

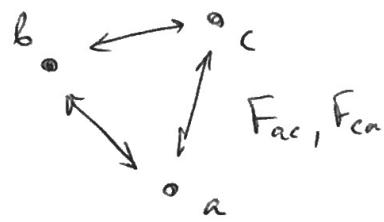
① Classical mechanics

(1)

Point mass (particle)

Coordinates: $r = (x, y, z) \in \mathbb{R}^3$, time: $t \in \mathbb{R}$, mass $m > 0$

System: r_a, r_b, r_c, \dots or r_α , $\alpha \in \{a, b, c, \dots\}$
 (n particles)



Equation of motion (2^d Newton's law)

$$m_\alpha \ddot{r}_\alpha = F_\alpha = \sum_{\beta \in \{a, b, \dots\} \setminus \alpha} F_{\alpha\beta}$$

↑ ↑ ↗

mass acceleration force

$(\bullet = \frac{d}{dt})$

force applied to
particle α from
particle β .

~~Masse~~
 3^d Newton's law: $F_{\alpha\beta} = -F_{\beta\alpha}$ ($F_{\alpha\alpha} = 0$)

Momentum: $p_\alpha = m_\alpha \dot{r}_\alpha \Rightarrow \dot{p}_\alpha = F_\alpha$

Total momentum: $p = \sum_\alpha p_\alpha$

$$\dot{p} = \sum_\alpha \dot{p}_\alpha = \sum_{\alpha, \beta} F_{\alpha\beta} = 0.$$

(momentum is
conserved)
(for isolated
system.)

Potential Force

(2)

$$F_\alpha = -\frac{\partial U}{\partial r_\alpha}, \quad U(r_1, r_2, \dots) \text{ potential energy}$$

$$F_\alpha = \begin{pmatrix} F_{\alpha 1} \\ F_{\alpha 2} \\ F_{\alpha 3} \end{pmatrix} = \begin{pmatrix} \partial U / \partial x_\alpha \\ \partial U / \partial y_\alpha \\ \partial U / \partial z_\alpha \end{pmatrix}.$$

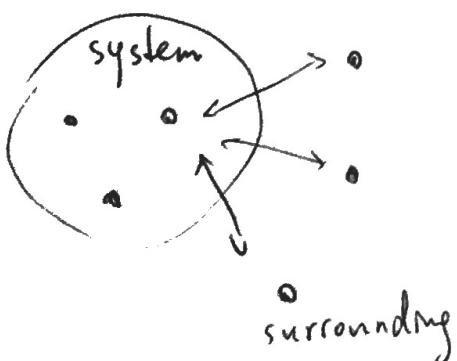
$$T = \sum_{\alpha} \frac{1}{2} m_\alpha |\dot{r}_\alpha|^2 \quad \text{kinetic energy}$$

$$E = T + U \quad \text{total energy.} \quad \swarrow \text{scalar product}$$

$$\begin{aligned} \frac{dE}{dt} &= \dot{E} = \dot{T} + \dot{U} = \sum_{\alpha} m_\alpha (\ddot{r}_\alpha \cdot \ddot{r}_\alpha) + \sum_{\alpha} \left(\frac{\partial U}{\partial r_\alpha} \cdot \dot{r}_\alpha \right) = \\ &= \sum_{\alpha} F_\alpha \cdot \dot{r}_\alpha - \sum_{\alpha} F_\alpha \cdot \dot{r}_\alpha = 0 \end{aligned}$$

Energy of isolated system is conserved.

External forces



$$F_\alpha = \underbrace{\sum_{\beta_1 \text{ in system}} F_{\alpha \beta_1}}_{\text{int. } F_\alpha} + \underbrace{\sum_{\beta_2 \text{ outside system}} F_{\alpha \beta_2}}_{\text{ext. } F_\alpha}$$

$$m_\alpha \ddot{r}_\alpha = F_\alpha^{int} + F_\alpha^{ext}$$

$$\sum_{\alpha \in S} m_\alpha \ddot{r}_\alpha = \dot{P}_S$$

~~total~~ momentum of
a system
(S = system)

$$\dot{P}_S = \sum_{\alpha \in S} m_\alpha \ddot{r}_\alpha = \sum_{\alpha \in S} (F_\alpha^{int} + F_\alpha^{ext}) =$$

$$= \underbrace{\sum_{\alpha \in S} \sum_{\beta_1 \in S} F_{\alpha \beta_1}}_{=0 \quad (3^d N.L.)} + \sum_{\alpha \in S} F_\alpha^{ext}$$

$$\Rightarrow \dot{P}_S = \sum_{\alpha \in S} F_\alpha^{ext} \quad (\text{external force applied to particle } \alpha)$$

Energy ~~total~~

Kinetic $T_S = \sum_{\alpha \in S} \frac{m_\alpha}{2} |\dot{r}_\alpha|^2$

Potential : $U = U_S + U_{ext}$, U_S depends only on \vec{r}_α !

where $F_\alpha^{int} = -\frac{\partial U_S}{\partial r_\alpha}$, $F_\alpha^{ext} = -\frac{\partial U_{ext}}{\partial r_\alpha}$.

