3 + o(x_0^3), \quad u = u_0'(0)x_0 + o(x_0). \tag{5}$$

Equivalently,

$$x = ut - cu^3 + o(u^3), \quad c = \frac{u_0'''(0)}{6(u'(0))^4} > 0. \tag{6}$$

In particular, at $t = 0$, we obtain the well-known singular dependence $u \sim -x^{1/3}$.

Consider now the function

$$u_\lambda(t,x) = \mathcal{G}_\lambda u(t,x) \equiv \lambda^{1/3} u(\lambda^{-2/3}t,\ \lambda^{-1}x). \tag{7}$$

It is easy to see that u_λ is a new (scaled) solution of (1), so (7) represents one of the symmetries of (1). Multiplying both sides of (6) by λ and making the substitution $t \mapsto \lambda^{-2/3}t$, $x \mapsto \lambda^{-1}x$ yields the equation for $u_\lambda(t,x)$ as

$$x = u_\lambda t - cu_\lambda^3 + \lambda\, o\Big((\lambda^{-1/3}u_\lambda)^3\Big). \tag{8}$$

The last (correction) term contains powers $(\lambda^{-1/3}u_\lambda)^n$ with $n > 3$. Hence, it vanishes in the limit of large λ, and (8) takes the exact form

$$x = wt - cw^3 \tag{9}$$

for the limiting function

$$w(t,x) = \lim_{\lambda\to\infty} u_\lambda(t,x). \tag{10}$$

Equation (9) was found earlier in [10] and its linear stability analysis for the inviscid Burgers equation is carried out in [5]. This equation determines the universal function $w(x,t)$ for $t \le 0$, which is independent of initial conditions. Note that different values of the coefficient c correspond to $w(t,x)$ determined up to scaling $\sqrt{c}w(t, x/\sqrt{c})$, which is a symmetry of the Burgers equation (1). The function $w(t,x)$ is a self-similar solution,

$$\mathcal{G}_\lambda w = w. \tag{11}$$

A numerical example of convergence of $u(t,x)$ to the universal function $w(t,x)$ is shown in Fig. 1a.

Now we are going to generalize the above theory by showing that the same limiting function $w(t,x)$ appears in the blowup for a generic scalar conservation law

$$\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = 0 \tag{12}$$

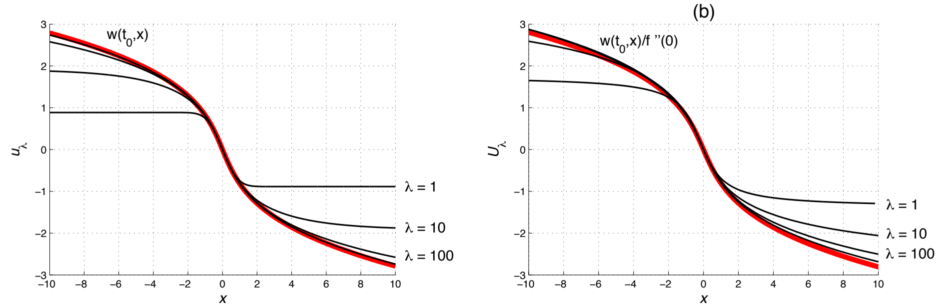

FIGURE 1. (a) Convergence of scaled profiles $u_\lambda(t_0, x)$ (thin black curves) to the universal function $w(t_0, x)$ (bold red curve) for the inviscid Burgers equation (1). The initial condition is $u_0(x) = -(\sqrt{\pi}/2)\mathrm{erf}\, x$ at $t_0 = -1$, and $\lambda = 1,\, 10,\, 10^2,\, 10^3$. (b) Similar convergence results (15) for blowup of a simple wave in ideal polytropic gas. The bold red curve represents the scaled universal function $w(t_0, x)/f''(0)$, and thin back curves are the scaled profiles $U_\lambda(t_0, x) = \mathcal{G}_\lambda U(t_0, x)$ corresponding to the density variation U determined by (16), (18), (21).

