Problem Set 1

April 7, 2021

<u>Deadline</u>: April 22nd

All problems from Lax's *Linear Algebra and its Applications*, as well as the problems from this set, are recommended.

These problems from Lax should be **<u>sent by e-mail</u>** (scanned, typed or photographed) to juliahdomingues@hotmail.com:

Chapter 3: 12, 14

Chapter 4: 7, 8

Chapter 5: 4, 7

From this set: 2, 4

Exercise 1

Let V be a vector space and let T be a linear transformation from V into V.

a) If V is an n-dimensional vector space and the range and null space of T are identical, then prove that n is even.

b) Prove that the following statements are equivalents:

  1) $R_T \cap N_T = \{0\}$

  2) If $T(T(u)) = 0$, then $T(u) = 0$.

c). Let S be a subset of a vector space V. We shall say that S is **convex** if for every $u, v \in S$ we have $tu + (1-t)v \in S$ for all $t \in [0,1]$. Suppose that $u_0 \in V$ is fixed and $T(u) = u + u_0$ for all $u \in V$. Prove that if S is convex then $T(S)$ is convex.

Exercise 2

Let T be the linear transformation on $\mathbb{R}^3$, the matrix of T in the canonical basis is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$$

Find a basis for the range of T and a basis for the null space of T.

Exercise 3

Let T be a linear transformation on $\mathbb{R}^2$ and suppose that the matrix of T in the canonical basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Prove that $T^2 - (a+d)T + (ad - bc)I = 0$.

Exercise 4

Let V the space of all infinitely differentiable functions, and let $D : V \longrightarrow V$ be the derivative, that is, $D(f) = f'$.

a) Let $T = D - I$ where I is the identity mapping. What is the null space of T?

b) Same question if $T = D - aI$, where a is a number.

Exercise 5

Let V be a vector space over K and assume that $V = X \oplus Y$ for subspaces X and Y of V, i.e., for every $u \in V$, there are unique vectors $x \in X$ and $y \in Y$ such that $u = x + y$. We define the functions $P_1(u) = x$ (the projection map of V onto X relative to Y) and $P_2(u) = y$ (projection map of V onto Y relative to X).

a) Prove that P_1 and P_2 are linear transformations from V to V.

b) $P_i \circ P_i = P_i$ for $i = 1, 2$

c) $P_1 + P_2 = I_V$

d) $P_1 \circ P_2 = P_2 \circ P_1 = 0_V$

e) Let U be a vector space over K and $T : U \longrightarrow V$ a map. Assume that $P_1 \circ T$ and $P_2 \circ T$ are linear transformations. Prove that T is a linear transformation.