Linear Algebra and Applications
List 2
Deadline: April 15, 2020

The list below includes exercises from the textbook and some additional problems. Solving all problems is strongly recommended. The problems that must be written and sent to the monitor (scanned or photographed) by email **jose14manuel14@gmail.com by April 15** are: **Exercises: 12,14 (Chapter 3), 3,7 (Chapter 4), 1,4 (Chapter 5) and Problems: 3,6,7.**

List of Problems

Textbook Exercises: 10,11,12,13,14,15 (Chapter 3), 1,3,4,7,8 (Chapter 4) and 1,4,6 (Chapter 5).

Problem 1. *Which of the following functions T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations?* ***Justify your answer.***

a). $T(x, y) = (1 + x, y)$

b). $T(x, y) = (\sin x, -y)$

c). $T(x, y) = (y, x)$

d). $T(x, y) = (2x + y, 3x - 5y^2)$

Problem 2. *If*

$$\alpha_1 = (1, -1), \qquad \beta_1 = (1, 0)$$
$$\alpha_2 = (2, -1), \qquad \beta_2 = (0, 1)$$
$$\alpha_3 = (-3, 2), \qquad \beta_3 = (1, 1)$$

is there a linear transformation T from $\mathbb{R}^2$ into $\mathbb{R}^2$ such that $T(\alpha_i) = \beta_i$ for $i = 1, 2, 3$.

Problem 3. *Let V be a vector space and let T be a linear transformation from V into V.*

a). If V is an n-dimensional vector space and the range and null space of T are identical, then prove that n is even.

b). Prove that the following statements are equivalents:

b.1) $R_T \cap N_T = \{0\}$

b.2) If $T(T(u)) = 0$, then $T(u) = 0$.

c). Let S be a subset of a vector space V. We shall say that S is ***convex*** *if for every $u, v \in S$ we have $tu + (1 - t)v \in S$ for all $t \in [0, 1]$. Suppose that $u_0 \in V$ is fixed and $T(u) = u + u_0$ for all $u \in V$. Prove that if S is convex then $T(S)$ is convex.*

Problem 4. *Let T be the linear transformation on $\mathbb{R}^3$ defined by*

$$T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$$

a). What is the matrix of T in the standard ordered basis of $\mathbb{R}^3$?

b). Prove that T is invertible and give a rule for T^{-1} like the one which defines T.

Problem 5. *Let T be a linear transformation on $\mathbb{R}^2$ and suppose that the matrix of T in the standard basis of $\mathbb{R}^2$ is*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Prove that $T^2 - (a + d)T + (ad - bc)I = 0$.

Problem 6. *Let T be the linear transformation on $\mathbb{R}^3$, the matrix of T in the standard basis is*

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$$

Find a basis for the range of T and a basis for the null space of T.

Problem 7. *Let V the space of all infinitely differentiable functions, and let $D : V \longrightarrow V$ be the derivative, that is, $D(f) = f'$.*

a). Let $T = D - I$ where I is the identity mapping. What is the null space of T?

b). Same question if $T = D - aI$, where a is a number.

Problem 8. *Let V be a vector space over K and assume that $V = X \oplus Y$ for subspaces X and Y of V. For every $u \in V$, there are unique vectors $x \in X$ and $y \in Y$ such that $u = x + y$. We define the functions $P_1(u) = x$ (**projection map of** V **onto** X **relative to** Y) and $P_2(u) = y$ (**projection map of** V **onto** Y **relative to** X).*

a). Prove that P_1 and P_2 are linear transformations from V to V.

b). $P_i \circ P_i = P_i$ for $i = 1, 2$

c). $P_1 + P_2 = I_V$

d). $P_1 \circ P_2 = P_2 \circ P_1 = 0_V$

e). Let U be a vector space over K and $T : U \longrightarrow V$ a map. Assume that $P_1 \circ T$ and $P_2 \circ T$ are linear transformations. Prove that T is a linear transformation.

Problem 9. *Assume $T_i : V \longrightarrow V$ are linear transformations for $i = 1, 2$ such that $T_1 + T_2 = I_V$ and $T_1T_2 = T_2T_1 = 0_V$. Set $X = R_{T_1}$ and $Y = R_{T_2}$. Prove that $V = X \oplus Y$.*