with an analytic flux function $f(U)$ and initial condition $U_0(x)$ at $t = t_0$. This fact can be shown as follows. Solution of equation (12) has the form

$$x = x_0 + f'(U_0(x_0))(t - t_0), \quad U = U_0(x_0), \tag{13}$$

which reduces to solution (2) of the Burgers equation by the substitution $u = f'(U)$. We assume that the coordinates for time, space and state are chosen such that (4) holds for $u_0(x) = f'(U_0(x))$, and $u = f'(U)$ is locally invertible as $U = g(u)$ with $g(0) = f'(0) = 0$. This implies the blowup at $t = x = 0$ with $U = u = 0$. We will use the relation

$$\mathcal{G}_\lambda u^n(t, x) = \lambda^{1/3} u^n(\lambda^{-2/3} t,\, \lambda^{-1} x) = \lambda^{(1-n)/3} u_\lambda^n(t, x). \tag{14}$$

Expanding $g(u)$ in Taylor series and using (10), (14), we see that all terms with $n > 1$ vanish for large λ. Since $g(0) = 0$ and $g'(0) = 1/f''(0)$, the remaining terms yield

$$\lim_{\lambda\to\infty} \mathcal{G}_\lambda U(t, x) = \lim_{\lambda\to\infty} \mathcal{G}_\lambda g(u(t, x)) = w(t, x)/f''(0). \tag{15}$$

Conditions (10) and (15) with the function $w(t, x)$ given by (9) demonstrate strong universal character of the blowup of a classical solution for a scalar 1D conservation law. We see that, when the singular point is approached, a part of the wave profile $u(t, x)$ becomes universal, i.e., independent both of the initial condition and of the flux function up to the scaling transformation.

As an example, let us consider formation of a shock wave in a simple wave solution for one-dimensional flow of ideal polytropic gas. The density $\rho(t, x)$ in this wave is described implicitly by

$$\frac{x - x_0}{t - t_0} = \frac{\gamma + 1}{\gamma - 1}\sqrt{A\gamma}\rho_0^{(\gamma-1)/2}(x_0), \quad \rho = \rho_0(x_0), \tag{16}$$

where $\rho_0(x)$ is the initial condition at $t = t_0$, see, e.g., [2]. We will use the values $\gamma = 5/3$, $A = 3/5$, and $\rho_0(x) = 2 - \arctan x$. Then expressions (16) take the form

$$\begin{aligned} x &= x_0 + 4(2 - \arctan x_0)^{1/3}(t - t_0), \\ \rho &= 2 - \arctan x_0, \end{aligned} \tag{17}$$

which can be written in the form (13) for the density variation in moving frame

$$U(t, x) = \rho(t, x + x_1 + v_1(t - t_0)) - \rho_1 \tag{18}$$

with

$$\begin{aligned} f'(U) &= 4(U + \rho_1)^{1/3} - v_1, \\ U_0(x) &= 2 - \arctan(x + x_1) - \rho_1. \end{aligned} \tag{19}$$

The quantities t_0, x_1, ρ_1 and v_1 are chosen such that $f'(0) = 0$ and the function

$$u_0(x) = f'(U_0(x)) = 4(2 - \arctan(x + x_1))^{1/3} - v_1 \tag{20}$$

satisfies conditions (4) corresponding to blowup at $t = x = 0$ with $U = 0$. Values of these quantities (with first three decimal digits) are

$$t_0 = -1.155,\ x_1 = 0.183,\ \rho_1 = 1.818,\ v_1 = 4.882. \tag{21}$$

Convergence (15) of the scaled wave profile $U_\lambda(t, x) = \mathcal{G}_\lambda U(t, x)$ to the universal function is demonstrated in Fig. 1b.

