

Motivation

[AT]

Turbulence

- N. S. eq 3D Incompr.
- $t \rightarrow \infty$ average
- $\nu \rightarrow 0$ ($Re \rightarrow \infty$)
- $l \rightarrow 0$ small scales

?
theory
→

Observations

- probability
- intermittency
- scaling anomaly
- ... etc.



Probabilistic models

- symmetries
- conservation laws

→
motivated

- gaussian KIT
- multifractal
- et..

How to make the connections rigorously?

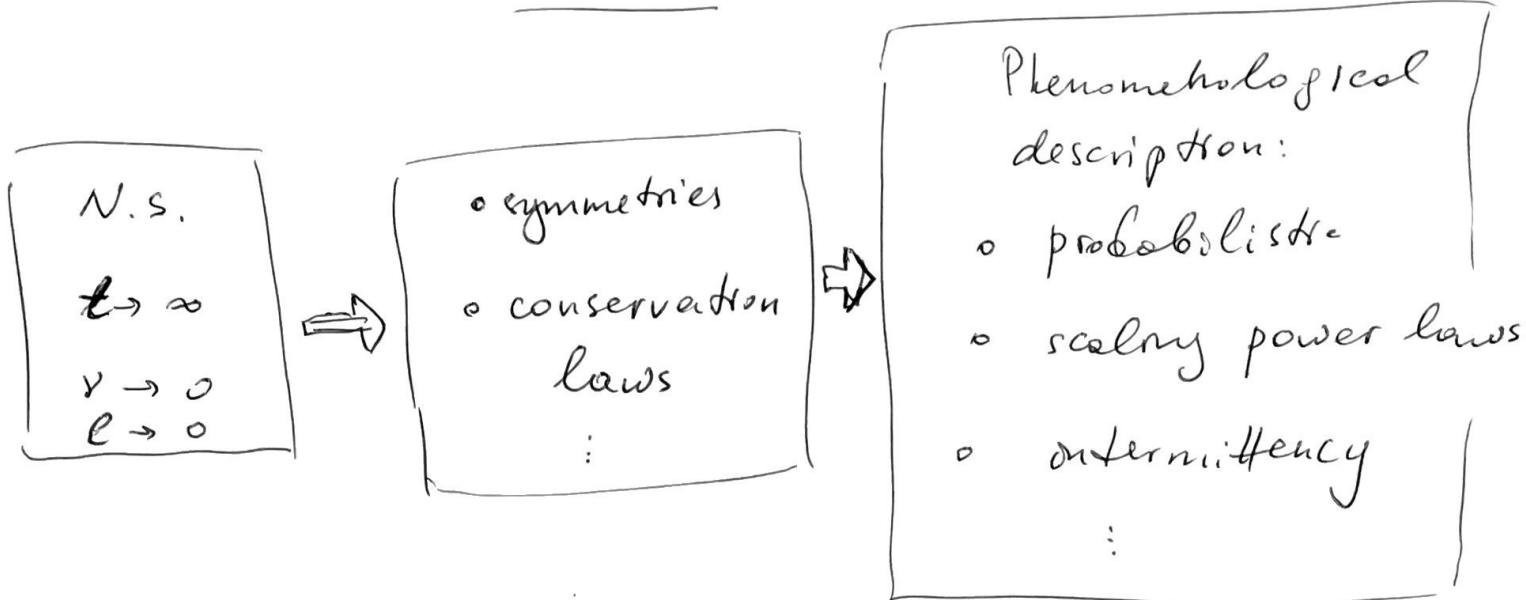
Hypothesis: Properties of turbulence (intermittency) apply to a large class of systems (not just Ns), which share similar properties like symmetries, conservation laws, etc.

Plan

- solvable intermittent model of turbulence
- hidden scaling symmetry in shell models of turbulence \Rightarrow scaling anomaly
- hidden symmetries in NS
- hidden symmetries explained by group theory.
- other topics : HS in dissipation range

$\ell \sim \eta$ vs. $\eta \ll \ell \ll L$
 (dissipation) \uparrow (inertial)
 Kolmogorov scale.

and HS in turbulent transport.

Turbulence

How to make this connection rigorous?

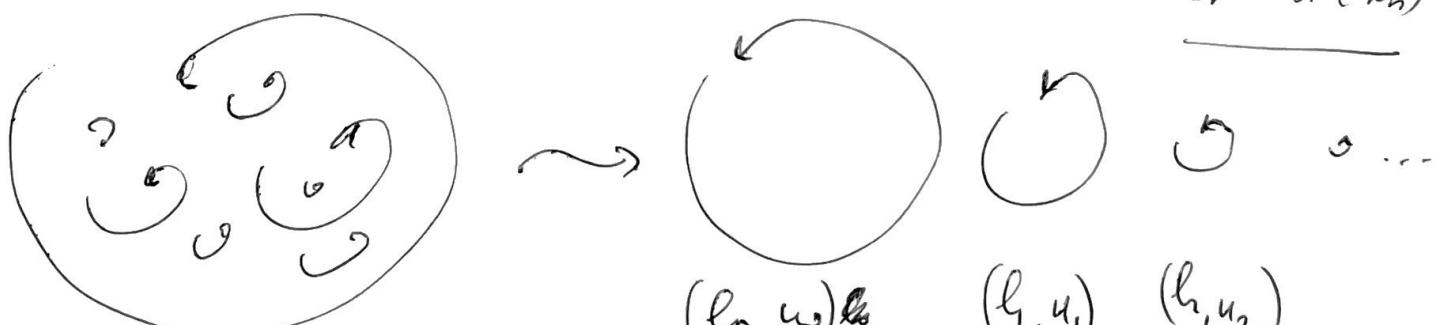
H: Properties of turbulence apply to a larger class of models (not just N.S.), which share similar properties like symmetries, conserv. laws, etc.

