Course 2 : (Data-driven) turbulence.

O

or " a gaussian representation of NS turbulence "

(1) Steady-state NS turbulence. (A Hell of a dissipative machine).

- Turbulent measure.
- Observations.

(2) Random functions - Short survey.

- Random variables, Law, Equivalence in Law
- Independence
- Processes - FDD - Equivalence in Laws
- Gaussian process.
- Stationary (Homogeneous) processes.

(3) Gaussian HIT representation of NS turbulence.

- Two-point correlator.
- Increments, power law, self-similarity

(4) NS vs HIT: Kolmogorov - Frisch scenario.

- Statistical restoring of symmetries.
- Mono scaling symmetry.

(5) $\nu > 0$: probing HIT with NS.

- Running Reynolds number
- Inertial range.

(6) Openings : Gaussian HIT faithfull

① Steady-state (NS) turbulence.

- The NS equation provides an efficient way of generating "turbulent state" as a limit $t \to \infty$, $\nu \to 0$, of physically reasonable solutions.

- "Turbulent state": ~~stationary long-time~~ "measure"

  φ observable of the space of suitable velocity field.

$$\langle \varphi[v] \rangle_{t,\nu} = \frac{1}{T} \int_0^T \varphi[u_\nu]\, ds .$$

$$\text{"Turbulent state"} = \lim_{\nu \to 0} \lim_{t \to \infty} \langle \ \rangle_{t,\nu} = \langle \rangle_* .$$

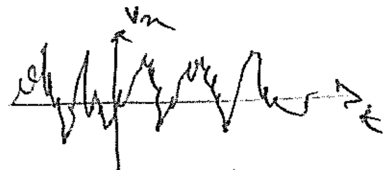
with the scenario: $\underbrace{\langle \varepsilon_\nu \rangle}_{\langle \nu |\nabla u|^2 \rangle} = \langle I \rangle_* \longrightarrow I = \langle f, u \rangle .$

- Numerical observations from triply periodic numerical simulations.

Definition of the "turbulent state"

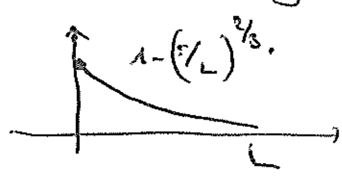
For sufficiently high Reynolds, the velocity field behaves as a random function

(I) stationary.



(II) Power-law correlations on a certain range of scale.

$$C(r) \simeq \left[1 - \left(\frac{r}{L}\right)^{2/3}\right]$$
$$\eta \ll r \ll L$$



(III) Finite-variance: $\langle u_\nu^2 \rangle_* \longrightarrow U^2_{\text{rms}} > 0 .$

(IV) Finite-Dissipation $\langle \varepsilon_\nu \rangle_* \xrightarrow[\nu \to 0]{} \varepsilon > 0 .$

OBS: (III) & (IV) are non trivial!

② Random functions. Vocabulary

Jacob. Rosa. Tois Frisch Durrott, Stoch Processes Applications

Probability Space:

$$(\Omega, \mathcal{F}, \mathbb{P})$$

events.

Probability.

outcomes / realization

$\mathcal{F}$ σ-algebra, $\mathbb{P}: \mathcal{F} \to \mathbb{R}^+$

$\mathbb{P}[\Omega] = 1$ $\quad \mathbb{P}[\emptyset] = 1$

$\mathbb{P}[\bigcup_{i \in \mathbb{N}} A_i] = \sum_i \mathbb{P}[A_i]$ $\quad A_i \cap A_j$

Event with probability 1: "almost sure"

Random variable: $X: \Omega \to \mathbb{R}^n$ such that $\forall I$ interval of $\mathbb{R}^n$ $X^{-1}(I) \in \mathcal{F}$

$I = \bigotimes_{i=1}^n]x_i ; y_i]$

$$\mathbb{P}_X [I] := \text{"} \mathbb{P}[X(\omega) \in I] \text{"} = \mathbb{P}[\underbrace{X^{-1}(I)}_{\in \mathcal{F}}].$$

$\mathbb{P}_X$: pushforward of $\mathbb{P}$ by X., sometimes written $X \# \mathbb{P}$.

Averages: Let $\varphi[X]$ a functional of x [observable].

