Problem Set 5

May 22, 2021

Deadline: June 4th

All problems from Lax's *Linear Algebra and its Applications*, as well as the problems from this set, are recommended.

All problems from this set should be **sent by e-mail** (scanned, typed or photographed) to juliahdomingues@hotmail.com:

Exercise 1

Let A be a $n \times n$ matrix and denote the eigenvalues of the matrix $B = A^T A$ by λ_i $i = 1, 2, ..., n$. Then,

$$\| A \|_2 = \max_{i=1}^{n} \lambda_i^{1/2},$$

where $\| \, . \, \|_2$ is the standard Euclidean norm in $\mathbb{R}^n$: $\| x \|_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$.

Exercise 2

Let V be a finite-dimensional inner product space, and let E be an idempotent linear operator on V, i.e., $E^2 = E$. Prove that E is self-adjoint if and only if $EE^* = E^*E$.

Exercise 3

If T is a unitary ans S is a self-adjoint operator, prove that TST^{-1} is a self-adjoint operator.

Exercise 4

Definition: A *group* is a set G together with a binary operation "$\cdot$" that satisfy:

1. $\forall a, b, c \in G$, it is true that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This is called *associativity*.
2. There is unique $e \in G$ such that $\forall a \in G$, $e \cdot a = a \cdot e = a$. Such e is called *neutral element*.
3. For each $a \in G$ there is unique $b \in G$ such that $a \cdot b = e$ and $b \cdot a = e$. Such b is usually denoted a^{-1} and is called *inverse element*.

Let G be a finite set of $n \times n$ matrices $\{M_i\}$, $1 \le i \le r$, which form a group under matrix multiplication. Suppose that

$$\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0. \tag{4.1}$$

Prove that $S = \sum_{i=1}^{r} M_i$ has no ones in its Jordan normal form. Conclude that $S = 0$ necessarily.

Exercise 5

Definition: An $n \times n$ matrix A is said to be *orthogonally diagonalizable* when an orthogonal matrix P can be found such that $P^{-1}AP = P^T AP$ is diagonal.

Prove that the following conditions are equivalent for an $n \times n$ matrix A.

1. A has an orthonormal set of n eigenvectors
2. A is orthogonally diagonalizable
3. A is symmetric.

Exercise 6

Assume to be true that every $m \times n$ matrix A with linearly independent columns has a factorization $A = QR$, where Q has orthonormal columns and R is upper triangular with positive diagonal entries. Prove that the QR factorization is unique, i.e., if $A = QR$ and $A = Q_1 R_1$, then $Q = Q_1$ and $R = R_1$.