Simpler (shell) models

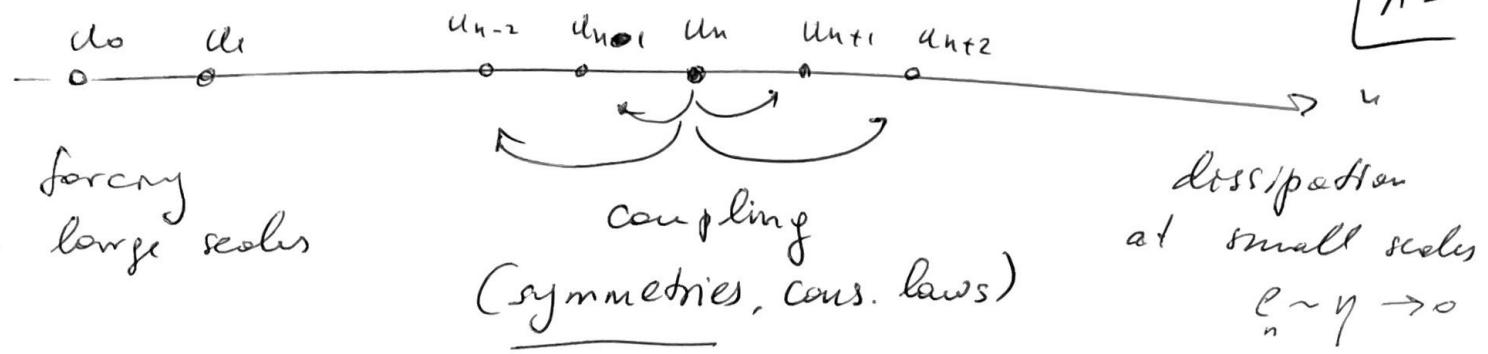
Scales : $l_0 = L = 1, l_1 = 1/2, l_2 = 1/4, \dots, l_n = 1/2^n$.

Wave numbers : $k_n = 1/l_n, k_n = 2^n$.

Shell variable : $u_n \in \mathbb{R}^d \subset \mathbb{C}$. $\sim \sum_{n=1}^{\infty} u(l_n)$
or $\hat{u}(k_n)$



Harmonic analysis: Littlewood-Paley decomposition ($u_n \in$ Banach space).



We keep the scaling symmetry: ~~all g.~~

NS with $\nu=0$ (Euler) : $u_t + u \cdot \nabla u = -\nabla p, \nabla \cdot u = 0$.

$$t, x, u \mapsto \underbrace{\lambda^{1-h}}_{t'}, \underbrace{\lambda x_1}_{x'}, \underbrace{\lambda^h u}_{u'} \quad (\lambda > 0, h \in \mathbb{R})$$

$$u(t, x) \rightarrow u'(t', x') = \underbrace{\lambda^h u}_{\begin{matrix} t \\ x \end{matrix}} \left(\frac{t'}{\lambda^{1-h}}, \frac{x'}{\lambda} \right).$$

Shell model : Take $\lambda = 2, x \sim l_n$

we associate $u(t, x)$ with $u_n(t)$

$$\text{Then } u'(t', x') \rightsquigarrow \lambda^h u_{n+1} \left(\frac{t'}{\lambda^{1-h}} \right). \quad \left(\frac{l_n}{2} = l_{n+1} \right)$$

Scaling symmetry : $t, u_n \mapsto \lambda^{1-h} t, \lambda^h u_{n+1}$ $(\lambda = 2)$
 $(h \in \mathbb{R})$

Constructing scale-invariant equations:

$$u_t + u \cdot \nabla u = \dots$$

$$\downarrow \quad \downarrow \quad \searrow$$

$$\frac{du_n}{dt} + u_n k_n = \frac{1}{l_n} u_n \quad u_n \sim k_n u_n^2$$

type of terms.

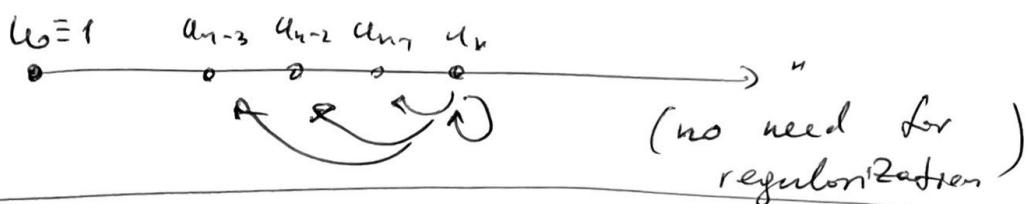
Solvable shell model:

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$$\bullet \quad u_n \in \mathbb{C} \setminus \{0\}, \quad n = 0, 1, 2, \dots$$

Forcing condition: $u_0(t) \equiv 1$.

Coupling:



$$\boxed{\frac{du_n}{dt} = k_n |u_n|^2 F_n + u_n \sum_{m < n} \operatorname{Re}(k_n u_m^* F_m)}$$

$$F_n \text{ (scale invariant factor)} = \left| \frac{u_{n-1}}{u_n} \right|^2 f \left(\frac{u_n}{|u_{n-1}|} \right)$$

$f(z)$ is a C^2 function in $\mathbb{C} \setminus \{0\} \subset \mathbb{R}^2$.

~~Observe~~ Ex: verify scale invariance.

Rescaled equations:

$$\text{multiplier: } w_n = \frac{u_n}{|u_{n-1}|}$$

$$\text{rescaled time: } d\tau_n = \underbrace{k_n |u_{n-1}| dt}_{|u_{n-1}| / \ell_n} \quad \text{Calibration}$$

$$= \frac{dt}{T_n}, \quad T_n = \frac{\ell_n}{|u_{n-1}|} \quad \begin{pmatrix} \text{turnover} \\ \text{time} \end{pmatrix}$$

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$$\begin{aligned}
 \frac{dw_n}{d\tau_n} &= \frac{1}{k_n |u_{n-1}|} \frac{d}{dt} \frac{u_n}{|u_{n-1}|} = \\
 &= \frac{1}{k_n |u_{n-1}|^2} \frac{du_n}{dt} - \frac{u_n}{k_n |u_{n-1}|^3} \frac{d|u_{n-1}|}{dt} \\
 &= \frac{1}{k_n |u_{n-1}|^2} \frac{du_n}{dt} - \frac{u_n}{k_n |u_{n-1}|^3} \operatorname{Re} \left(u_{n-1}^* \frac{du_{n-1}}{dt} \right) \\
 &= \dots = f(w_n)
 \end{aligned}$$

$$\Rightarrow \boxed{\frac{dw_n}{d\tau_n} = f(w_n)} \quad \text{for all } n = 1, 2, \dots$$

Relation between times : $d\tau_1 = k_1 |u_0| dt = 2dt$

$$\frac{d\tau_{n+1}}{d\tau_n} = \frac{k_{n+1} |u_n| dt}{k_n |u_{n-1}| dt} = 2 |w_n|$$



$$(\tau_1 = 2t)$$

This system is scale invariant $\tau_n, w_n \rightarrow \tau_{n+1}, w_{n+1}$.
This is a hidden scaling symmetry (H.S.).

