8/1/2024.
Inya.

Course 1 : Navier-Stokes, Basics & turbulence.

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① Incompressible fluid dynamics.

- Navier-Stokes / Euler.
- Domain
- Rationale.
- Pressure field.
- Physically reasonable solutions.
- History.

② Reynolds number.

- Dimensional analysis.
- Reynolds number.
- Similarity principle.

③ Dynamical Symmetries ($f = 0$)

- Discrete.
- Continuous.

④ Energy conservation

- Finite Reynolds.
- Euler.
- Dissipation anomaly.

⑤ Openings. / Concluding remarks.

Navier-Stokes pose problem both for Math (Einstein) & Physicists (should then be turbul

Understanding the properties of these equations, we will even rely caricature them. minimal models!

(i) Incompressible fluid dynamics.

To the best of our knowledge, the motion incompressible fluids are faithfully captured by the

Navier Stokes $\quad \partial_t \vec{u} + \vec{u}\cdot\nabla u + \nabla p + \frac{1}{\rho_0}\nabla p = \nu \Delta u + f.$

$$\nabla . u = 0.$$

with $\quad \vec{x} \in \mathcal{D} \subset \mathbb{R}^3.$

$\vec{u} = (u_1, u_2, u_3) \quad$ 3-dimensional vector field. $t \times \mathcal{D} \to \mathbb{R}^3$.

$p(t, \vec{x}) \quad$ (scalar) pressure field.

$\nu \geqslant 0 \quad$ [kinematic viscosity].

Boundary conditions: in principle solid boundaries

$\mathcal{D}$ compact $\quad \partial D : \vec{n} \cdot \vec{u} = 0$.

To simplify the discussion, we will consider $\mathcal{D} = \mathbb{R}^3$

but enforce periodicity for $\vec{u}, \vec{P}$, ie:

$$\begin{cases} \vec{u}(\vec{x}_1 + n_x e_1, y + n_y L \vec{e}_y, z + n_z L \vec{e}_z) = \vec{u}(\vec{x}). \\ \end{cases}$$

We recover the case $\mathcal{D} = \mathbb{R}^3$ by letting $L \to \infty$.

More than often, we will use $L = 2\pi$.

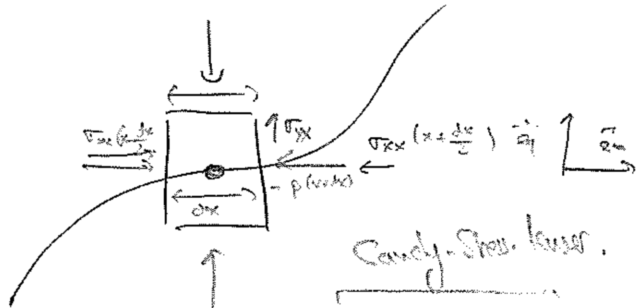
The case $\nu = 0$: with the same periodic BC provide the

Euler equations.

Rationale: [Lagrangian viewpoint].

The NS describe the motion of a fluid particle subject to internal forces, volume force and external ones.

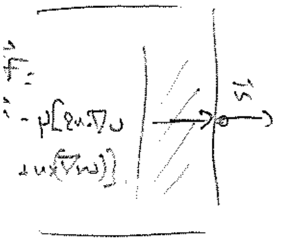
$$\begin{cases} \dfrac{dx}{dt} = u(x,t), \\ x(t=0) = x_0. \end{cases}$$



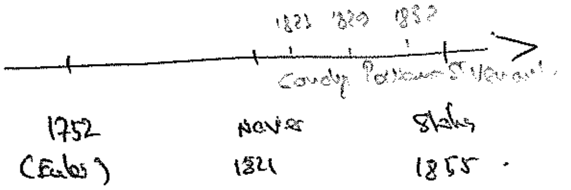
Cauchy-Stress tensor.

with $\vec{u}$: $\rho_0 \dfrac{d}{dt} \vec{u}[x(t),t] = \boxed{\begin{matrix}-\nabla p \\ + \rho_0 \nu \nabla\cdot\tau\end{matrix}} = -\nabla\cdot\sigma \qquad \sigma_{ij} = p\delta_{ij} + \mu(\partial_j u_i + \partial_i u_j)$

$+ f$ (gravity, magnetic field, coriolis).



History



'World of flows', Darrigol.

