Stochastic Differential Equations
List 02

Due: 20/09

In the exercises below, all the random variables are defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, unless stated otherwise.

1) Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces, Q a probability measure in $(\Omega', \mathcal{F}')$ and $\{\mathbb{P}_{\omega'}\}_{\omega' \in \Omega'}$ a family of probability mesures in $(\Omega, \mathcal{F})$ such that for all $A \in \mathcal{F}$, the mapping $\omega' \mapsto \mathbb{P}_{\omega'}(A)$ is $\mathcal{F}'$-measurable. Show that $\mathbb{P} : \mathcal{F} \mapsto [0,1]$ given by:

$$\mathbb{P}(A) = \int_{\Omega'} \mathbb{P}_{\omega'}(A) dQ(\omega'), \quad A \in \mathcal{F},$$

is a probability measure on $(\Omega, \mathcal{F})$. $\mathbb{P}$ is called the *annealed* probability measure and $\mathbb{P}_{\omega'}$ is called the *quenched* probability measure.

2) Two processes X and Y are said to be *indistinguishable* if

$$\mathbb{P}(X_t = Y_t, \text{ for all } t \geq 0) = 1.$$

Give an example of two processes X, Y such that X is a version of Y but they are not indistinguishable.

3) A real valued stochastic process $(X_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Gaussian process if all the finite dimensional distributions of X are Gaussian random variables.

a) Assume that we are given functions $m : \mathbb{R}_+ \mapsto \mathbb{R}$ and $K : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$, where K is symmetric and positive-semidefinite function, in the sense that for any sequence $a_1, \dots, a_n \in \mathbb{R}$ and $t_1, \dots, t_n \in \mathbb{R}_+$, we have

$$\sum_{1 \leq i,j \leq n} a_i a_j K(t_i, t_j) \geq 0.$$

Show that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Gaussian process $(X_t)_{t \geq 0}$ defined on it, whose mean function is m and whose covariance function is K.

b) Now assume that the process has zero mean and there exists $\alpha \in (0,1]$ and $C > 0$ such that $K(t,s) = C(1 - |t-s|^{\alpha}) \mathbb{1}_{|t-s| \leq 1}$, for any $t, s \in \mathbb{R}_+$. What can you conclude about the regularity of the process X in part a)?

4) Let α, ε, $c > 0$. Let $(X_t)_{t \in [0,1]}$ be a continuous Gaussian process such that for $s, t \in [0,1]$,

$$\mathbb{E}(|X_t - X_s|^{\alpha}) \leq c|t-s|^{1+\varepsilon}.$$

Show that for every $\gamma \in [0, \varepsilon/\alpha)$, there is a positive random constant η such that $\mathbb{E}(\eta^p) < \infty$, for every $p \geq 1$ and such that for every $s, t \in [0,1]$,

$$|X_t - X_s| \leq \eta |t-s|^{\gamma}, \quad \text{a.s.}$$

Hint: You may use the Garsia-Rodemich-Rumsey inequality: Let $p \geq 1$ and $\alpha > p^{-1}$, then there exists a constant $C_{\alpha,p} > 0$ such that for any continuous function f on $[0,T]$, and for all $t, s \in [0,T]$ one has:

$$|f(t) - f(s)|^p \leq C_{\alpha,p} |t-s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + 1}} dx dy.$$

5) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Prove the following statements.

a) $(-B_t)_{t\geq 0}$ is a Brownian motion.

b) For any $h \geq 0$, $(B_{t+h} - B_h)_{t\geq 0}$ is a Brownian motion.

c) For any $c > 0$, the process $(B_{ct})_{t\geq 0}$ has the same law as $(\sqrt{c}B_t)_{t\geq 0}$.

d) $\lim_{t\to+\infty} \frac{B_t}{t} = 0$, a.s.

e) The process $(tB_{\frac{1}{t}})_{t\geq 0}$ is a Brownian motion.

6) Show that

$$\mathbb{P}\left(\sup_{s,t\in[0,1]} \frac{|B_t - B_s|}{\sqrt{t-s}} = +\infty \right) = 1.$$

In particular, this implies that Brownian motion is not locally Hölder with exponent 1/2, almost surely. Explain why this is not a contradiction with the results presented in the class, about the "slow points" of Brownian motion.