Original variables:

[A5]

$$u_n = w_n |u_{n-1}| = w_n |w_{n-1}| |u_{n-2}| = \dots = w_n |w_{n-1}| \dots |w_1|.$$

~~Elliptic Market Equilibrium & Economics~~

$$dt = \frac{dt_n}{k_n |u_{n-1}|} = \frac{dt_n}{k_n |w_{n-1}| |w_{n-2}| \dots |w_1|}.$$

We solve these equations for

$$f(z) = (R(\theta) + \frac{dR}{d\theta} - \rho + i\rho) e^{i\theta}, \quad z = \rho e^{i\theta}.$$

where $R(\theta)$ is a C^2 function $R: S^1 \rightarrow \mathbb{R}$

with the properties $R(\theta) > 0$, $R(\theta) + \frac{dR}{d\theta} > 0$.

$$w_n = \rho_n e^{i\theta_n}, \quad \frac{dw_n}{dt_n} = f(w_n)$$

$\xrightarrow{\text{(ex)}}$ $\frac{d\rho_n}{dt_n} = \operatorname{Re} \left(e^{-i\theta_n} f(w_n) \right) = R(\theta_n) + \frac{dR(\theta_n)}{d\theta_n} - \rho_n.$

$$\frac{d\theta_n}{dt_n} = \frac{\operatorname{Im} \left(e^{-i\theta_n} f(w_n) \right)}{\rho_n} = 1.$$

~~Elliptic Market Equilibrium & Economics~~

$$\stackrel{(ex)}{\Rightarrow} w_n(\tau_n) = [r_n e^{i\tau_n} + R(\tau_n + \varphi_n)] e^{i(\tau_n + \varphi_n)} \quad \boxed{AG}$$

$$(l_n = \arg w_n(0), \quad r_n = |w_n(0)| - R(l_n)).$$

Theorem: For any initial conditions $u_n(0) \in C^1(\log)$, the shell model has a unique global-in-time solution $u_n(t)$. Every component $u_n(t)$ is bounded, $0 < c_n \leq |u_n(t)| \leq C_n < \infty$ for all times.

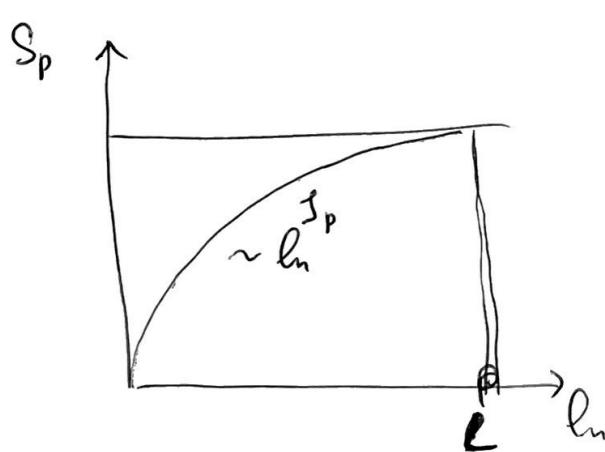
Scaling laws

$$u_n \sim S_p u(l_n)$$

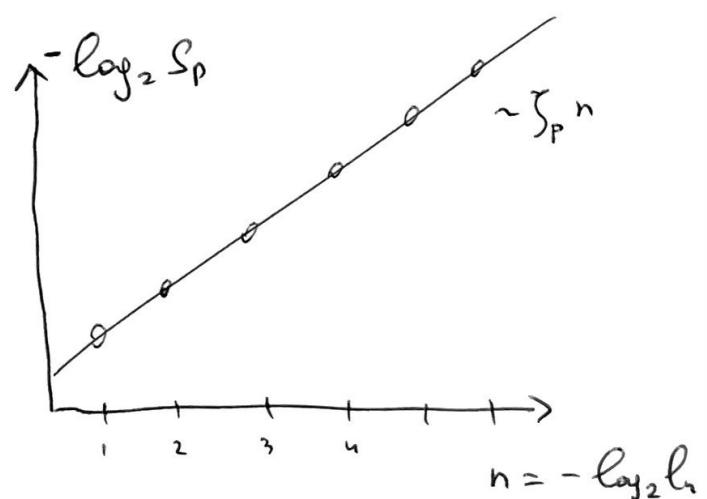
$$\text{Structure functions: } S_p(l_n) = \langle |u_n(t)|^p \rangle_t$$

$$\langle \cdot \rangle_t = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \cdot dt. \quad (\text{time avg})$$

$$\text{Power-law scaling: } S_p(l_n) \underset{?}{\sim} l_n^{-\zeta_p}$$



or



Th Let $R_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(\theta) d\theta$ be AT

transcendental. Then, for any $p \in \mathbb{R}$,

$$S_p(l_n) = l_n^{S_p}$$

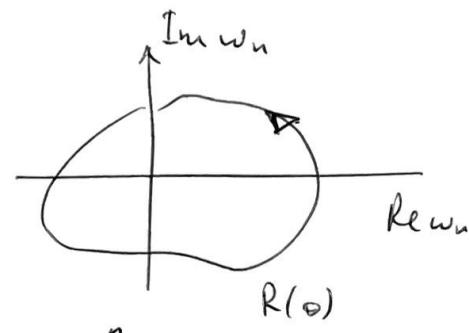
in \mathcal{O}_n

$$J_p = -\log_2 \langle R^p(\theta) \rangle_\theta.$$

Obs Explicit scaling laws for all orders p !
 "Any" J_p can be obtained by choosing $R(\theta)$.

