Stochastic Differential Equations
List 04

Due: 18/10

1) Let $(M_t)_{0\leq t\leq T}$ be a continuous martingale such that

$$\sup_{\Delta n[0,T]} \sum_{k=0}^{n-1} |M_{t_{k+1}^n} - M_{t_k^n}| < +\infty.$$

Show that $(M_t)_{0\leq t\leq T}$ is constant.

2) Let $f : \mathbb{R}_{\geq 0} \times \mathbb{R} \mapsto \mathbb{C}$ be a function that is once continuously differentiable with respect to its first variable and twice continuously differentiable with respect to its second variable that satisfies

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} = 0.$$

Show that if $(B_t)_{t\geq 0}$ is a Brownian motion, then $(f(t, B_t))_{t\geq 0}$ is a continuous martingale. Deduce that for $\lambda \in \mathbb{C}$, the process

$$(e^{\lambda B_t - \frac{1}{2}\lambda^2 t})_{t\geq 0}$$

is a martingale.

3) The Hermite polynomial of order n is defined as

$$H_n(x) = (-1)^n \frac{1}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

a) Compute H_0, H_1, H_2, H_3.

b) Show that if $(B_t)_{t\geq 0}$ is a Brownian motion, then the process $(t^{n/2} H_n(\frac{B_t}{\sqrt{t}}))_{t\geq 0}$ is a martingale.

c) Show that

$$t^{n/2} H_n(\frac{B_t}{\sqrt{t}}) = \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} dB_{t_1} \dots dB_{t_n}.$$

4) Let $(B_t)_{t\geq 0} = (B_t^1, \dots B_t^n)_{t\geq 0}$ be a n-dimensional Brownian motion with $n \geq 2$. For $a > 0$ and $x \in \mathbb{R}_n$, we consider the stopping time $T_a^x = \inf\{t \geq 0, ||B_t + x|| = a\}$.

a) Show that for $a < ||x|| < b$,

$$\mathbb{P}(T_a^x < T_b^x) = \begin{cases} \dfrac{\log b - \log ||X||}{\log b - \log a}, & \text{if} \quad n = 2 \\ \dfrac{||x||^{2-n} - b^{2-n}}{a^{2-n} - b^{2-n}}, & \text{if} \quad n \geq 3. \end{cases}$$

b) Show that for $a < ||x||$,

$$\mathbb{P}(T_a^x < +\infty) = \begin{cases} 1, & \text{if} \quad n = 2 \\ \dfrac{||x||^{2-n}}{a^{2-n}}, & \text{if} \quad n \geq 3. \end{cases}$$

5) Let $(B_t)_{t\geq 0}$ be a 2-dimensional Brownian motion.

a) Show that B_t is neighborhood recurrent, that is, for every non-empty open set $\mathcal{O} \subset \mathbb{R}^2$,

$$\mathbb{P}\left(\exists t \geq 0; B_t \in \mathcal{O}\right) = 1.$$

b) Show that points are always polar, that is, for every $x \in \mathbb{R}^2$, $\mathbb{P}(\exists t > 0, B_t = x) = 0$.

6) Let $n \in \mathbb{R}$ and consider the one-dimensional SDE

$$dX_t = \frac{n-1}{2}\frac{dt}{X_t} + dW_t,$$

with initial condition $X_0 = x > 0$. Define also the hitting time $T_0 = \inf\{t \geq 0; X_t = 0\}$. Prove the following.

a) There exists a unique solution to the SDE above for times $t < T_0$.

b) For $n < 2$, show that $T_0 < +\infty$ almost surely.

c) For $n \geq 2$, show that $T_0 = +\infty$ almost surely.

d) If n is a positive integer then $X_t \overset{\text{law}}{=} ||B_t||$, where B_t is a n-dimensional Brownian motion.

The process X is called *Bessel process*.