Th (spectral)

For a complex $n \times n$ matrix $A$, one can find a basis consisting of genuine or generalized eigenvectors.

Construction: distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

$$\mathbb{C}^n = N_1 \oplus \cdots \oplus N_k$$

where $N_i$ is nullspace for $(A - \lambda_i I)^{d_i}$.

(All vectors in $N_i$ are gener. eigenvect. of $\lambda_i$).

Th (similarity)

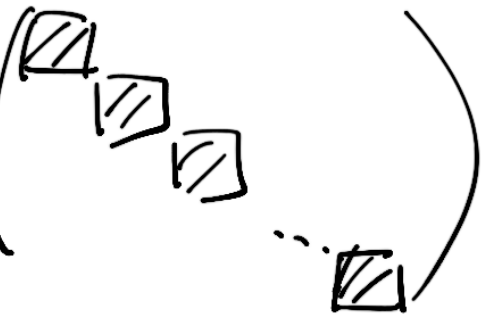
$A \sim B$ $(A = S^{-1} B S)$ iff

① $A$ and $B$ have same eigenvalues

② $\dim \text{null}(A - \lambda_i I)^j = \dim \text{null}(B - \lambda_i I)^j$

for all $i$ and $j$.

**Th (Jordan normal form)**

$A \sim$  , where all blocks

are Jordan blocks of the form

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}.$$

This form is unique up to permutation of blocks.

**Th** $A \sim A^T$.

Check cond. ① and ②.

① Eigenvalues:

$$\det(A - \lambda I) = 0, \quad \det(A^T - \lambda I) = 0$$

identical.

② $\dim \text{null}\,(A-\lambda_i I)^j \overset{?}{=} \dim \text{null}\,(A^T-\lambda_i I)^j$

We know that $\dim \text{null}\, B = \dim \text{null}\, B^T$.

Take $B=(A-\lambda_i I)^j$, $B^T=(A^T-\lambda_i I)^j$.



Th Let $u$ be eigenvector of eigenvalue $\lambda_1$ of matrix $A$, and $v$ be eigenvector of eigenvalue $\lambda_2$ of $A^T$. If $\lambda_1 \neq \lambda_2$, then

$$v^T u = 0.$$

Pr

$$Au = \lambda_1 u \qquad (\text{I})$$

$$A^T v = \lambda_2 v \;\Big|^T \Rightarrow\; v^T A = \lambda_2 v^T \qquad (\text{II})$$

$v^T$ (I) $-$ (II) $u$

$$v^T A u - v^T A u = \lambda_1 v^T u - \lambda_2 v^T u$$

$$0 = \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} v^T u \Rightarrow v^T u = 0$$

□

**Th** Consider an $n \times n$ matrix $A$, and assume that all its eigenvalues are simple (distinct). Then a set of eigenvectors $u_1, \dots, u_n$ is a basis, and also a set of eigenvectors $v_1, \dots, v_n$ of $A^T$ is also basis. We claim that

(1) $v_i^T u_i \neq 0$. Remember, that $v_i^T u_j = 0$ if $i \neq j$.

(2) $$x = \sum_{i=1}^{n} k_i u_i, \quad k_i = \frac{v_i^T x}{v_i^T u_i}.$$

**Pr**. Exercise.

# Euclidean structure of a linear space.

$X$ is a linear vector space (over $\mathbb{R}$) or (over $\mathbb{C}$).

Scalar (inner) product

(R)

$(\cdot,\cdot): X^2 \to \mathbb{R}$

$x, y \in X$, $(x,y) \in \mathbb{R}$

(C)

$(\cdot,\cdot): X^2 \to \mathbb{C}$

$(x,y) \in \mathbb{C}$

① Scalar product is linear w.r.t first argument $x$.

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y); \quad (kx, y) = k(x,y).$$

② symmetric

$(x,y) = (y,x)$

conjugate symm.

$(x,y) = \overline{(y,x)}$

③ positive : $(x,x) > 0$ for any $x \neq 0$.