Pr. Take $t \rightarrow \infty$ (attractor.)

$$w_n = R(\theta_n) e^{i\theta_n}, \quad \theta_n = \theta_0 + \varphi_n.$$



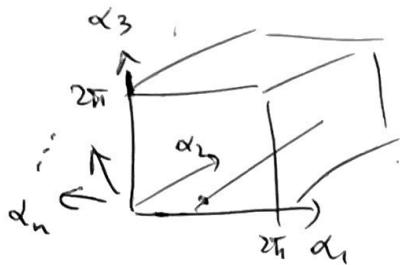
$$\Rightarrow u_n = w_n / |w_n| = |w_n|^{-1} w_n = e^{i\theta_n} \prod_{m=1}^n R(\theta_m).$$

$$\frac{d\theta_n}{dt} = \frac{d\varphi_n}{dt} = k_n |w_{n-1}| \dots |w_1| = k_n \prod_{m=1}^{n-1} R(\theta_m)$$

Lemma \exists change of angles $\alpha_n = \theta_n + h_n(\theta_1, \dots, \theta_{n-1})$

such that $\frac{d\alpha_n}{dt} = k_n R_0^{n-1}$. (ex).

(quasi-periodic motion) $\alpha_n \in S^1$.



$$(\alpha_1, \dots, \alpha_N) \in \underbrace{S' \times \dots \times S'}_N = \overline{\Pi}^N$$

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R_o is transcendental $\Rightarrow k_1, k_2 R_o, k_3 \dots, k_N R_o^{N-1}$

are rationally independent for any N

\Rightarrow (Kronecker - Weyl th.) dynamics of

$\alpha_1, \dots, \alpha_N$ is uniquely ergodic with a uniform inv. measure in $\overline{\Pi}^N$.

Same is true for $\theta_1, \dots, \theta_n$ because the Jacobian of the transformation is unity.

By ergodic th. ($\langle \cdot \rangle_t = \langle \cdot \rangle_{\text{inv. measure}}$):

$$\begin{aligned} S_p(l_n) &= \left\langle |u_n|^p \right\rangle_t = \left\langle \prod_{m=1}^n R^p(\theta_m) \right\rangle_\theta = \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} \prod_{m=1}^n R^p(\theta_m) d\theta_m \right)^n = \left(\frac{1}{2\pi} \int_0^{2\pi} R^p(\theta) d\theta \right)^n \\ &= \left(\langle R^p(\theta) \rangle_\theta \right)^n = \cancel{\text{Diagram}} \frac{-n}{2} J_p^n = l_n^{J_p}. \end{aligned}$$

□

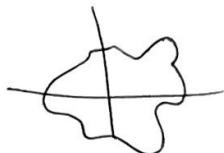
Example $R(\theta) = R_0 \left(1 + \frac{\cos \theta}{2} \right)$, $R_0 = \frac{2\pi}{g}$.

Conclusions

LA9

Invariant probability measure μ_*

$$u_n = e^{i\theta_n} R(\theta_n) R(\theta_{n-1}) \dots R(\theta_1), \quad n = 1, 2, \dots$$



where ~~all~~ phases $\theta_1, \theta_2, \theta_3, \dots$ are i.i.d uniformly distributed. $\theta \xrightarrow{f} u \quad \mu_* = f_* \mu_{\text{unit}}(\theta)$

Ergodic Theorem: For any observable $\varphi(u)$

$$\langle \varphi \rangle_t = \langle \varphi \rangle_{\mu_*}.$$

Multiplicators $|w_n| = \left| \frac{u_n}{u_{n-1}} \right| = R(\theta_n)$

distributed with μ_* are i.i.d. random variables.

\Rightarrow moments $\langle |u_n|^p \rangle_t$ are determined by moments of $\langle R^p \rangle_\theta$.

Symmetries:

all scaling symmetries $t, u_n \mapsto \lambda^{-h} t, \lambda^h u_{n+1}$
are broken. $(\lambda = 2, h \in \mathbb{R})$

But the symmetry of multipliers $w_{n,T_n} \mapsto w_{n+1,T_{n+1}}$ is restored.

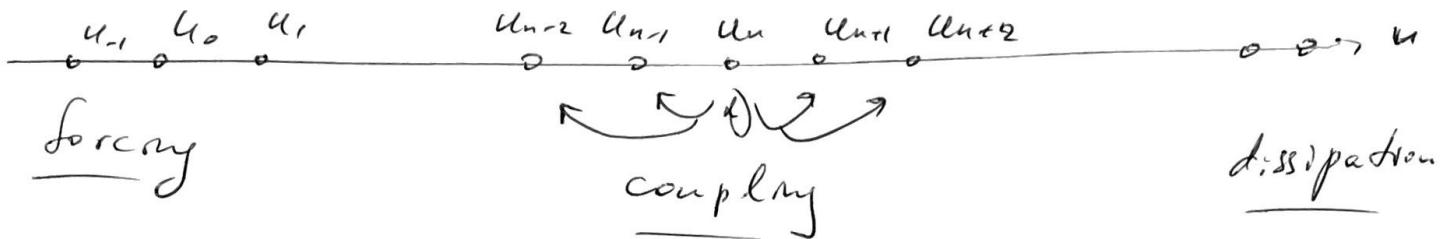
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Shell model of Turbulence (Sabra)

PRF 8,054605 (2023); 7,034604 (2022); 6, L012601 (2021)

Scales: $l_n = 2^{-n}$, Wave numbers $k_n = 1/l_n = 2^n$.

Velocities: $u_n \in \mathbb{C} (\sim \delta u(l_n))$, Shells ~~for~~, $n \in \mathbb{Z}$.



Properties we keep:

Scaling symmetry: $t, u_n \rightarrow \lambda^{1-h} t, \lambda^h u_{n+1}$ ($\lambda = 2, h \in \mathbb{R}$)

Conservation of energy: $E = \sum_n |u_n|^2$

Obs There are some other properties, e.g. cons. of helicity.

Ex Is it possible to make a solvable model?
(open problem)

Sabra model:

$$\frac{du_n}{dt} = i k_n \left[2 u_{n+2} u_{n+1}^* - \frac{u_{n+1} u_{n-1}^*}{2} + \frac{u_{n-1} u_{n-2}}{4} \right] - \nu k_n^2 u_n$$

$$\text{N.S. } u_t = -u \cdot \nabla u - \nabla p + \nu \Delta u$$

Ex: Verify the symmetry and conservation of energy when $\nu = 0$.

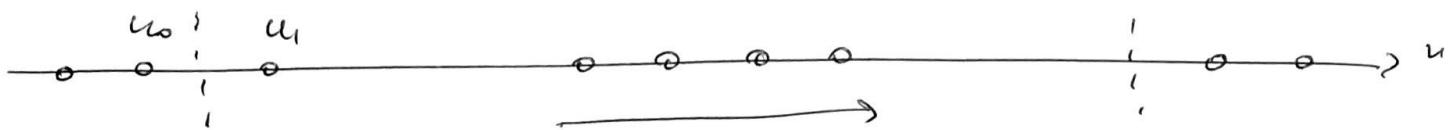
Boundary (forcing) conditions: $u_0(t) \equiv 1$, $u_n(t) \equiv 0$, $n < 0$.