System energy : $E_S = T_S + U_S$.

$$\begin{aligned} \dot{E}_S &= \dot{T}_S + \dot{U}_S = \sum_{\alpha \in S} m_\alpha (\ddot{r}_\alpha, \dot{r}_\alpha) + \sum_{\alpha \in S} \left(\frac{\partial U_S}{\partial r_\alpha}, \dot{r}_\alpha \right) = \\ &= \sum_{\alpha \in S} (F_\alpha^{int} + F_\alpha^{ext}, \dot{r}_\alpha) - \sum_{\alpha \in S} (F_\alpha^{int}, \dot{r}_\alpha) = \sum_{\alpha \in S} (F_\alpha^{ext}, \dot{r}_\alpha) \end{aligned}$$

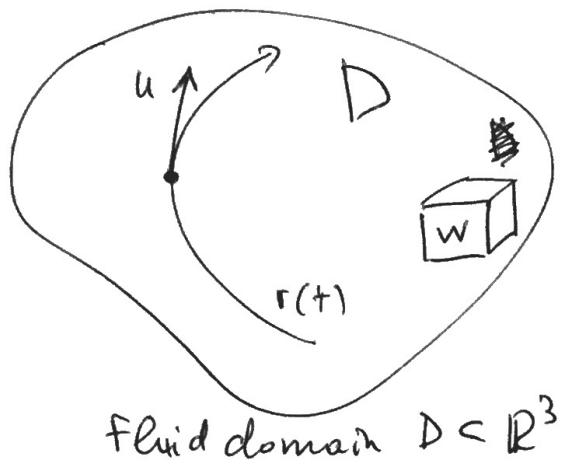
$$\dot{E}_S = \sum_{\alpha \in S} (F_\alpha^{ext}, \dot{r}_\alpha)$$

change of ~~the~~ system energy
or a work done by external forces

II Ideal fluid

Continuum motion

in \mathbb{R}^3 (can be \mathbb{R}^2 or \mathbb{R})



Fluid particle (point)

$$r = (x, y, z) \in \mathbb{R}^3$$

$r = r(t)$ trajectory

$$u = \dot{r}(t) \quad \text{velocity} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Mass: $\exists \rho(x, t)$ density function

$$m(w, t) = \int_w \rho(x, t) dV \quad \text{is the mass on the domain } w \subset D$$

Basic principles:

I Mass is conserved (never created or destroyed)

II $\frac{d}{dt} (\text{momentum}) = (\text{force}) \quad 2^{\text{d}} \text{ Newton's law}$

III Energy is conserved (never created or destroyed)

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Conservation of mass

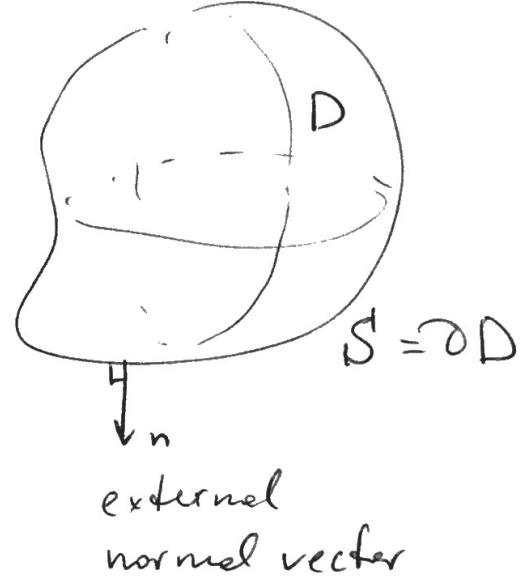
$$\text{grad } \rho = \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial y} \\ \frac{\partial \rho}{\partial z} \end{pmatrix}, \quad dV u = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Divergence theorem :

$$\int_D dV F = \oint_S (F \cdot n) dA$$

D volume element S surface area element.

for any D and vector-function $F(r)$.



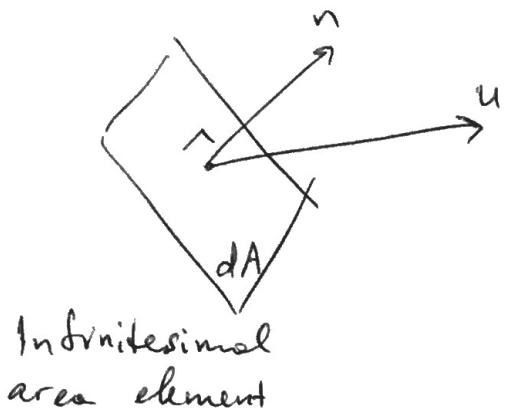
Other useful notations :

$$\text{grad } \rho = \nabla \rho, \quad dV u = \nabla \cdot u, \quad \nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\int_D \nabla \cdot F dV = \oint_S (F \cdot n) dA$$

Change of mass in the volume $W \subset D$:

$$\frac{d}{dt} m(W, t) = \frac{d}{dt} \int_W \rho(x, t) dV = \int_W \frac{\partial \rho}{\partial t}(x, t) dV.$$



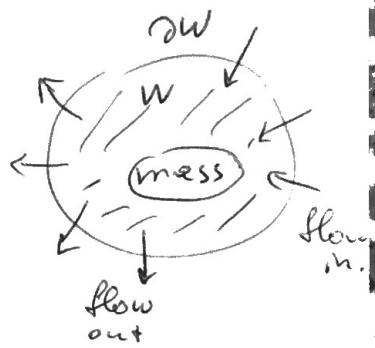
Flow rate (per unit time):
 $(u \cdot n)$

Mass flow rate :

$$(\rho u \cdot n)$$

Mass balance :

$$\frac{d}{dt} \underbrace{\int_W \rho dV}_{\text{change}} = - \underbrace{\int_{\partial W} \rho u \cdot n dA}_{\text{flow through the boundary}}$$



$$\int_W \frac{\partial \rho}{\partial t} dV = - \int_W \operatorname{div}(\rho u) dW$$

$$\boxed{\int_W \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) \right) dW = 0} \quad \text{for any volume } W.$$

integral form

Continuity equation :

