Stochastic Differential Equations
List 1

Due: 06/09

In the exercises below, all the random variables are defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1) Let X be a random variable with finite p-moment for some $p \geq 1$. Show the following inequality

$$\mathbb{P}(|X| \geq \alpha) \leq \frac{E(|X|^p)}{\alpha^p},$$

for any $\alpha > 0$.

2) Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables and let $\overline{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$.

a) Under the assumption $\mathbb{E}(X_1^2) < \infty$, show that $\overline{X}_n \to \mathbb{E}(X_1)$ in probability, as $n \to \infty$.

b) Show that the same conclusion holds under the weaker assumption $\mathbb{E}(X_1) < \infty$.

3) Give one example of a continuous and discrete random variable Y with finite exponential moments of any order, i.e., $\mathbb{E}(e^{\lambda Y}) < \infty$, for all $\lambda \in \mathbb{R}$.

4) Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables and let $\overline{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$. In what follows, assume that $\mathbb{E}(e^{\lambda X_1}) < \infty$, for all $\lambda \in \mathbb{R}$.

a) Show that for $x \geq \mathbb{E}(X_1)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \geq x) \leq -\sup_{\lambda\in\mathbb{R}}\{\lambda x - M(\lambda)\}$$

where $M = \log \mathbb{E}(e^{\lambda X_1})$.

b) Show the corresponding lower bound

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \in (x-\delta, x+\delta)) \geq -\sup_{\lambda\in\mathbb{R}}\{\lambda x - M(\lambda)\},$$

for any $\delta > 0$.

5) (a) Show that if $(Y_n)_{n\geq 1}$ is a sequence of Gaussian random variables which converges in probability to Y, then Y is Gaussian and the convergence actually holds in $L^p(\Omega)$, for $p \geq 1$.

(b) A *Gaussian space* is a closed linear subspace of $L^2(\Omega)$ consisting only of centered Gaussian random variables. Let $(G_i)_{i\in I}$ be a countable family of closed subspace of a given Gaussian space. Show that the σ-algebras $\sigma(G_i)$ are independent if and only if the spaces G_i are pairwise orthogonal.

6) Let $(X_j)_{j\geq 1}$ be a sequence of independent (not necessarily identically distributed!) random variables, such that $\mathbb{E}(X_j) = 0$ and $\mathbb{E}(X_j^2) = \sigma_j^2 < \infty$ for all $j \geq 1$. Assume also that $s_n^2 = \sum_{j=1}^n \sigma_j^2$ goes to infinity as n goes to infinity. Show that if for all $\varepsilon > 0$,

$$\frac{1}{s_n^2} \sum_{j=1}^{n} \mathbb{E}\left[\mathbb{1}\{|X_j| \geq \varepsilon s_n\} X_j^2\right] \xrightarrow{n\to\infty} 0,$$

then $\frac{1}{s_n}(X_1 + \cdots + X_n)$ converges in law to a standard Gaussian random variable.