**Obs** in real case, $(x,y)$ is also linear w.r.t second argument $y$.

in complex case, it is conjugate linear w.r.t $y$:

$$(x, y_1+y_2) = (x,y_1)+(x,y_2)$$

$$(x, ky) = \bar{k}\,(x,y)$$

**Def** Norm $\|x\| = \sqrt{(x,x)}$.

**Th** (Schwarz Inequality)

$|(x,y)| \le \|x\|\,\|y\|$ for any $x,y \in X$.

**Pr** Consider real case.

$$q(t) = \|x+ty\|^2 \ge 0, \quad t \in \mathbb{R}$$

$$q(t) = (x+ty, x+ty) = \|x\|^2 + 2t(x,y) + t^2\|y\|^2.$$

If $y=0$ $\Rightarrow$ Schw. ineq becomes $0 \le 0$. ok.

If $y \neq 0$, take $t = -\dfrac{(x,y)}{\|y\|^2}$.

$$\|x\|^2 - \frac{(x,y)^2}{\|y\|^2} \geq 0 \Rightarrow (x,y)^2 \le \|x\|^2\|y\|^2. \blacksquare$$

Ex: check complex case.

**Th (Triangle inequality)**

$$\|x+y\| \le \|x\| + \|y\|.$$

**Pr** (real case)

$$\|x+y\|^2 = (x+y, x+y) = \|x\|^2 + 2(x,y) + \|y\|^2$$

$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

$\blacksquare$

**Def:** $x \perp y$ (orthogonal) if $(x,y) = 0$.

**Def** A basis $x_1, \dots, x_n$ of $X$ is called orthonormal if

$$(x_i, x_j) = \delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq 0. \end{cases}$$

**Th** (Gram-Schmidt)

Given a basis $y_1, \dots, y_n$ of a linear space $X$ ($n = \dim X$). There exists an orthonormal basis $x_1, \dots, x_n$ such that each $x_k$ is a linear combination of $y_1, \dots, y_k$.

**Pr** Take $x_1 = \dfrac{y_1}{\|y_1\|}$.

By induction, assuming that we have found $x_1, \dots, x_{k-1}$, take

$$x_k = c\left(y_k - \sum_{j=1}^{k-1} c_j x_j\right)$$

① orthogonality:

$$(x_k, x_i) = c\Big((y_k, x_i) - c_i(x_i, x_i)\Big) = 0$$

$i < k$

$$\Rightarrow \quad c_i = \frac{(y_k, x_i)}{\|x_i\|^2}.$$

Then, choose $c$ such that $\|x_k\| = 1$.

∎

**Th**

Any linear function $l(x)$ on a finite-dim linear space $X$ can be written as

$$l(x) = (y, x) \quad \text{for some } y \in X.$$

Pr Take some orthonormal basis $x_1, \dots, x_n$.

Denote $b_k = \ell(x_k)$.

Set $y = \sum_{k=1}^{n} b_k x_k$.

Now, for any $x = x_j$, we have

$$(y, x) = \Big( \sum_k b_k x_k, x_j \Big) \underset{\text{orthonorm}}{=} b_k = \ell(x)$$

We checked that $(y, x) = \ell(x)$ for every basis vector $\Rightarrow$ true for any vector. ∎

Corollary $\ell \to y$ by previous Th. defines isomorphism of $X'$ and $X$.

Def Given linear subspace $Y$ of $X$, the orthogonal complement $Y^\perp = \{z \in X : z \perp y, y \in Y\}$.

**Th** For any subspace $Y$ of $X$,

$$X = Y \oplus Y^{\perp}.$$

This means that any $x \in X$ is uniquely written as $x = y + z$, $y \in Y$, $z \in Y^{\perp}$.

**Pr** exercise. ▨

**Def** The orthogonal projector of $Y$ is a linear map $P_Y : X \to Y$ such that

$$x = P_Y x + z, \quad z \in Y^{\perp}.$$

**Th** ① $P_Y$ is a linear map

② $P_Y^2 = P_Y$.