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0} \quad \text{differential form}$$

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Conservation of momentum;

$\mathbf{r}(+) = (x(+), y(+), z(+))$ fluid particle.

$$\mathbf{u}(\mathbf{r}, +) = \frac{d\mathbf{r}}{dt} \quad (\text{at particle velocity any point and time}).$$

$$\mathbf{a}(+) = \frac{d^2\mathbf{r}}{dt^2} \quad (\text{particle acceleration}) .$$

$$\mathbf{a}(+) = \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt} \mathbf{u}(\mathbf{r}(+), +) = \frac{\partial u}{\partial x} \dot{x} + \frac{\partial v}{\partial y} \dot{y} + \frac{\partial w}{\partial z} \dot{z} + \frac{\partial u}{\partial t}$$

$$\text{Other notations : } \frac{\partial u}{\partial x} = \partial_x u = u_x .$$

$$\begin{aligned} \mathbf{a}(+) &= \cancel{\frac{\partial \mathbf{u}}{\partial t}} \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ &= \frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \left(\cancel{\frac{\partial}{\partial t}} + \mathbf{u} \cdot \nabla \right) \mathbf{u} . \end{aligned}$$

$$\text{Here } \mathbf{u} \cdot \nabla = u \partial_x + v \partial_y + w \partial_z .$$

In general, for any function $f(\mathbf{r}, +)$, a derivative along particle trajectory is

$$\frac{d}{dt} (f(\mathbf{r}(+), +)) = \cancel{\frac{\partial f}{\partial t}} + \cancel{\mathbf{r} \cdot \nabla f} = \cancel{\frac{\partial f}{\partial t}} + \mathbf{u} \cdot \nabla f$$

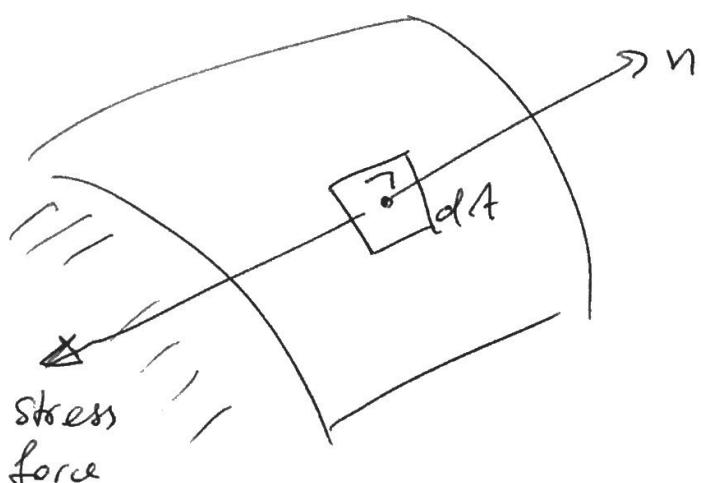
or $\frac{d}{dt} f(\mathbf{r}(+), +) = \frac{Df}{Dt} , \boxed{\frac{D}{Dt} = \cancel{\frac{\partial}{\partial t}} + \mathbf{u} \cdot \nabla} \quad (\text{material derivative})$

Forces : stress (between fluid particles)
— applied to a surface

body forces (e.g. gravity).

For ideal fluid, stress forces are given by
the pressure.

Pressure



Force per ~~unit~~ area dA
is given by

$$- p(r, t) \vec{n} dA$$

with a scalar function

$$p(r, t)$$

celled pressure

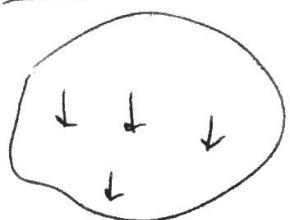
Obs Force is the same in all directions \vec{n}
(isotropy of the fluid).

~~James B. Woods~~

Body Forces :

$$F_{ext} = \int_W g \vec{b} dV$$

where $\vec{g}_B(r,t)$ is the force field.



For example: $\vec{b} = \vec{g}$ (gravity)

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Total force on a fluid element $w \subset D$

$$F = - \int_{\partial w} p n dA + \int_w g b dV =$$

$$= - \int_w \nabla p dV + \int_w g b dV =$$

$$= \int_w (-\nabla p + g b) dV$$

\Rightarrow force applied to a fluid element dV
 is equal to $(-\nabla p + g b) dV$

Mass of fluid element is ρdV

Acceleration : $\frac{Du}{Dt}$

2^d Newton's law: $\int_w \frac{Du}{Dt} dV = (-\nabla p + g b) dV$

$$\Rightarrow \int_w \frac{Du}{Dt} = -\nabla p + g b \Rightarrow \frac{Du}{Dt} = -\frac{\nabla p}{\rho} + b$$

OR

$$\left[\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{\nabla p}{\rho} + b \right] \text{The Euler equations (1757)}$$

Momentum (balance) equation

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Let us fix some volume $W \subset D$.

$$\frac{d}{dt} \int_W \rho u \, dV = \int_W \frac{\partial}{\partial t} (\rho u) \, dV = \int_W \left(\frac{\partial \rho}{\partial t} u + \rho \frac{\partial u}{\partial t} \right) \, dV =$$

w is fixed! ↑
continuity eq. Euler eq.

$$= \int_W \left(- \nabla \cdot (\rho u) u - (\rho u \cdot \nabla) u - \nabla p + \rho b \right) \, dV$$

1st 2d terms: $\int_W (\nabla \cdot (\rho u) u \cancel{+} (\rho u \cdot \nabla) u) \, dV = (*)$