Then
$$\langle \varphi[X] \rangle_{\mathbb{P}} \doteq \int_{\Omega} \varphi[X(\omega)] \, \mathbb{P}[d\omega].$$
$$= \int_{x \in \mathbb{R}^n} \varphi[x] \, \mathbb{P}_X [dx].$$
$$= \int_{x \in \mathbb{R}^n} \varphi[x] \, (X \# \mathbb{P})(dx).$$
$$= \langle \varphi \rangle_{X \# \mathbb{P}}. \quad = \langle 1 \rangle_{\varphi \circ X \# \mathbb{P}}.$$

$\mathbb{P}_X = X \# \mathbb{P}$ "Push forward of $\mathbb{P}$ by X"

$\langle \varphi \rangle_X$

$\langle \varphi[X] \rangle_{\mathbb{P}} = \langle \varphi \rangle_{X \# \mathbb{P}} = \int_{\mathbb{R}^n} \varphi(x) \mathbb{P}_X(dx)$
$= \int_{\Omega} \varphi(X(\omega)) \, \mathbb{P}(d\omega)$

$\varphi(t) = \langle e^{itx} \rangle$

Voice

Notations: $\mathbb{P}[X \in dx] = \mathbb{P}[X \in]x, x+dx[]$. $\quad n = 1$

or : $\mathbb{P}[X_1 \in]x_1, x_1+dx_1[\ \dots \ X_n \in]x_n, x_n+dx_n[]$

Examples:

Average: $\mu_X = \langle X \rangle$

Correlator: $\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle$.

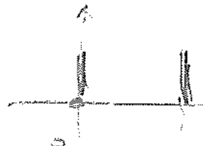
Characteristic Function: $\varphi(t) = \langle e^{i t \cdot X} \rangle$, etc.

Density: $$f(x_1 \dots x_n) = \lim_{dx_1 \dots dx_n \to 0} \frac{\mathbb{P}[X \in]x; x+dx[]}{dx_1 \dots dx_n}$$

Equality in law: $$\mathbb{P}_X = \mathbb{P}_Y \iff X \overset{L}{=} Y .$$

Example:

- Discrete density: $f(x) = \frac{1}{2}\delta(x) + \delta(x-1)\frac{1}{2}$



- Gaussian density

1D $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$X \sim \mathcal{N}(\mu, \sigma^2)$.

nD $$f(\vec{x}) = \frac{1}{[(2\pi)^n \det C]^{1/2}} e^{-\frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)}$$

↳ symmetric non-negative definite.

- Characteristic function of

1D gaussian variable $X \sim \mathcal{N}(0, \sigma^2)$.

$$\varphi(t) = \langle e^{itX} \rangle = e^{-\frac{t^2}{2}\sigma_X^2} .$$

nD $$\varphi(t) = e^{-\frac{t_i t_j}{2}\langle X_i X_j \rangle} . \qquad X \sim \mathcal{N}(0, C)$$

Conditional probability. X, Y two ($\mathbb{R}$) random variables.

$$\mathbb{P}_{X|Y} = \frac{\mathbb{P}_{(X,Y)}[X \in]x; x+dx[; Y \in]y; y+dy[]}{\mathbb{P}[Y \in]y; y+dy[]}$$

Notations: $$\mathbb{P}_{X|Y}[dx|y] = \frac{\mathbb{P}_X[dx, dy]}{\mathbb{P}_Y[dy]}$$ Not needed

Independence: $$\mathbb{P}_{X|Y}[dx|y] = \mathbb{P}_X[dx] \iff X \perp\!\!\!\perp Y .$$

$$\mathbb{P}_{X,Y}[dx, dy] = \mathbb{P}_X[dx]\,\mathbb{P}_Y[dy]$$

- Stochastic process.

collection of random variables parametrized by continuous parameter ex: $(X(t,\omega), t \in \mathbb{R})$

$(u(t,\vec{x},\omega), t \in \mathbb{R}, \vec{x} \in \mathbb{R}^3)$

For fixed realization $\omega \in \Omega$, $t \mapsto u(t)$ is the "path".

- Equivalence in Law.

$$(X_t, t \in \mathbb{R}) \overset{L}{\sim} (Y_t, t \in \mathbb{R})$$

iff

$$FDD(X) = FDD(Y)$$

with $FDD(X) = \{ \mathbb{P}_{X_{t_1}, \dots, X_{t_n}}, t_i \in \mathbb{R}, n \in \mathbb{N} \}$

- Gaussian process

$(X_t, t \in \mathbb{R})$ is gaussian

iff

$\forall k \in \mathbb{N}\ \forall t_1 \dots t_n \in \mathbb{R}\ \exists \mu_n, C_k$ (non-negative defin.)