Phenomenology

LAII

We consider the limit $\nu \ll 0$ or

$$Re = \frac{u_0 l_0}{\nu} = \frac{1}{\nu} \gg \infty$$



Forcing range

$$\begin{aligned} l_h &\sim l_0 \\ \text{energy input} & \end{aligned}$$

merbral interval

$$\begin{aligned} l_0 &\ll l_h \ll \eta \\ \text{energy transfer} & \end{aligned}$$

dissipation

$$\begin{aligned} \text{range} \\ l_h &\sim \eta = \nu^{3/4} \\ (\text{K41}) & \end{aligned}$$

Properties of the merbral interval (observations):

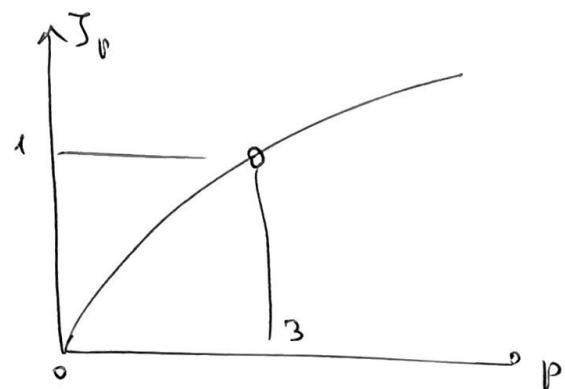
- viscous transfer of energy from large to small scales.
- intermittency



$$S_p(l_h) = \langle |u_{nl}|^p \rangle_t \propto l_h^{-\beta_p} \quad (\approx c_p l_h^{-\beta_p})$$

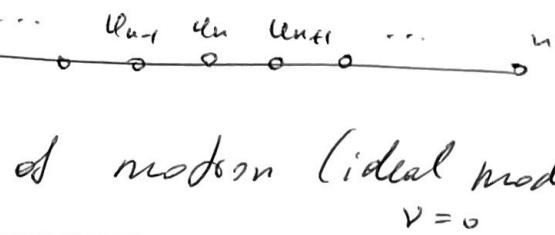
Questions:

- why scaling laws
and what are β_p 's?
- ~~what characterizes~~
why energy transfer?
- what else happens in the merbral range?



Inertial interval dynamics

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Eq. of motion (ideal model): terms are negligible

Assumptions (checked a posteriori)

- dissipation and forcing / bound.
- interactions are local
- ($\sum_{j>0}$ series converge)

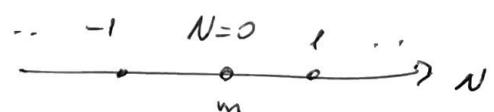
$$\frac{du_n}{dt} = B_n[u], \quad n \in \mathbb{Z}_{\geq 0}$$

$$u = (u_n)_{n \in \mathbb{Z}} = (\dots, u_{n-1}, u_n, u_{n+1}, \dots)$$

$$B_n[u] = ik_n \left[2u_{n+2} u_{n+1}^* - \frac{u_{n+1} u_{n-1}^*}{2} + \frac{u_{n-1} u_{n-2}}{4} \right].$$

Rescaled equations

Fix $m \in \mathbb{Z}$ (reference shell)



Velocity amplitude at shell m :

$$A_m[u] = \sqrt{|u_m|^2 + \alpha |u_{m-1}|^2 + \alpha^2 |u_{m-2}|^2 + \dots}$$

(small $\alpha > 0$)

{just $|u_m|$ yields divergences, other choices yield equivalent theory}

$$A_m[u] = \sqrt{\sum_{j>0} \alpha^j |u_{m-j}|^2}, \quad T_m[u] = \frac{l_m}{A_m[u]}$$

(velocity amplitude) (turnover time)

Rescaled variables:

$$U_N = \frac{u_{m+N}}{A_m[u]}, \quad d\bar{t} = \frac{dt}{T_m[u]}.$$

normalizing velocity & stretching time.

Prop $\mathcal{A}_0[\mathcal{U}] = 0$.

$$\text{Pr: } \mathcal{A}_0\left(\frac{\sum_{n \in \mathbb{Z}} (u_{m+n})_n}{\mathcal{A}_m[u]}\right) = \frac{\mathcal{A}_0[(u_{m+n})_n]}{\mathcal{A}_m[u]} = \frac{\mathcal{A}_m[(u_n)]}{\mathcal{A}_m[u]} = 1.$$

$\mathcal{U} = (u_n)$ are ratios and do not change if $u \rightarrow cu, co$.
 \Rightarrow rescaling is a "projection" (more later)

Rescaled equations (ex.):

$$\frac{d u_n}{dt} = B_n[\mathcal{U}] - u_n \sum_{j \geq 0} \alpha^j \operatorname{Re}(u_{-j} B_{-j}[\mathcal{U}]).$$

No dependence on m !

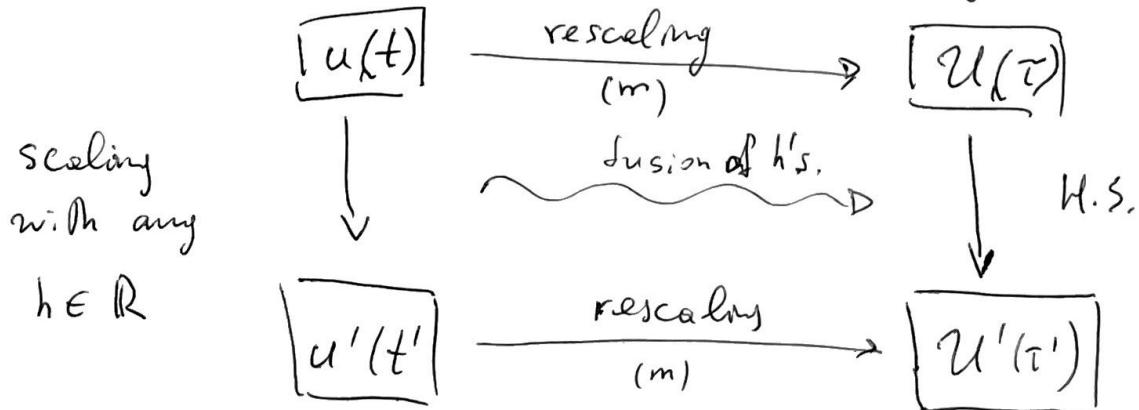
Ex: let $m' = m+1$ and $\mathcal{U}'(\tau')$ is a corresp. rescaled field.