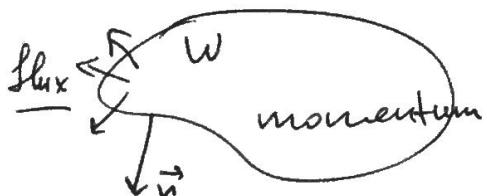
For any ~~fixed~~ $e \in \mathbb{R}^3$:

$$\begin{aligned} (*) \cdot e &= \int_W [\nabla \cdot (\rho u)(u \cdot e) + (\rho u \cdot \nabla)(u \cdot e)] \, dV = \\ &= \int_W \nabla \cdot [\cancel{\rho u}(u \cdot e)] \, dV = \int_{\partial W} \rho(u \cdot n)(u \cdot e) \, dA \\ \Rightarrow (*) &= \int_{\partial W} \rho u (u \cdot n) \, dA. \end{aligned}$$

3d term: $\int_W \nabla p \, dV = \int_{\partial W} \vec{p} \cdot \vec{n} \, dA.$

All together yields:

$$\frac{d}{dt} \int_W \rho u \, dV = - \int_{\partial W} [p n + \rho u (u \cdot n)] \, dA + \int_W \rho b \, dV$$



momentum flux
due to pressure
and flow

body
force.

Incompressible flow

Variables ρ, u, p (5 in total)

Equations continuity + Euler (4 in total) } 1 eq.
is missing!

Incompressibility condition: $\rho = \text{const}$ along every trajectory of fluid particle $r(t) \Rightarrow \frac{D\rho}{Dt} = 0$

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho = 0 \quad \cancel{\text{cancel}}$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) - \rho \nabla \cdot \mathbf{u}}_{=0 \text{ (continuity)}} = 0$$

If $\rho \neq 0 \Rightarrow \nabla \cdot \mathbf{u} = 0$ or

$$\boxed{\nabla \cdot \mathbf{u} = 0}$$

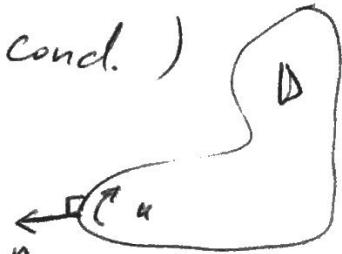
incompressibility
condition.

Complete system in the case $\rho = \text{const}$

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = - \frac{\nabla p}{\rho} + \mathbf{g} \quad (\text{Euler eqs.}) \\ \nabla \cdot \mathbf{u} = 0 \end{array} \right.$$

(incomp. cond.)

4 eqs., 4 variables (\mathbf{u}, p)



Boundary conditions (rigid wall): $\mathbf{u} \cdot \mathbf{n} = 0$ on ∂D

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Compressible (isentropic) flow

$$\rho = \rho(p) \quad \text{or} \quad p = P(\rho)$$

Let us define a function such that

$$dw = \frac{dp}{\rho} \Rightarrow \frac{dw}{dp} = \frac{P'(p)}{\rho} \Rightarrow \boxed{w = \int \frac{P'(p)}{\rho} dp}$$

enthalpy

Complete system:

$$\left. \begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\nabla w + f \\ \frac{\partial p}{\partial t} + \nabla \cdot (pu) &= 0 \end{aligned} \right\} \text{in } D$$

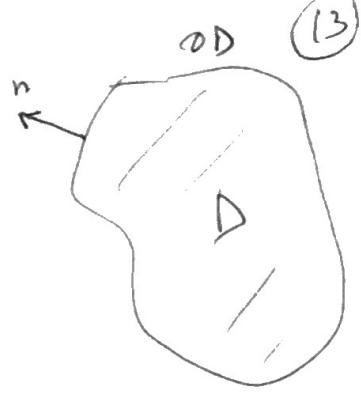
Boundary condition ~~for~~: $u \cdot n = 0$ on ∂D .

Example: $p = A\rho^\gamma$ (ideal gas)

$$w = \int \frac{\gamma A p^{\gamma-1}}{\rho} dp = \gamma A \int p^{\gamma-2} dp = \frac{\gamma A p^{\gamma-1}}{\gamma-1}$$

Ideal fluid

Compressible (isentropic) $P = P(\rho)$.



$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = - \frac{\nabla P}{\rho} + f \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \end{array} \right.$$

in D

Boundary conditions : $u \cdot n = 0$ in ∂D

Incompressible ($\frac{D\rho}{Dt} = 0$)

In the case $\rho \equiv \text{const}$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + (u \cdot \nabla) u = - \frac{\nabla P}{\rho} + f \\ \nabla \cdot u = 0 \end{array} \right.$$

in D

Boundary cond. : $u \cdot n = 0$ in ∂D

Pressure in incompressible flow?

Take $\nabla v = \nabla \cdot$ of the Euler eq. : $\nabla \cdot \nabla p = \Delta p$

$$\Rightarrow \Delta p = \rho (\nabla \cdot f - \nabla \cdot [(u \cdot \nabla) u]) \quad \text{Poisson's eq.}$$

Take $(n \cdot *)$ of the Euler eq. : $n \cdot \nabla p = \partial_n p$
(normal derivative)

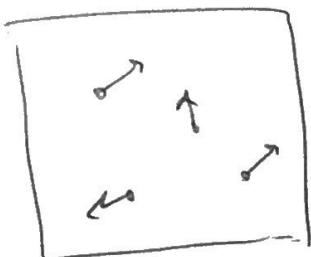
$$\partial_n p = \rho n \cdot (f - (u \cdot \nabla) u) \text{ in } \partial D$$

Newmann-type
boundary condition

Solution can be expressed using Green's functions (PDE theory). Nonlocality!