$(X_{t_1} \dots X_{t_n}) \sim \mathcal{N}(\mu_n, C_n)$.

OBS gaussian process is fully specified by

the knowledge of $C(t_1,t_2), (t_1,t_2) \in \mathbb{R}, \mu(t), t \in \mathbb{R}$.

How

(Stationary) process.

A process whose finite dimensional distribution are invariant under arbitrary ~~time~~ shift of indexing parameter.

$$X_{t_1} \dots X_{t_n} \overset{L}{=} X_{t_1+\tau} \dots X_{t_n+\tau}, \quad \tau \in \mathbb{R}.$$

Or

$$a_\tau : X_t \longrightarrow X_{t+\tau}.$$

$$\underbrace{X_t, t \in \mathbb{R}}_{X} \overset{L}{=} a_\tau [X]$$

$$\mathbb{P}_X = \mathbb{P}_{a_\tau \circ X}.$$

Obs: t "time like" $\to$ stationary.

Observation: For stationary process, the average is constant over time and the covariance / second order structure function only depends on the time lag.

$$\langle X(t_1) \rangle = \langle X(0) \rangle = \mu_0.$$

i.e.: $$\langle X(t_1) X(t_2) \rangle = \langle X(t_1 - t_2) X(0) \rangle = C(t_1 - t_2)$$
$$= C(t_2 - t_1).$$

$C : \tau \to \mathbb{R}$: Correlation function of the process.

OBS.

- Continuity of sample path is guaranteed by continuity at 0 for the correlation function.

- Ergodic theorem holds for C that decays fast enough.

<u>Ex</u>: if $\int_0^\infty |C| < \infty$ then

$$\mathbb{E} \left| \frac{1}{T} \int_0^T X_t - \mu \right|^2 \xrightarrow[t \to \infty]{} 0 .$$

For stationary process, we may identify long-time average with ensemble averages!

- <u>Homogeneity</u>: stationarity when the indexing parameter is 'space-like'.

③ Gaussian "HIT" representation of turbulence.

A simple representation of the spatial structure of a gaussian field is to prescribe it as a <u>gaussian random field</u>, homogeneous in space, fully prescribe by the two-point correlator:

$$\vec{x}' = \vec{x} + \vec{r}$$

$$\langle V_i(x)\, V_j(x') \rangle = \langle V_i(0)\, V_j(r) \rangle, \quad i, j \in \{1,2,3\}$$
$$= C_{ij}(\vec{r}).$$

with C_{ij} prescribed by:

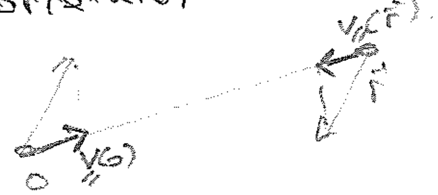
$$C_{ij}(\vec{r}) = \frac{f(r) - g(r)}{r^2}\, r_i r_j + g(r)\, \delta_{ij}.$$

with $$f(r) = U_0^2 \left[1 - \left(\frac{r}{L}\right)^{2/3} \right] \mathbb{1}_{r<L}.$$

$$g(r) = f(r) + \frac{r}{2} f'$$
$$= U_0^2 \left[1 - \left(\frac{4}{3}\right)\left(\frac{r}{L}\right)^{2/3} \right] \mathbb{1}_{r<L}.$$

Prphen.

- $f(r)$ represents a parallel correlator



$$f(r) = \langle v_\parallel(0)\, v_\parallel(\vec r)\,\rangle = \hat r_i\, C_{ij}'\, \hat r_j$$

- $g(r)$ ——— a $\perp$ correlator.

$$g(r) = \hat n_i\, C_{ij}'\, \hat n_j$$

- The relation between f and g is mediated through incompressibility $\partial \cdot C_{ij}' = 0$.

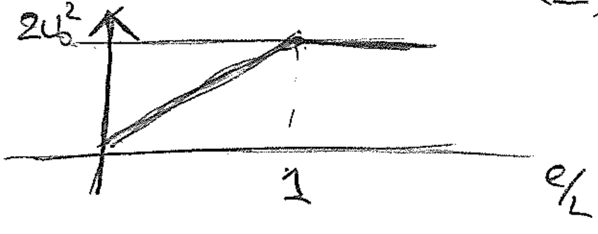
- Beyond homogeneity, the gaussian vector field is isotropic, ie the distribution is invariant under arbitrary $O \in SO_3(\mathbb{R})$ and parity invariant!