3. **Renormalization group approach.** The universality just described can also be explained using the renormalization group approach. For this purpose, we consider $\mathcal{G}_\lambda$ in (7) as an operator acting in the space of solutions of the Burgers equation (1) with initial conditions satisfying (4). Then the system dynamics can be seen as an action of $\mathcal{G}_\lambda$ combined with the scaling of time, space and state. This operator defines a differentiable group with the property $\mathcal{G}_{\lambda_1}\mathcal{G}_{\lambda_2} = \mathcal{G}_{\lambda_1+\lambda_2}$. The universal function $w(t, x)$ represents a stationary point (11) of the renormalization group operator. Our analysis showed that this stationary point (more precisely, a set of stationary points $w(t, x)$ determined by (9) up to a scaling constant c) is asymptotically stable in the sense of Lyapunov (for λ considered as "time"). From this property, the universality result (10) follows. Note that the linear stability analysis was carried on explicitly in [5].

In the case of a general conservation law (12), the function $U_\lambda(t, x) = \mathcal{G}_\lambda U(t, x)$ is a solution for a conservation law with a different flux function $f_\lambda(U) = \lambda^{2/3} f(\lambda^{-1/3}U)$. Thus, $\mathcal{G}_\lambda$ is a renormalization group operator acting in the functional space $(U, f) \mapsto (U_\lambda, f_\lambda)$ of solutions and fluxes. The universality (15) of the blowup is explained by the fact that $\mathcal{G}_\lambda$ has the asymptotically stable stationary point $(U, f) = (w(t, x), U^2/2)$ corresponding to the universal solution (9) of the Burgers equation (1).

The role of the renormalization group operator $\mathcal{G}_\lambda$ is similar to those in other, much more sophisticated theories. For the inner scale, $x \sim ut \sim t^{3/2}$ in (6), the wave dynamics is governed by the universal function, which is a stationary point of $\mathcal{G}_\lambda$. This is analogous, e.g., to the stationary point of the renormalization group operator, which determines critical phenomena in second-order phase transitions [12]. At larger spatial scales, there is no universality and the solution depends on the initial condition $u_0(x)$ as well as on the flux function $f(u)$. This is analogous, in turn, to the phenomenological Landau theory of second-order phase transitions valid at larger (though still small) deviations of temperature from a critical value [8].

4. **Blowup as a solitary wave in Fourier space.** Finite time blowup implies rapid increase of solution in a short wavelength range. We will see now that the universality of the blowup in x-space induces a solitary wave moving with constant speed to large wave numbers in logarithmic coordinates. Consider the Fourier transform of the solution, $u(t,k) = \mathcal{F}_x[u(t,x)]$. Since $u(t,x)$ is real, we have $u(t,-k) = u^*(t,k)$, and we assume $k \geq 0$ in the analysis that follows. For Fourier transformed functions, the relation (10) yields

$$w(t,k) = \lim_{\lambda\to\infty} u_\lambda(t,k), \tag{22}$$

with the k-space renormalization group operator

$$u_\lambda(t,k) = \mathcal{G}_\lambda u(t,k) \equiv \lambda^{4/3} u(\lambda^{-2/3} t,\, \lambda k). \tag{23}$$

Expression (23) can be checked using Fourier transform of (7). Note that the function $w(t,x)$ grows as $w \sim x^{1/3}$ for large x, see (9), and its Fourier transform (regularized by adding a small imaginary part to x) behaves as $w \sim k^{-4/3}$ for small k.