Pressure field. $P(x,t)$ enforces the incompressibility condition as a "Lagrange multiplier" through the Poisson equation: $-\Delta \dfrac{P}{\rho_0} = Tr(\nabla u)^2 = \nabla\cdot(u\cdot\nabla u)$.

Obs • The NS can be formally written as:

$$\partial_t u + P_L[u\cdot\nabla u] = \nu\Delta u + \vec{f}.$$

with $P_L[\sigma] = \sigma - \nabla\Delta^{-1}\nabla\cdot\sigma$. "Leray Projector"

• The Leray P_L entails non local interaction:

$\mathbb{R}^3$ $\quad p = G_{3D} \circledast Tr(\nabla u)^2 \qquad P_L[\sigma] = \sigma - \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^3} \dfrac{x-y}{\|x-y\|^3}\,\sigma(y)\,dy.$

Periodic $\quad \hat{P}_L[\hat{\sigma}][\vec{k}] = \left(\delta_{ij} - \dfrac{k_i k_j}{|k|^2}\right)\hat{\sigma}_j. \qquad p = \dfrac{1}{4\pi}\displaystyle\int \dfrac{1}{\|x-y\|}\,Tr(\nabla u)^2(y)\,dy.$

Physically reasonable solutions (PRS)

Solutions that do not grow large as $|x| \to \infty$

Periodic case: u^0 smooth

$p, u \in C^\infty(\mathbb{R}^3 \times [0;\infty[)$.

$\mathbb{R}^3$: (1) Bounded energy. $\int_{\mathbb{R}^3} |u|^2 < C$ for all t.

Whether physically reasonable solutions exist is a notoriously hard problem, that we here take for granted :)

cf Fefferman "Existence and smoothness of the NS"

Tao " why global existence existence is hard"

OBS:

For the modeling of turbulence, we will need to consider arbitrary long solutions to define statistically steady state - we take the existence of physically reasonable solutions for granted [and this is supported by the numerics].

② Reynolds number.

- As a (physical) equation, the terms featured in NS have a physical dimension!

Clearly: $[\partial_t u] = [u.\nabla u] = [\nabla p] = [\nu \Delta u] = [f] = \frac{U}{T} = \frac{U^2}{L}$.

kinematic viscosity $[\nu] = \frac{L}{T^2}$ (m/s^2) : diffusion coefficient.

- Typical values.

Air	Water	Honey	Asphalt.
10^{-5}	10^{-6}.	10^{-3}	10^{5}

- Similarity principle. (first symmetry).

Under: $x \to x' = \frac{x}{L}$. $\quad T \to T' = \frac{t\, U_0}{L}$. $\quad u \to u' = \frac{u}{U_0}$.

$p \to p' = \frac{p}{\rho U_0^2}$ $\quad f \to f' = \frac{U_0^2}{L} f$, the NS equations rescale into

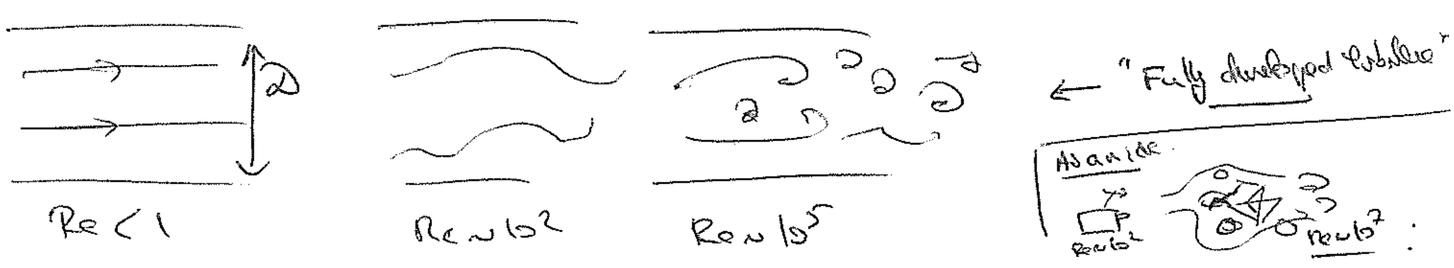
$$\partial_t u' + u'.\nabla' u' + \nabla' p' = \frac{1}{Re} \Delta' u' + f'$$

with $Re = \frac{UL}{\nu}$ (Reynolds number)

with new domain $\mathcal{D} \to \mathcal{D}/L$.

Physical meaning: the feature of the fluid depend only on Re (+ geometry).

