Problem Set 4

May 10, 2021

<u>Deadline</u>: May 20th

All problems from Lax's *Linear Algebra and its Applications*, as well as the problems from this set, are recommended.

<u>All problems from this set should be **sent by e-mail**</u> (scanned, typed or photographed) to juliahdomingues@hotmail.com:

Exercise 1

Let $A = (a_{ij})$ be a $n \times n$ invertible matrix such that

$$\sum_{j=1}^{n} a_{ij} = a, \qquad i = 1, 2, ..., n$$

a) Show that a must be an eigenvalue of A and that $a \neq 0$..

b) Show that if $A^{-1} = (b_{ij})$ then $\sum_{j=1}^{n} b_{ij} = 1/a$, for $i = 1, 2, ..., n$.

Exercise 2

Consider the equations for exposed and infected individuals derived form a linearized version of a SEIR model. Their evolution is given by

$$\begin{aligned} \dot{E} &= r\gamma_2 I - \gamma_1 E \\ \dot{I} &= \gamma_1 E - \gamma_2 I \end{aligned} \tag{2.1}$$

where $1/\gamma_1$ is the time an individual remains infected, γ_2 is the rate of removal and $r > 0$.

a) Re-write the problem above as a discrete evolution depending on the initial data, i.e., find a matrix A such that $[E_i, I_i]^T = A^i[E_0, I_0]^T$

b) Compute the eigenvalues of A

c) Decide whether the eigenvalues are real or not

d) See that the solution behaves differently when $r > 1$, $r < 1$ and $r = 1$. Give an interpretation for the parameter r.

Exercise 3

Let $A = (a_{i,j})$ be an $n \times n$ matrix with $a_{ij} > 0$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$ real-valued differentiable functions satisfying $\dot{\mathbf{x}} = A\mathbf{x}$ and $x_i(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for every $i = 1, \dots, n$. Is it true that all functions $x_1, x_2, \dots, x_n$ are necessarily linearly independent?

Exercise 4

Consider the real vector space $\mathbb{R}^2$ and define the function

$$< x, y >= x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$$

for $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

a) Show that $<,>$ is an inner product on $\mathbb{R}^2$.

b) Show that $|x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2| \leq \sqrt{(x_1 - x_2)^2 + 3x_2^2}\sqrt{(y_1 - y_2)^2 + 3y_2^2}$

Exercise 5

Let V be a vector space equipped with the inner product $<,>$.

a) Let S be a non-empty subset of V. Show that $S^\perp$ is a subspace of V and $S \subset (S^\perp)^\perp$.

b) Let S_1 and S_2 be two non-empty subsets of V. If $S_1 \subset S_2$ then $S_2^\perp \subset S_1^\perp$.

c) Let U, W be two subspaces of V. Show that $(U + W)^\perp = U^\perp \cap W^\perp$.

Exercise 6

Prove that if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isomorphism that maps a base $\mathcal{B}$ to another base $\mathcal{D}$, then $< Tx, Ty >_{\mathcal{B}} \; = \; < x, y >_{\mathcal{D}}$ where $< \cdot, \cdot >$ is the inner product.

Conclude that T maps the unit ball $S_{\mathcal{D}}^{n-1}$ to $S_{\mathcal{B}}^{n-1}$