Laws of Thermodynamics

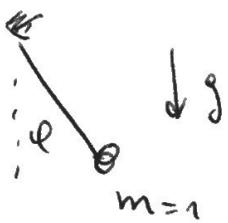
(16)
AFTER 13



Fluid as interacting particles
(microscopic description)
requires statistical theory (probabilities)

Thermodynamic equilibrium is a stationary state of the system in a statistical sense.

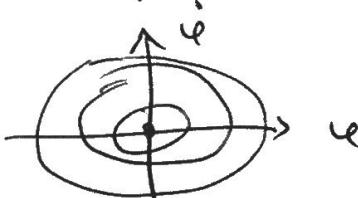
Illustrative example (pendulum)



$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

$$E = \frac{\dot{\theta}^2}{2} - \frac{g}{l} \cos \theta = \text{const.}$$

Phase space $(\theta, \dot{\theta})$ dynamics



(a) Each trajectory belongs to $E = \text{const}$ level.

(b) Stationary ~~stationary~~ state:

$(\theta, \dot{\theta})$ belongs to $E = \text{const}$ surface with some probability.

Thermod. equilibrium of a large system is described by two parameters: volume (V) and energy (E).

This description ~~is~~ is not convenient, because E is hard to measure \Rightarrow we need another variable!

Large system: n particles

Phase space: $\bigodot_{\text{all particles}}^{(r) \quad (mv)}$ (coordinate, momentum).

We can define: V_{PS} = statistical mean value of phase-space volume occupied by the system.

$$\underline{\text{Def}} \quad \text{Entropy} \quad S = \log \left[\frac{1}{(2\pi\hbar)^n} V_{PS} \right]$$

↑
factor to match quant. mech. def.

Entropy measures a "disorder" in a system.

2^d law of thermodynamics: $S(t)$ cannot decay in time

$$\left(\frac{ds}{dt} > 0 \right)$$

Adiabatic (reversible) process: $S = \text{const}$, $\frac{ds}{dt} = 0$,

Obs If system is divided into two parts (1) and (2),

~~they~~ V_{PS} which are independent, then $V_{PS} = V_{PS}^{(1)} \cdot V_{PS}^{(2)}$

$$\Rightarrow S = S^{(1)} + S^{(2)}, \text{ an additive function!}$$

In thermodynamic equilibrium $S = S(E, V)$
 (solving for E) $\Rightarrow E = E(S, V)$.

(1) $dS = 0, dV \neq 0$ (reversible compression)

$$dE = \text{work} = -PdA \cdot dx = -PdV$$

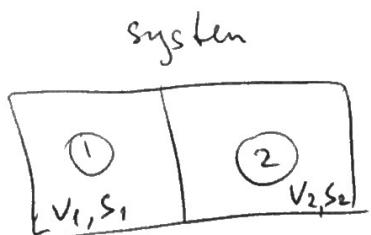


$$\Rightarrow P = -\frac{\partial E}{\partial V} \quad (\underline{\text{pressure}})$$

(2) $ds \neq 0, dV = 0$.

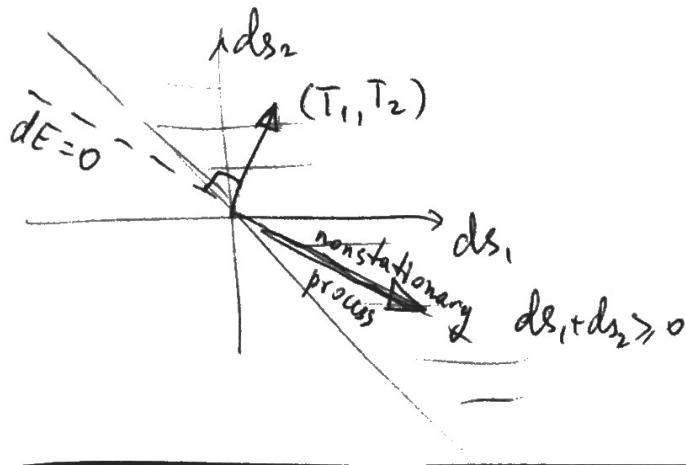
$$dE = Tds, \quad T \stackrel{\text{(def)}}{=} \frac{\partial E}{\partial S} \quad (\underline{\text{temperature}})$$

What is the temperature for?



Let us consider a process with no volumetric changes $dV_1 = 0, dV_2 = 0$ for isolated system (1) + (2).

$$\left. \begin{array}{l} dE = dE_1 + dE_2 = T_1 ds_1 + T_2 ds_2 = 0 \\ ds = ds_1 + ds_2 \geq 0. \end{array} \right\} \quad \begin{array}{l} \text{(conservation)} \\ \text{of energy} \end{array}$$



If $T_1 \neq T_2 \Rightarrow \exists$ nonstationary process with $ds > 0$.

$\Rightarrow T_1 = T_2$ in thermodynamic equilibrium

(Easy to measure!)

Thermodynamics of fluid

Ideal fluid assumes that every ^{small} fluid element ~~is~~ is locally in thermodynamic equilibrium. The flow that keeps system at therm. equilibrium is adiabatic ("slow"), which means that it does not change the ~~E~~ entropy of every small fluid element : isentropic flow. All processes are reversible.

$$d\delta = 0 \Rightarrow dE = -PdV$$

For unit mass: $V = \frac{1}{P} \Rightarrow dV = -\frac{dp}{P^2}$

$$\boxed{dE = \cancel{\frac{P}{P^2}} dp.} \quad E = E(P)$$

energy per unit mass.