Show that $u'_n = \frac{u_{n+1}}{\sqrt{\alpha + |u_1|^2}}, \quad d\tau' = 2\sqrt{\alpha + |u_1|^2} dt$.

Hidden scaling symmetry:

~~#~~ $u_n \rightarrow \frac{u_{n+1}}{\sqrt{\alpha + |u_1|^2}}, \quad dt \rightarrow 2\sqrt{\alpha + |u_1|^2} dt$.

Relation to original scaling symmetry: $t, u_n \rightarrow \lambda^h t, \lambda^h u_{n+1}$
 $\lambda = 2, h \in \mathbb{R}$.



No $h \in \mathbb{R} \rightarrow !$

$$\dots \xrightarrow{\text{HS}} \boxed{U^{(m-1)}(\tau^{(m-1)})} \xleftrightarrow{\text{HS}} \boxed{U^{(m)}(\tau^{(m)})} \xleftrightarrow{\text{HS}} \boxed{U^{(m+1)}(\tau^{(m+1)})} \xrightarrow{\text{HS}} \dots$$

Conjecture: H.S. is restored statistically for rescaled variables $U_n(\tau)$, i.e., their statistics do not depend on m .

→ numerical verification + ~~symbols~~ orig. vs. rescaled dynamics

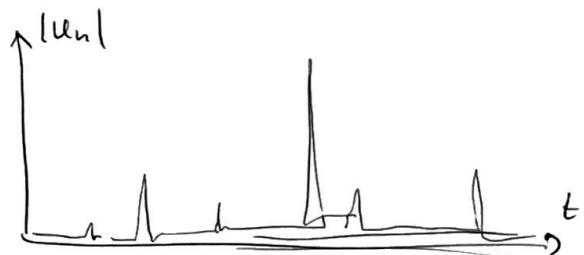
(Kolmogorov) multipliers

$$X_N[u] := \frac{d_n[u]}{d_{n-1}[u]} \quad (\text{ex.}) \quad \frac{d_n[u]}{d_{n-1}[u]}, \quad n = m+N$$

ratio of velocity amplitude

HS \Rightarrow statistics of multipliers does not depend on m → numerical verification.

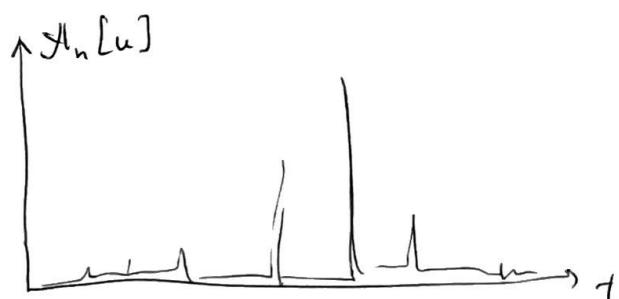
Intermittency



Structure functions

$$S_p(l_n) \stackrel{\text{def}}{=} \left\langle d_n^p[u] \right\rangle$$

$$S_p(l_n) \propto l_n^{\beta_p}$$



Questions:

- Why power law: $S_p(l_m) \propto l_m^{-\zeta_p}$?
- What are ζ_p 's ?
- ζ_p 's \rightarrow intermittency ?

We assume

- ergodicity for measures $\mu^{(m)}(u)$ of rescaled field ~~of~~ $u^{(m)}(\tau^{(m)})$: $\langle \cdot \rangle_{\substack{\text{time} \\ \tau^{(m)}}} = \langle \cdot \rangle_{\text{measure}}$
- statistical H.S.: $\mu^{(m)} \approx \mu^*$ does not depend on m .

Structure functions in terms of μ^*

$$\mathcal{A}_0[u] = 1, \quad \mathcal{A}_m[u] = \frac{\mathcal{A}_1[u]}{\mathcal{A}_{m-1}[u]} \dots \frac{\mathcal{A}_r[u]}{\mathcal{A}_0[u]} = \prod_{j=0}^{m-1} \chi_{-j}[u^{(m)}].$$

$$\frac{dt}{d\tau^{(m)}} = \frac{l_m}{\mathcal{A}_m[u]} = l_m \left(\prod_{j=0}^{m-1} \chi_{-j}[u^{(m)}] \right)^{-1}.$$

We denote $\mathbf{x} = (\chi_0[u], \chi_1[u], \chi_2[u], \dots)$ and the corresponding measure $\mu^*(\mathbf{x}) = \chi_{\#} \mu^*(u)$.

$$S_p(l_m) = \langle \mathcal{A}_m^p[u] \rangle_+ = \left\langle \prod_{j=0}^{m-1} \chi_{-j}^{p-1}[u^{(m)}] \right\rangle_+$$

$$= \frac{\left\langle \prod_{j=0}^{m-1} \chi_{-j}^{p-1} [\cancel{u^{(m)}}] \right\rangle_{\tau^{(m)}}}{\left\langle \prod_{j=0}^{m-1} \chi_{-j}^{-1}[u^{(m)}] \right\rangle_{\tau^{(m)}}} = \frac{\left\langle \prod_{j=0}^{m-1} \chi_{-j}^{p-1} \right\rangle_*}{\left\langle \prod_{j=0}^{m-1} \chi_{-j}^{-1} \right\rangle_*}.$$

$$\Rightarrow S_p(l_m) = \int d\mu_p^{(m)}(\mathbf{x}), \quad d\mu_p^{(m)}(\mathbf{x}) = \frac{\prod_{j=0}^{m-1} \mathbf{x}_{-j}^{p-1} d\mu^*(\mathbf{x})}{\left\langle \prod_{j=0}^{m-1} \mathbf{x}_{-j}^{-1} \right\rangle_*}.$$

Lemma $d\mu_p^{(m+1)}(\mathbf{x}) = x_0^p \underbrace{p^*(x_0 | \mathbf{x}_1, \mathbf{x}_2, \dots)}_{\text{conditional prob. density}} d\mathbf{x}_0 d\mu_p^{(m)}(\mathbf{x}).$

$\mathbf{x}_- = (x_1, x_2, \dots)$, $S_p(\mathbf{x}) = (x_-, x_{-2}, \dots)$ - shift

In short, $d\mu_p^{(m+1)} = L_p d\mu_p^{(m)}$

linear positive operator. positive measure
to
positive measure

Pr Use H.S. $\mu^{(m)}(\mathbf{x}) = \mu^*(\mathbf{x})$ and relations between $\mu^{(m)}$ and $\mu^{(m+1)}$. \rightarrow papers □

Corr. $\mu_p^{(m)} = L_p d\mu_p^{(m-1)} = L_p^2 d\mu_p^{(m-2)} = \dots = L_p^{m-m_0} d\mu_p^{(m_0)}$

$(m_0$ is a large-scale end of mental interval).

