# Stochastic Differential Equations
# List 05

Due: 02/12

**1**) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $\lambda, \alpha > 0$. We denote

$$\nu(x) = \mathbb{E}\left(\int_0^\infty \exp\left(-\lambda t - \alpha \int_0^t 1_{[0,+\infty]}(B_s + x)\, ds\right) dt\right).$$

(a) Show that $\nu$ is the unique solution of the differential equation

$$y'' - (\alpha 1_{[0,+\infty)} + \lambda) y = -1$$

that satisfies,

$$\lim_{x\to+\infty} \nu(x) = \frac{1}{\alpha+\lambda}, \quad \lim_{x\to-\infty} \nu(x) = \frac{1}{\lambda}.$$

(b) Deduce that

$$\int_0^\infty e^{-\lambda t}\mathbb{E}(e^{-\alpha t A_t})\, dt = \frac{1}{\lambda}\frac{\lambda + \sqrt{\lambda(\alpha+\lambda)}}{\alpha + \lambda + \sqrt{\lambda(\alpha+\lambda)}},$$

where $A_t = \frac{1}{t}\int_0^t 1_{[0,+\infty)}(B_s)\, ds$.

(c) Conclude that the density of $A_t$ is given by

$$s \mapsto \frac{1}{\pi\sqrt{s(1-s)}}, \quad s \in (0,1).$$

This is the *arc-sine law* for Brownian motion.

**2**) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $\mu > 0$. We denote

$$A_\infty = \int_0^\infty e^{-2(B_t+\mu t)} dt.$$

(a) Let $h(x) = e^{\mu x}\mathbb{E}\left(\exp\left(-\frac{1}{2}e^{-2x}A_\infty\right)\right)$. Show that $h$ is the unique solution of the differential equation

$$\frac{d^2h}{dx^2} - e^{-2x}h = \mu^2 h,$$

such that $h(x) \overset{x\to+\infty}{\sim} e^{\mu x}$.

(b) Deduce that

$$h(x) = \frac{2^{1-\mu}}{\Gamma(\mu)} K_\mu(e^{-x}),$$

where $K_\mu$ is the McDonald function

$$K_\mu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\mu \int_0^\infty \frac{e^{-\frac{x^2}{4t}-t}}{t^{1+\mu}} dt.$$

(c) Compute the density of $A_\infty$.

(d) Consider the process $(\rho_t)_{t\geq 0}$ such that for every $t \geq 0$,

$$e^{-B_t-\mu t} = \rho_{\int_0^t e^{-2(B_s+\mu s)}}$$

Show that the process

$$\rho_t - \int_0^t \frac{-\mu + \frac{1}{2}}{\rho_s} ds$$

is a Brownian motion.

(e) By using $A_\infty$, compute the density of the stopping time

$$T_0 = \inf\{t \geq 0 : \rho_t = 0\}.$$