In this case: $P = \cancel{P^2} \frac{dE}{dp} = P(E).$

Total energy:

$$E = T + U, \quad T = \sum_{\text{particles}} \frac{m}{2} v_i^2.$$

For ~~the~~ a small fluid element: $\langle v \rangle = \sum v_i,$

$$T = \sum \frac{m}{2} v_i^2 = \sum \frac{m}{2} (\langle v \rangle + \Delta v_i)^2 =$$

$$= \underbrace{\frac{1}{2} (2m) \langle v \rangle^2}_{\text{total mass}} + \underbrace{\frac{1}{2} \sum \frac{m}{2} (\Delta v_i)^2}_{T_{\text{int}}} + m \langle v \rangle \sum \Delta v_i$$

$\Delta v_i = v_i - \langle v \rangle$
 Speeds in reference frame of fluid

We decompose the energy as

$$E = \underbrace{\frac{1}{2} m \langle v \rangle^2}_{\text{macroscopic kinetic energy}} + \underbrace{T^{int}}_{E^{int}} + U$$

$\langle v \rangle = u$, fluid speed

For a small volume dV :

$$\frac{1}{2} m \langle v \rangle^2 \rightarrow \frac{1}{2} \rho u^2 dV$$

$$E^{int} \rightarrow \rho \varepsilon^{int} dV, \quad \varepsilon^{int} \text{ is the internal energy per unit mass}$$

Total energy in Volume W :

$$E = \int_W \frac{1}{2} \rho u^2 dV + \int_W \rho \varepsilon^{int} dV$$

$$\varepsilon^{int} = \varepsilon^{int}(\rho) \quad \text{for isentropic flow.}$$

$$\frac{d\varepsilon^{int}}{d\rho} = \frac{P}{\rho^2} \quad (\text{thermodynamics})$$

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$$\frac{dE}{dt} = \frac{d}{dt} \int_w \left(\rho \frac{|u|^2}{2} + \rho \varepsilon^{int} \right) dV$$

$$\frac{\partial \rho \varepsilon^{int}}{\partial p} = \varepsilon^{int} + \rho \frac{\partial \varepsilon^{int}}{\partial p} = \varepsilon^{int} + \underbrace{\frac{P}{\rho}}_{\text{enthalpy}} (E + PV)$$

$$w \stackrel{\text{def}}{=} \varepsilon^{int} + \frac{P}{\rho}$$

$$\frac{dw}{dp} = \frac{P}{\rho^2} + \left(\frac{\partial P}{\partial \rho} - \rho \frac{\partial P}{\partial \rho^2} \right) = \frac{1}{\rho} \frac{\partial P}{\partial \rho}$$

$$\frac{dE}{dt} = \int_w \left(\frac{\partial \rho}{\partial t} \frac{|u|^2}{2} + \rho u \cdot \frac{\partial u}{\partial t} + \frac{\partial \rho \varepsilon^{int}}{\partial p} \frac{\partial p}{\partial t} \right) dV \quad (=)$$

cont. eq.

Euler eq.

thermod. + cont. eq.

$$-\cancel{\frac{|u|^2}{2} \nabla \cdot (\rho u)} + \rho u \cdot \left(-(\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{\nabla P}{\rho} + b \right) \rightarrow w \nabla \cdot (\rho u)$$

$$(\mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} (\mathbf{u} \cdot \nabla) |\mathbf{u}|^2)$$

verify!

$$= \int_w \left\{ -\frac{|u|^2}{2} \nabla \cdot (\rho u) - (\rho u \cdot \nabla) \frac{|u|^2}{2} - u \cdot \nabla p + \rho u \cdot b - w \nabla \cdot (\rho u) \right\} dV$$

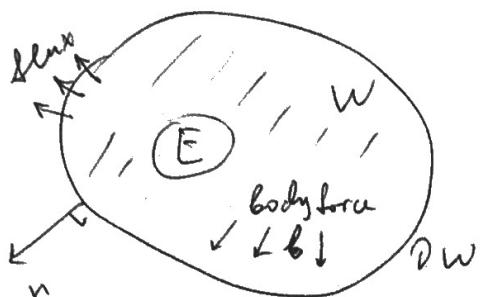
$$\left(\nabla p = \frac{\partial P}{\partial \rho} \nabla \rho = \rho \frac{\partial w}{\partial \rho} \nabla p = \rho \nabla w \right)$$

$$= \int_w -\nabla \cdot \left(\frac{|u|^2}{2} \rho u \right) - \nabla (w \rho u) + \rho u \cdot b$$

$$\frac{dE}{dt} = \frac{d}{dt} \int_w \left(\rho \frac{|u|^2}{2} + \rho \varepsilon^{int} \right) dV$$

$$= - \oint_{\partial W} \left(\rho \frac{|u|^2}{2} + w \right) \rho u \cdot n dA + \underbrace{\int_w b \cdot \rho u dV}_{\text{work of external forces}}$$

energy flux.



$$\text{Energy flux: } \left(\frac{|u|^2}{2} + w \right) \rho u$$

Includes flux of energy with fluid and work ~~done~~^{done} by pressure.

Obs 1:

In the case of incompressible flow : $\frac{D\rho}{Dt} = 0$.

If also $\rho \equiv \text{const}$ $\Rightarrow \varepsilon^{int}(\rho) = \text{const.}$

In this case, the flow energy is ~~only~~ simply

$$E = \int \rho \frac{|u|^2}{2} dV \quad \text{(kinetic energy).}$$

and $\varepsilon^{int} \rightarrow 0$, $w \rightarrow \frac{P}{\rho}$.