Let us consider the wave profile $u(t,k)$ transformed to logarithmic coordinates

$$\tau = -\log(t/t_0), \quad \xi = \log k, \tag{24}$$

which we denote by the same letter $u(\tau,\xi)$. Here the blowup at $t = 0$ corresponds to $\tau \to \infty$. Using (23), we find

$$u_\lambda(\tau,\xi) = e^{4a/3} u\left(\tau + \frac{2}{3}a,\, \xi + a\right), \quad a = \log\lambda. \tag{25}$$

Because of (22), we have

$$e^{4a/3} u\left(\frac{2}{3}a,\, \xi + a\right) \to w(0,\xi) \quad \text{as} \quad a \to \infty, \tag{26}$$

where we put $\tau = 0$, and $w(\tau,\xi)$ denotes the universal function $w(t,k)$ written in coordinates (24). Now we substitute a by $3\tau/2$ and write (26) in the form

$$u\left(\tau,\, \xi + \frac{3}{2}\tau\right) \to e^{-2\tau} w(0,\xi) \quad \text{as} \quad \tau \to \infty. \tag{27}$$

Therefore, $u(\tau,\xi)$ forms a wave in the ξ-space for large τ with the profile described by the universal function $w(0,\xi)$. This wave moves with the constant speed 3/2 and decays as $e^{-2\tau}$.

Fourier transforms of derivatives $u^{(n)}(t,x) = \partial^n u/\partial x^n$ have the form $u^{(n)}(t,k) = (ik)^n u(t,k)$. Using logarithmic coordinates (τ,ξ), derivations similar to (24)–(27) yield

$$u^{(n)}\left(\tau,\, \xi + \frac{3}{2}\tau\right) \to i^n e^{(3n/2-2)\tau} e^{n\xi} w(0,\xi) \quad \text{as} \quad \tau \to \infty. \tag{28}$$

Recall that $w \sim k^{-4/3} = e^{-4\xi/3}$ for $\xi \to -\infty$ (small k). Thus, $e^{n\xi} w(0,\xi) \to 0$ as $\xi \to -\infty$ when $n > 4/3$. In this case the wave derivative $u^{(n)}$ forms a solitary wave in the space (τ,ξ), which moves with speed 3/2, grows exponentially as $e^{(3n/2-2)\tau}$, and has the universal shape determined by $e^{n\xi} w(0,\xi)$. Formation of such a wave is shown in Fig. 2 for the second derivative, $n = 2$. Note that this solitary wave can be related to renormalized blowup solutions in shell models of turbulence, see [4, 9].

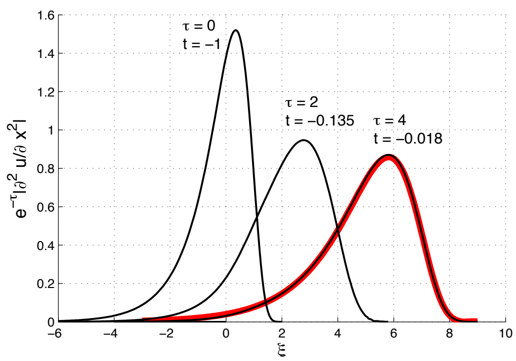

FIGURE 2. Profiles of the scaled second derivative $e^{-\tau}|\partial^2 u/\partial x^2|$ in logarithmic Fourier coordinates (τ, ξ) defined by (24) for the solution shown in Fig. 1(a). Initial condition corresponds to $\tau = 0$, and the blowup corresponds to $\tau \to \infty$. The profile forms a solitary wave traveling with constant speed $3/2$, which reflects the universality of the blowup in x- and k-spaces. The bold red curve shows the exact limiting shape of the wave determined by the universal function $w(t,x)$.

5. **Conclusion.** We showed that solutions of conservation laws in one space dimension possess universal structure when approaching the finite-time singularity (blowup). The limiting wave profile is described by a scaled universal function in x-space, or by a solitary wave moving with constant speed in logarithmic coordinates of (wave number) k-space. The universal function is independent both of initial condition and of flux function. The universal properties are confirmed numerically in Figs. 1 and 2. It is interesting to extend the obtained results to systems of conservation laws and to higher dimensional spaces. In particular, the renormalization technique may be used as an effective tool for the analysis of finite-time singularities for inviscid conservation laws in 2D and 3D.

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