Reynolds Experiment (1883) "The circumstance which determine whether the motion shall be direct or sinuous."



3 Dynamical symmetries.

- To understand the turbulent state, a mediation between observations, NS and minimal models will be the concept of symmetry.

- NS is invariant over various symmetry groups: G.

> G is a symmetry group $\iff \forall g \in G$ u_j sol to NS $\implies g\,u_j$ sol to NS.

- Symmetry group of the NS under $f=0$ and $\mathcal{D} = \mathbb{R}^3$.

Discrete: $x \to -x$, $t \to t$, $u \to -u$, $\nu \to \nu$.

Continuous:

Space translation	$x \to x + \rho$	$t \to t$	$u \to u$	$\nu \to \nu$	$\rho \in \mathbb{R}^3$
Time translation	$x \to x$	$t \to t + T$	$u \to u$	$\nu \to \nu$	$T \in \mathbb{R}$
Rotation	$x \to Ox$	$t \to t$	$u \to Ou$	$\nu \to \nu$	$O \in SO_3(\mathbb{R})$
Galilean invariance	$x \to x + U_0 t$	$t \to t$	$u \to u + U_0 t$	$\nu \to \nu$	$U_0 \in \mathbb{R}^3$
Scaling	$x \to \underbrace{\lambda x}_{x'}$	$t \to \lambda^{1-h} t$	$u \to \lambda^h u$	$\nu \to \lambda^{1+h} \nu$	$\lambda \in \mathbb{R}, h \in \mathbb{R}$

OBS: this means (Ex scaling) $u_j(x,t)$ sol $\;\; u'_j(x',t') = \lambda^h u_j\left[\frac{x'}{\lambda}, \frac{t'}{\lambda^{1-h}}\right]$ idem.

OBS: $h = -1$: similarity principle!

(4) Energy conservation.

Energy budget

Local: $\partial_t \frac{u^2}{2} + \nabla\cdot\left[u\left(\frac{u^2}{2}+p\right) - \nu\nabla\frac{u^2}{2}\right] = -\nu(\nabla u)^2 + f\cdot u$

global: $\partial_t\langle\frac{u^2}{2}\rangle_\nu = -\nu\langle(\nabla u)^2\rangle_\nu + \langle f\cdot u\rangle_\nu$.

où $\langle\ \rangle_\nu = \frac{1}{L^3}\int_{x\in\Omega}$

∂/ Local

$$u\cdot(u\cdot\nabla u) = u_i u_j \partial_j u_i = u_j\partial_j\frac{u^2}{2} = \partial_j u_j\frac{u^2}{2}.$$

$$u\cdot\nabla p = \partial_i(u_i p)$$

$$u\cdot\nabla^2 u = u_i\partial^2_{jj}u_i = \underbrace{\partial_j(u_i\partial_j u_i)}_{\partial_{jj}\frac{u^2}{2}} - \underbrace{(\partial_j u_i)(\partial_j u_i)}_{(\nabla u)^2}$$

$$= \Delta\frac{u^2}{2} - \|\nabla u\|^2$$

OBS
- $I_\nu = \langle f\cdot u\rangle_\nu$ injection
- $\varepsilon_\nu = -\nu\langle(\nabla u)^2\rangle$ dissipation.

• For pr. to Euler, $\nu = 0$ entails $\partial_t\langle\frac{u^2}{2}\rangle_\nu = \cancel{I}$.
The only steady state which is allowed is the vanishing one $I = 0$ (Eq).
Alternative syte if $I = \cancel{\ldots}$ $\langle\frac{u^2}{2}\rangle \nearrow$: no steady-state.

• Observation shows that $\begin{cases}\varepsilon_\nu \to \varepsilon_\infty > 0 \\ I_\nu \to I\end{cases}$ for fixed forcing.

In the limit $\nu\to 0$ the pr NS flows dissipate without viscosity.

This defines the NS turbulent limit as the limit $\mathcal{T}\to\infty, \nu\to 0$, at fixed forcing.

- (PRS) to Euler are not good candidate for the turbulent limit!

- Anomalous dissipation implies emergence of rough structures

with $\langle (\nabla v)^2 \rangle \xrightarrow[\nu \to 0]{} \infty$. We will explore

the statistical structures of NS turbulence in more details

in the next course.

- The aim of this minicourse is to provide

(qualitative) explanation for the key structures of NS

turbulence through symmetries and covariance of NS turbulence.