Obs 2:

$$\frac{dw}{dp} = \frac{1}{\rho} \frac{dp}{dp} \Rightarrow w = \int \frac{1}{\rho} \frac{dp}{dp} dp = \int \frac{P'(p)}{\rho} dp$$

(we used this before)

Exercises:

① prove $\mathbf{u} \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} (\mathbf{u} \cdot \nabla) |\mathbf{u}|^2$

② prove $\frac{D}{Dt} (f+g) = \frac{Df}{Dt} + \frac{Dg}{Dt}$

$$\frac{D}{Dt} (fg) = f \frac{Dg}{Dt} + g \frac{Df}{Dt} \quad (\text{Leibniz rule})$$

$$\frac{D}{Dt} (h \circ g) = (h' \circ g) \frac{Dg}{Dt} \quad (\text{composition chain rule})$$

- ③ Derive energy equation for incompressible fluid from ~~extended~~ Euler equations and incompressibility condition.

Eulerian vs. Lagrangian description

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Eulerian descr. is based on a specific location

in space: observer is fixed in space

$$u = u(x, t), \quad \rho = \rho(x, t), \quad P = P(x, t), \text{ etc.}$$

Lagrangian descr. follows an individual

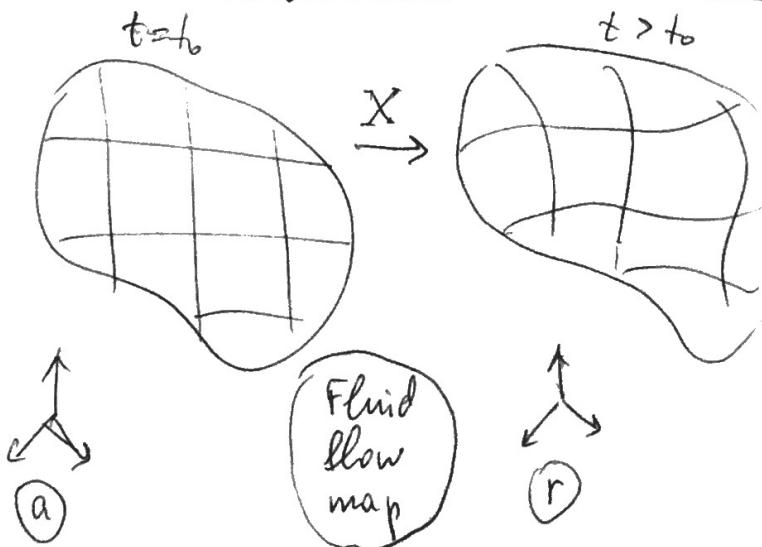
fluid element (parcel): observer is fixed in fluid.

Lagrangian description requires new variables (markers), which label each fluid element.

Usual way is to use the coordinates $a \in \mathbb{R}^3$ of the "Lagrangian particle" at $t = t_0$.

$\mathbf{x} = \mathbf{X}(a, t)$ is a particle trajectory with $\mathbf{X}(a, t_0) = a$.

$\mathbf{X}(a, t)$ maps initial position of ~~marker~~ Lagrangian particles at $t = t_0$ to their position at time t .



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All variables can be written in the Lagrangian representation: $u = u(a, t)$, $\rho = \rho(a, t)$, etc.

Velocity: $u = \frac{D\mathbf{r}}{Dt} = \frac{\partial \mathbf{X}}{\partial t}(a, t)$ for a particle "a".

Density:

$\rho(a, t_0) = \rho_0(a)$ is the initial density at $t=t_0$

How to find density for $t > t_0$?

Mass conservation: density * volume = const.

Volume at t_0 : $dV_0 = d^3a$

Volume at t : $dV = dx dy dz = \left| \det \frac{\partial \mathbf{X}}{\partial a} \right| dV_0$
 $r = \mathbf{X}(a, t)$

$$\rho(a, t) \left| \det \frac{\partial \mathbf{X}}{\partial a} \right| dV_0 = \rho_0(a) dV_0$$

$$\Rightarrow \boxed{\rho(a, t) = \frac{\rho_0(a)}{\left| \det \frac{\partial \mathbf{X}}{\partial a} \right|}}$$

The change of density is described by the Jacobian of the mapping $\mathbf{X}(a, t)$.

For incompressible flow: $\rho \equiv \text{const}$ along a trajectory

$$\Rightarrow \det \frac{\partial \mathbf{X}}{\partial a} = 1 \quad (\text{volume-preserving map}).$$

Equation of motion

$$\frac{\partial^2 \mathbf{X}}{\partial t^2} = \underbrace{\frac{\partial \mathbf{u}}{\partial t}(\mathbf{a}, t)}_{\text{Lagrangian}} = \underbrace{\frac{D\mathbf{u}}{Dt}(\mathbf{r}, t)}_{\text{Eulerian}} = - \frac{\nabla P}{\rho} + \mathbf{b}$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

Here:

$$\mathbf{g} = \mathbf{g}(\mathbf{a}, t)$$

$$\mathbf{b} = \mathbf{b}(\mathbf{r}, t) \quad \text{with} \quad \mathbf{r} = \mathbf{X}(\mathbf{a}, t)$$