Perron-Frobenius Th.: L_p has ~~all~~ eigenvalue $\lambda_p > 0$, which is ~~the~~ dominant (larger than abs. val. of other eig.).

~~Let's see~~ $L_p d\nu_p = \lambda_p d\nu_p$

eigenvector $\underbrace{\nu_p}_{\text{P.F. eigenvalue.}}$
 $\text{measure (normalized to unity).}$

$\Rightarrow d\mu_p^{(m)} \approx c_p \lambda_p^m d\nu_p \quad \text{for large } m.$

~~Let's see~~ $\sum d\mu_p^{(m)}$

$$S_p(\ell_m) = \int d\mu_p^{(m)} = C_p \lambda_p^m = C_p 2^{m \log_2 \lambda_p}, \quad \ell_m = 2^m.$$

$$\Rightarrow S_p(\ell_m) = e_p \ell_m^{I_p}, \quad I_p = -\log_2 \lambda_p$$

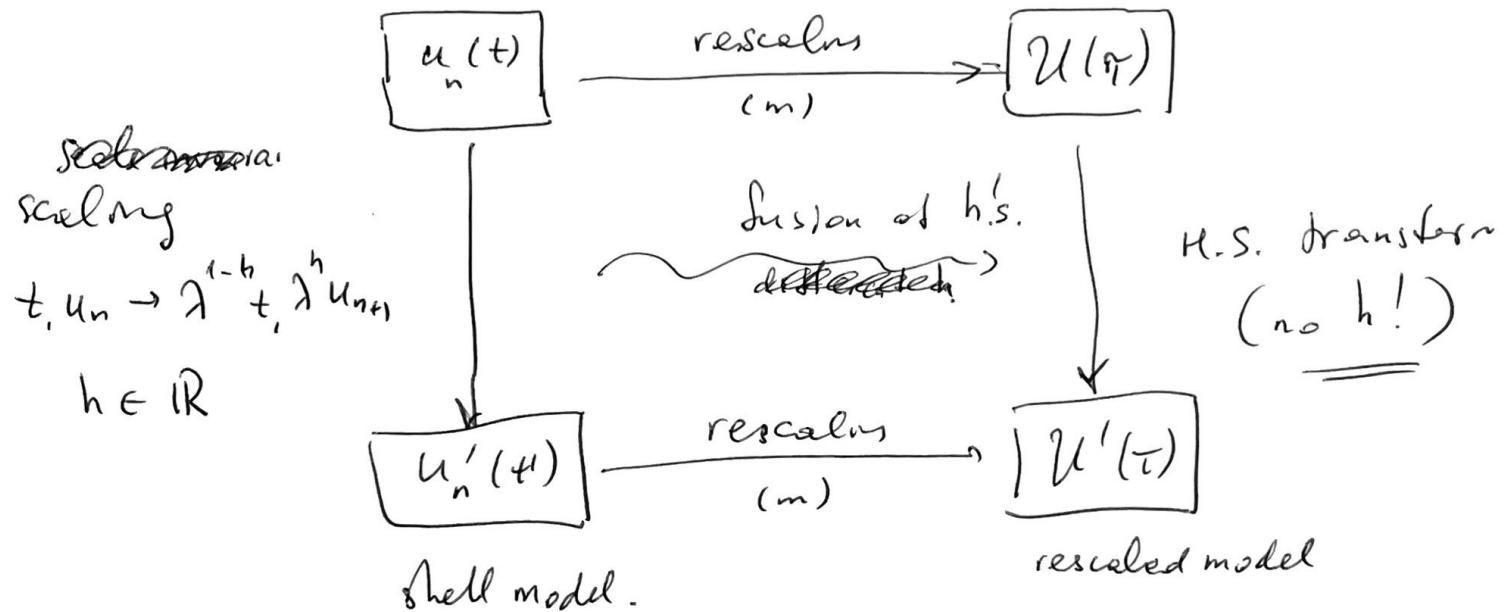
Summary

(A18)

Shell model: $\ell_n = 2^{-n}$, $u_n \in \mathbb{C}$.

Inertial interval: no forcing, no damping
(scale invariance)

① Hidden symmetry



② Structure functions.

$$\mathcal{I}_p(\ell_m) = \langle \mathcal{A}_m^p[u] \rangle = \int \mu_p^{(m)}(dx) \quad \begin{matrix} \text{amplitudes} \\ \text{Multipliers } x = (x_0, x_{-1}, x_{-2}, \dots) \end{matrix}$$

Iterative relation: $\mu_p^{(m+1)}(dx) = \mathcal{L}_p[\mu_p^{(m)}] =$
 $= x_0^p \mathcal{P}_x(dx_0|x_-) \mu_p^{(m)}(dx_-)$, $x_- = (x_{-1}, x_{-2}, \dots)$
(like in multiplicative Markov process).

$$\Rightarrow \mu_p^{(m)} \rightarrow c_p \lambda_p^m \nu_p \text{ for P.F. mode } L_p \nu_p = \lambda_p \nu_p.$$

$$\Rightarrow \boxed{\mathcal{I}_p(\ell_m) = c_p \ell_m^{S_p}, \quad S_p = -\log_2 \lambda_p}$$

Conclusions

[A19]

- Power laws for $S_p(\ell_m)$ follow from H.S. and Perron-Frobenius th.

Numerical verification: compute $\gamma_p = \frac{\mu_p^{(m)}}{c_p \lambda_p^m} = \frac{\mu_p^{(m)}}{\int \mu_p^{(m)}}$

(must be independent of m).

