Stochastic Differential Equations
List 03

Due: 04/10

Exercises from Oksendal: 4.2, 4.8, 4.10.

1) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Let $t \geq 0$. Show that for every sequence $\Delta_n[0,t]$ of subdivisions of the interval $[0,t]$ such that

$$\lim_{n\to\infty} |\Delta_n[0,t]| = 0,$$

we have the following convergence in L^2

$$\lim_{n\to\infty} \sum_{k=1}^{n} (B_{t_k^n} - B_{t_{k-1}^n})^2 = t.$$

2) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Prove that there exists a sequence of subdivisions $\Delta_n[0,t]$ such that almost surely

$$\lim_{n\to\infty} \sum_{k=1}^{n} |B_{t_k^n} - B_{t_{k-1}^n}| = \infty.$$

3) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Show that for $t \geq 0$, almost surely

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} (B_{\frac{kt}{2^n}} - B_{\frac{(k-1)t}{2^n}})^2 = t.$$

4) We denote by $L^2(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ the set of processes $(u_t)_{t\geq 0}$ that are adapted with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and such that $\mathbb{E}(\int_0^\infty u^2(s)ds) < \infty$. Let $u, v \in L^2(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Show that

$$\mathbb{E}\left(\int_0^\infty u_s\, dB_s\right) = 0,$$

and

$$\mathbb{E}\left(\int_0^\infty u_s\, dB_s \int_0^\infty v_s\, dB_s\right) = \mathbb{E}\left(\int_0^\infty u_s v_s\, ds\right).$$