in the sense $\boxed{O_{i\ell}\, O_{j\ell'}\, C_{\ell\ell'}[O^{-1}\vec r] = C(\vec r)}$

$O \in (SO_3(\mathbb{R})$

- It produces scaling laws in the form

$$\langle \delta v_\parallel^2 \rangle = 2[C(0) - C(r)] = 2u_0^2 \left(\frac{\ell}{L}\right)^{2/3}$$



(4) NS - HIT : Kolmogorov Frisch - Scenario

$$\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + f$$

$$\nabla\cdot u = 0 .$$

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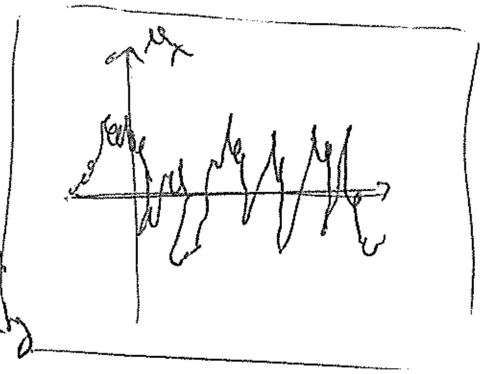
2 postulate of universality



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(A). Statistical restoring of symmetries ["Universality of statistics"].

The NS symmetries broken by the presence of forcing + viscosity are restored at the level of the turbulent measure for "small enough scales"

In other words:

In the limit $t\to\infty$, $\nu\to 0$, $\ell\to 0$

turbulent measures reflect a stochastic process $u(\vec{x},t)$ such that:

T	T	$u(\cdot,t) \overset{L}{=} u(\cdot,t+\tau)$
H	Hom.	$u(x+\rho,\cdot) \overset{L}{=} u(x,\cdot)$
I	Rot.	$u(\vec{x},\cdot) \overset{L}{=} O^{-1}u(Ox,\cdot)$
S_h.		$u(x,t) \overset{L}{=} \lambda^{-h}u(\lambda x,\lambda^{1-h}t)$.
G		$u(x+\cancel{U_0 t},t) \overset{L}{=} u(x+U_0 t,t)-U_0$.

Obs: • HIT: OK.

• G and SI are realized at the level of increments.

ie

$* \quad \delta u(\ell) \sim \mathcal{N}(0, \sigma_\ell^2) \qquad \sigma_\ell^2 = 2U_0^2 \left(\frac{\ell}{L}\right)^{2/3}$

$\sim \left(\frac{\ell}{L}\right)^{1/3} \delta u(L)$

D/ $\delta u(\ell) \sim \mathcal{N}(0, \sigma_\ell^2) \sim \left(\frac{\ell}{L}\right)^{1/3} \underbrace{\mathcal{N}(0, \sigma_L^2)} = \left(\frac{\ell}{L}\right)^{1/3} \delta u(L)$

$L = \lambda \ell$, with $\lambda = \frac{L}{\ell}$.

$h = 1/3$.

$* \quad \delta u(\ell) = u(\vec{\ell}) - u(0) \underset{\uparrow H \text{ (statistics)}}{=} u(\vec{\ell} + U_0 t) - u(U_0 t)$

$\Longrightarrow$ Gaussian HIT realizes the statistical restoring of universality with $h = \frac{1}{3}$.

(B). ~~Statistical~~ Scaling universality.

Dimensional analysis fully determine the value of h along with statistical properties at scale ℓ.

$$\boxed{\delta u^2(\ell) = C_K \, \varepsilon^{2/3} \, \ell^{2/3}.} \longrightarrow \langle u(\ell) u(0) \rangle = U_0^2 - \frac{1}{2} C \varepsilon^{2/3} \ell^{2/3}$$

Gaussian! $\quad L^2 T^{-3} \quad L^{2/3}$.

"Tabulated Gaussian"

ie $\frac{U_0^2}{L^{2/3}} = \varepsilon^{2/3} \Rightarrow \Big| \, U_0^2 = \varepsilon^{2/3} L^{2/3}$

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(5) $\nu > 0$

Can we probe HIT with NS at finite ν? $L = 1$ $F = 1$

Let us define $Re_\ell = \dfrac{S_2^{1/2}(\ell)\,\ell}{\nu}$. (Running number)

we define the inertial range of scales as:

$$1 \ll Re_\ell \ll Re_L.$$

$$\eta \ll \ell \ll L. \quad \text{with } \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}.$$

Power laws appear generically for $\eta \ll \ell \ll L$.

