Problem Set 5

June 10, 2021

<u>Deadline</u>: June 18th

All problems from Lax's *Linear Algebra and its Applications*, as well as the problems from this set, are recommended.

<u>All problems from this set should be **sent by e-mail**</u> (scanned, typed or photographed) to juliahdomingues@hotmail.com:

Exercise 1

Let a and b be real numbers.

a) If

$$K = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

Show that

$$e^{tK} = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$$

b) If

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Show that

$$e^{tA} = \begin{pmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ -e^{at}\sin(bt) & e^{at}\cos(bt) \end{pmatrix}$$

Exercise 2

Let A, B be $n \times n$ real matrices. Suppose that A and B are positive. Prove that the eigenvalues of AB are all positive.

Exercise 3

Compute a singular value decomposition for the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

Describe the algorithm you are using to compute such decomposition and give a brief argument of why it works.

Exercise 4

Let A be an $n \times m$ matrix, let $A = U\Sigma V^T$ be any SVD for A, with U, V orthogonal of size $m \times m$ and $n \times n$ respectively and

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

Here Σ is written in block form, $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_r)$ and $\lambda_i > 0$. Let us call $U = [\mathbf{u}_1, \dots, \mathbf{u}_r, \dots, \mathbf{u}_m]$ and $V = [\mathbf{v}_1, \dots, \mathbf{v}_r, \dots, \mathbf{v}_n]$ to see that the columns are orthonormal bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Prove that

a) The rank of A is r and $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_r}$ are the singular values of A.

b) $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthonormal basis of the column space of A.

c) $\{\mathbf{u}_{r+1}, \dots, \mathbf{u}_m\}$ is an orthonormal basis of the null space of A^T.

d) $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal basis of the null space of A.

e) $\{\mathbf{v}_r, \dots, \mathbf{v}_r\}$ is an orthonormal basis of the row space of A.

Exercise 5

Given a SVD for an invertible matrix A, find one for A^{-1}. How are Σ_A and $\Sigma_{A^{-1}}$ related?