Obs: similar to multiplicative Markov process, BUT involves

- Dependence of γ_p on p . a nonlinear change of time.

Example: $\rho^*(x_0 | x_{-1}, \dots) = \rho^*(x_0)$.

$$\Rightarrow d\mu_p^{(m+1)} = (x_0^p \rho^*(x_0) dx_0) d\mu_p^{(m)} (\underbrace{s(x)}_{x_{-1}, x_{-2}, \dots})$$

Integrate $\Rightarrow \gamma_p = \int x_0^p \rho^*(x_0) dx_0 = \langle x_0^p \rangle_*$.

Like in ~~ex~~ models we considered in Lect. 3 and 4.

- Usual structure functions:

$$\langle |\mu_m|^p \rangle_t = \int (x_0^2 - 1)^{p/2} d\mu_p^{(m)}(x) \approx \tilde{C}_p \ell_m^{\gamma_p}$$

with $\tilde{C}_p = c_p \int (x_0^2 - 1)^{p/2} d\nu_p(x)$.

Obs: power laws for measures \Rightarrow power laws for observables!

• Intermittency: ~~defn~~

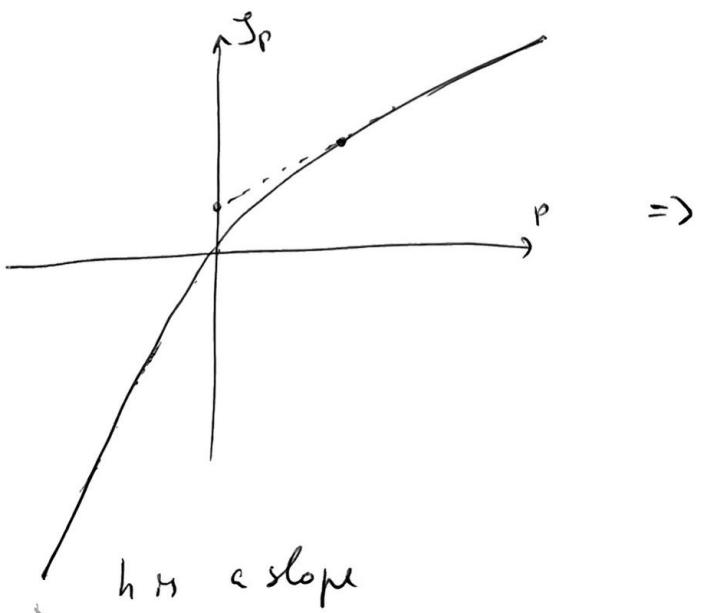
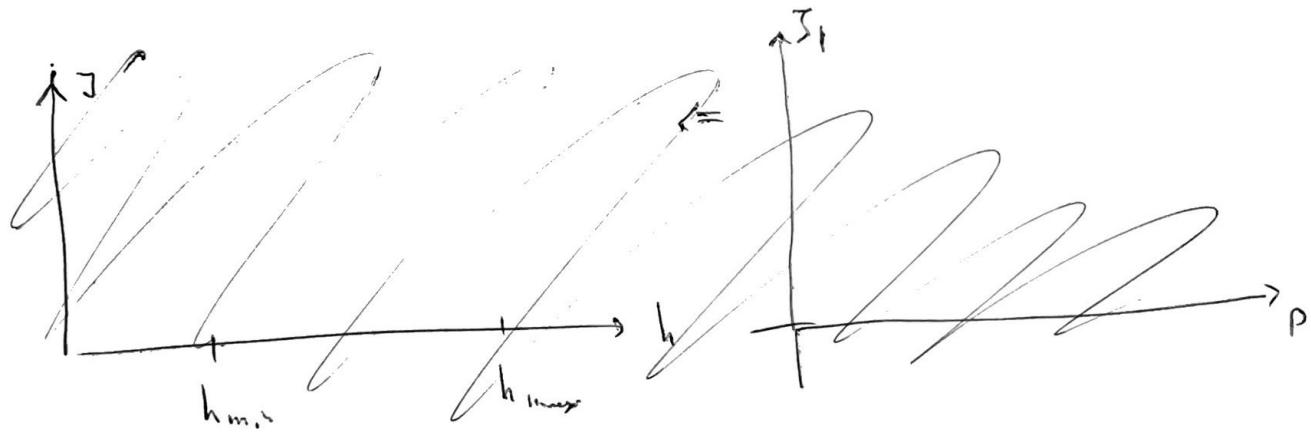
let $\langle A_m[u] \rangle \propto l_m^{J_p}$ with a differentiable exp.

J_p for all $p \in \mathbb{R}$. Then the Göttsche-Gärtner-Ellis theorem yields the Large Deviation Principle:

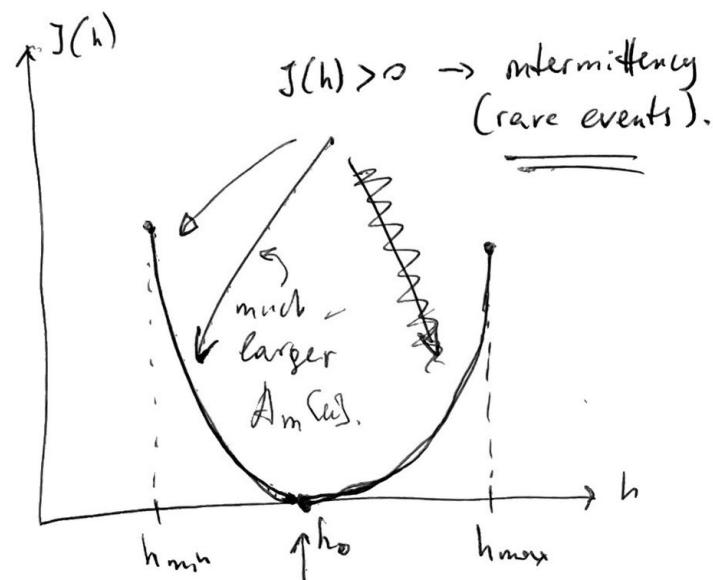
$$A_m[u] \sim l_m^{J(h)} \text{ with probability } P \sim l_m^{-J(h)}$$

where $J(h)$ is the rate function

$$J(h) = \sup_{p \in \mathbb{R}} (J_p - ph).$$



$J(h)$ is inters. with J_p -axis.



typical (probabilist 1)
Hölder exponent h .