Convergence

For $\mathbb{R}$ or $\mathbb{C}$, given a sequence

$a_1, a_2, \ldots$ $\{a_k\}$ of numbers

Def $\lim a_k = a$

means that $|a_k - a| \to 0$ as $k \to \infty$

($\forall \varepsilon > 0$, $\exists N > 0$ s.t. $|a_k - a| < \varepsilon$ for any $k > N$)

Def $\{a_k\}$ is a Cauchy sequence, if $|a_k - a_j| \to 0$ as $k, j \to \infty$

($\forall \varepsilon > 0$, $\exists N > 0$, s.t. $|a_k - a_j| < \varepsilon$ for any $j, k > N$).

In $\mathbb{R}$ and $\mathbb{C}$

completeness: every Cauchy sequence converges to a limit.

compactness: Every bounded sequence $\{a_k\}$ ($\exists c > 0$ s.t. $|a_k| < c$ for all k) contains a convergent subsequence.

Def A sequence of vectors $\{x_k\}$ in Euclidean space, $x_k \in X$, converges to a limit $x \in X$,

$$\lim_{k \to \infty} x_k = x$$

if $\| x_k - x \|$ tends to zero as $k \to \infty$.

Def Sequence $\{x_k\}$ is Cauchy if $\|x_k - x_j\| \to 0$ as $j, k \to \infty$.

Th (i) Every Cauchy sequence in finite-dimensional Euclidean space converges to a limit.

(ii) In finite dimensional Euclidean space, every bounded sequence ($\exists c > 0$ s.t. $\|x_k\| < c$ for all t) contains a convergent subsequence.

Obs: not true for infinite-dimensional spaces!!

Pr exercise.

Recall that

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$

$\|A\|$: matrix (lin. operator)

For matrices (finite-dim lin. spaces), we can substitute sup $\to$ max.

$\exists\, x \in X$ with $\|x\| = 1$ s.t. $\|Ax\| = \|A\|$.

Ex: show this using completeness.

Th Let X be a linear space with Euclidean structure. If X is locally compact (every bounded sequence $\{x_k\}$ has a convergent subsequence), then X is finite dimensional.

Pr Assume the converse: X is inf. dim.

Take any vector $y_1 \in X$.

Since the space is inf. dim $\Rightarrow$

$\Rightarrow \exists\, y_2 \in X$ s.t. y_1 and y_2 are linearly independent.

Similarly $\exists\, y_3 \in X$ s.t. y_1, y_2, y_3 are lin. indep. etc.

$\Rightarrow$ Infinite sequence $\{y_k\}$ of vectors such that $y_1, \dots, y_n$ are lin. indep. for any n.

Apply Gram-Schmidt process $\Rightarrow$

$\{x_k\}$ such that $(x_j, x_k) = \delta_{jk} = \begin{cases} 1, & j=k \\ 0, & j \neq k. \end{cases}$

But, $\|x_k - x_j\|^2 = \|x_k\| - 2(x_k, x_j) + \|x_j\|^2 = 2.$

($(x_k, x_j) \to 0$)

We obtained a sequence $\{x_k\}$ which is bounded and having no converged subsequences. Contradiction ∎

Def A sequence $\{A_k\}$ of linear mappings converges to A if

$$\lim_{k \to \infty} \|A_k - A\| = 0.$$

Consider a square $n \times n$ matrix A, and let us find lower bound for $\|A\|$.

Eigenvalue λ: $Au = \lambda u$ for some nonzero vector u. We can take $\|u\| = 1$.

$$\|A\| = \max_{\|x\|=1} \|Ax\| \geq |\lambda| \|u\| = |\lambda|.$$

$x = u$

Hence $\|A\| \geq |\lambda|$ for any eigenvalue.

$$\Rightarrow \boxed{\|A\| \geq r(A) := \max_{\text{all eigenvalues } \lambda} |\lambda|}.$$

$r(A)$ is called spectral radius of A.

Obs $\|A\| \neq r(A)$ in general.

For example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

Eigenvalues $\lambda_1 = \lambda_2 = 0$. $r(A) = 0$.

$\|A\| = 1$ $\left(x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ $\Rightarrow$ $\|A\| > r(A)$.

But consider A^j, its eigenvalues are λ^j

$$\Rightarrow \quad \| A^j \| \geq r(A^j) = \left(r(A)\right)^j$$

$$\Rightarrow \quad \| A^j \|^{1/j} \geq r(A). \qquad (\neq)$$

Th $\lim_{j \to \infty} \| A^j \|^{1/j} = r(A).$

Look of the proof in Lax, Appendix 10.

How about upper bound for $\|A\|$.

$$\|A\| = \max_{\|x\|=1} \|Ax\| =$$

$$= \max_{\|x\|=1} \sqrt{\sum_i \left(\sum_j a_{ij} x_j \right)^2} \leq$$

$$|x_j| \leq 1, \quad |a_{ij}| \leq \max_{ij} |a_{ij}|$$

$$\|A\| \leq n \max_{i,j} |a_{ij}|$$ for $n \times n$ matrix.

Similarly, you can show (ex.)

$$\|A\| \leq \underbrace{\left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}}$$

Hilbert - Schmidt norm.

Self-adjoint linear operators.

Consider Euclidean space X,
and a linear map $A : X \to X$.

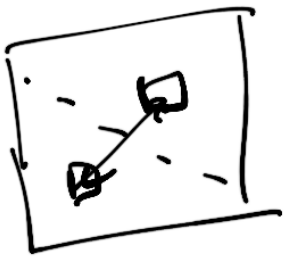
Adjoint $A^* : X \mapsto X$ s.t.

$$(Ax, y) = (x, A^* y)$$

A is self-adjoint if $A = A^*$.

In case of real $n \times n$ matrices

$$A^* = A^T \Rightarrow \quad A = A^T.$$



$$a_{ij} = a_{ji}$$

(symmetric matrix)

For complex matrices

$$A^* = \overline{A^T} \Rightarrow \quad A = \overline{A^T}$$

$$a_{ij} = \overline{a_{ji}}$$

(Hermitian matrix)

Application for symmetric matrices.

Consider a smooth function

$$f : \mathbb{R}^n \mapsto \mathbb{R}$$

Taylor series near point x.

$$f(x+y) = f(x) + \sum_i \frac{\partial f}{\partial x_i} y_i + \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} y_i y_j + o(\|y\|^2)$$

Linear function $\ell = (y, g)$

$$g = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

Matrix $H = [h_{ij}]$, $h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

$$f(x+y) = f(x) + (g, y) + \frac{1}{2}(y, Hy) + o(\|y\|^2)$$

H is a Symmetric matrix because

$$h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = h_{ji}$$

Symmetric matrices define second-order terms in Taylor expansion.