Rationale: Rescaling of the NS:

$$\partial_t u' + u' \cdot \nabla u' + \nabla' p' = \nu_\ell\, \partial^2_{xx} u' + f_\ell.$$

$$\text{with } \nu_\ell = \frac{1}{Re_\ell} \qquad f_\ell = \frac{\ell}{u_\ell^2} F = \frac{\ell}{S_2(\ell)} F.$$

with the scaling $S_2(\ell) \sim \ell^{2/3}$

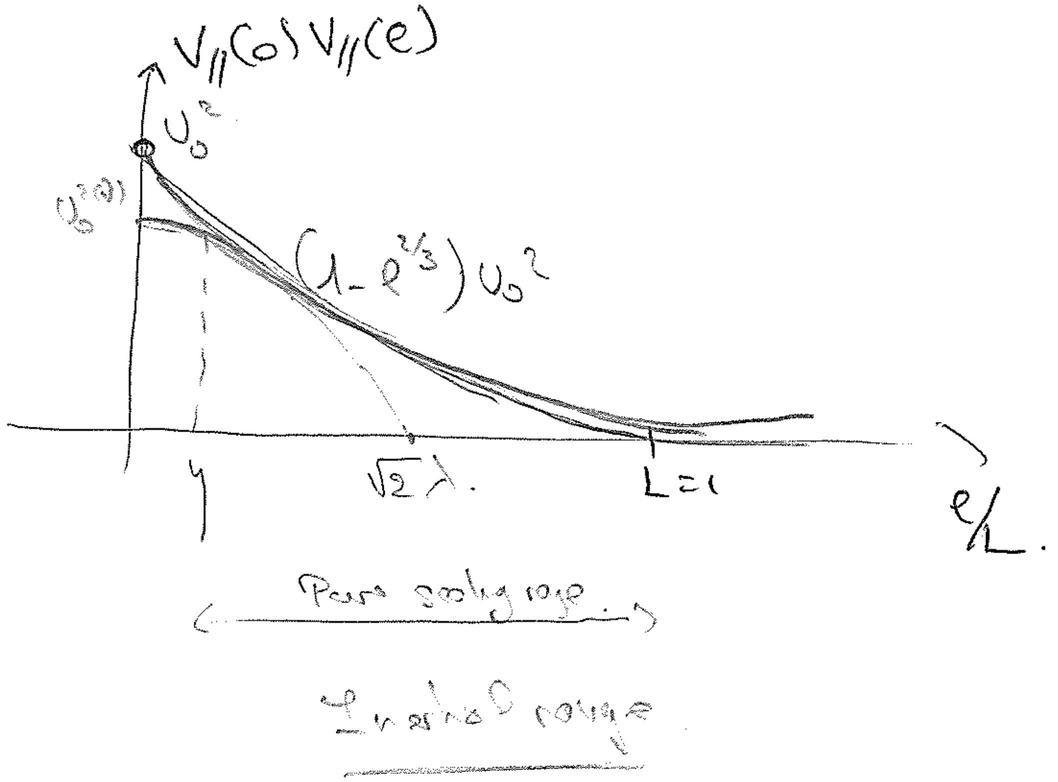
then, in the inertial range:

$$\nu_\ell = \frac{\nu}{S_2^{1/2}(\ell)\,\ell} = \frac{\nu}{\ell^{1/3}\varepsilon^{1/3}\ell} = \frac{\nu}{\ell^{4/3}\varepsilon^{1/3}} = \left(\frac{\nu^3}{\ell^4\varepsilon}\right)^{1/3} = \left(\frac{\eta}{\ell}\right)^{4/3} \ll$$

$$f_\ell = \frac{\ell}{\varepsilon^{2/3}\ell^{2/3}} F = \frac{\ell^{1/3}}{\varepsilon^{2/3}} F \ll \frac{F L^{1/3}}{\varepsilon^{2/3}} = O(1)$$

cst.

we formally recover the Euler!



<u>Obs</u>: • $Re = \left(\frac{L}{\eta}\right)^{4/3}$.

• $Re_\lambda = \frac{u\lambda}{\nu} = Re^{1/2}$.

<u>Calculation</u>: $\langle V_{\parallel}(0)\, V_{\parallel}(\ell) \rangle = \left[1 - \frac{r^2}{2\lambda^2}\right] v_0^2$

Taylor scale.

• $\varepsilon = 15\,\nu\, \frac{v_0^2}{\lambda^2}$.

Question: is the Gaussian HIT indeed realized in "true" steady state turbulence ??

→ answer "which is which"