For isentropic (compressible) flow: $P = P(\rho)$

$$\nabla P = \frac{dp}{d\rho}(\rho) \nabla \rho$$

$$\mathbf{r} = \mathbf{X}(\mathbf{a}, t) \rightarrow \mathbf{a} = \bar{\mathbf{X}}^{-1}(\mathbf{r}, t), \quad \mathbf{a} = (\xi, \eta, \zeta)$$

$$\text{Chain rule: } \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta}, \text{ etc.}$$

$$\Rightarrow \nabla = \left(\frac{\partial \mathbf{a}}{\partial \mathbf{r}} \right)^T \nabla_r, \quad \nabla_r = \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{pmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial \mathbf{r}} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{pmatrix} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{a}} \right)^{-1} = \left(\frac{\partial \bar{\mathbf{X}}(\mathbf{a}, t)}{\partial \mathbf{a}} \right)^{-1}$$

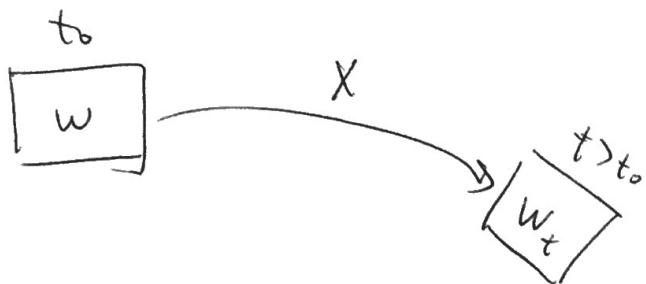
Complicated, ... but complete.

Similar derivation is possible for incompressible case

Material volume

(26)

$w_t = X(w, t)$ volume moving with fluid



Transport theorem for \mathcal{V} function $f(r, t)$,

$$\frac{d}{dt} \int_{w_t} f dV = \int_{w_t} \frac{\partial f}{\partial t} dV + \int_{\partial w_t} f(u \cdot n) f dA.$$

(valid for f scalar, vector, tensor etc.)

Proof

$$r = X(a, t), \quad \cancel{\text{det } J}$$

~~DEFORMATION~~

$$dV = J dV_0, \quad J = \det \frac{\partial X}{\partial a} \quad \left(J > 0 \text{ for orientation-preserving map} \right)$$

$$\hat{f}(a, t) = f(X(a, t), t)$$

$$\int_{w_t} f dV = \int_w \hat{f}(a, t) J dV_0.$$

Lemma

$$\frac{D J}{Dt} = \frac{\partial}{\partial t} J(a, t) = J (\nabla \cdot u)$$

$$(\nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})$$

Proof (Lemma)

$$J = \det \left(\frac{\partial \underline{X}}{\partial a} \right), \text{ where } r = \underline{X}(a, t) \text{ is the Lagrangian map.}$$

↓
Jacobi matrix

For a temporal derivative, we have

$$\frac{\partial}{\partial t} \left(\frac{\partial \underline{X}}{\partial a} \right) = \frac{\partial}{\partial a} \frac{\partial \underline{X}}{\partial t} = \frac{\partial u}{\partial a} = \left(\frac{\partial u}{\partial r} \right) \left(\frac{\partial \underline{X}}{\partial a} \right), \text{ a product of two Jacobi matrices.}$$

↑
chain rule
with $r = \underline{X}(a, t)$

Use Jacobi's formula: $\frac{d}{dt} \det A = (\det A) \cdot \text{tr} \left[A^{-1} \frac{dA}{dt} \right]$

(see linear algebra, or Wikipedia)

$$\begin{aligned} \frac{\partial J(a, t)}{\partial t} &= \frac{\partial}{\partial t} \left(\det \left(\frac{\partial \underline{X}}{\partial a} \right) \right) = \det \left(\frac{\partial \underline{X}}{\partial a} \right) \cdot \text{tr} \left[\left(\frac{\partial \underline{X}}{\partial a} \right)^{-1} \frac{\partial}{\partial t} \left(\frac{\partial \underline{X}}{\partial a} \right) \right] = \\ &= J \cdot \text{tr} \left[\left(\frac{\partial \underline{X}}{\partial a} \right)^{-1} \frac{\partial u}{\partial r} \frac{\partial \underline{X}}{\partial a} \right] = J \cdot \text{tr} \left(\frac{\partial u}{\partial r} \right) = \\ &= J (\nabla \cdot u). \quad \blacksquare \end{aligned}$$

$$\frac{\partial}{\partial t} \int_{W_t} f dV = \frac{d}{dt} \int_W \hat{f}(a,+) J dV_0 = \int_W \left(\frac{\partial \hat{f}}{\partial t} J + \hat{f} \frac{\partial J}{\partial t} \right) dV_0 =$$

$$= \int_W \left[\left(\frac{\partial f}{\partial t} + \nabla f \cdot \frac{\partial X}{\partial t} \right) J + f J (\nabla \cdot u) \right] dV_0 =$$

$\hat{f}(a,+) = f(r,+) \text{, } r = X(a,+)$

$$= \int_W \left[\frac{\partial f}{\partial t} + \nabla f \cdot u + f \nabla \cdot u \right] J dV_0 =$$

$$= \int_{W_t} \left[\frac{\partial f}{\partial t} + \nabla \cdot (fu) \right] dV = \int_{W_t} \frac{\partial f}{\partial t} dV + \oint_{\partial W_t} f f(u \cdot n) dA.$$

\uparrow divergence theorem

Example : $f(r,+) = \rho(r,+)$

$$\frac{d}{dt} \int_{W_t} g dV = \int_{W_t} \frac{\partial g}{\partial t} dV + \oint_{\partial W_t} g u \cdot n dA = \int_{W_t} \left[\frac{\partial g}{\partial t} + \nabla \cdot (gu) \right] dV$$

$$= 0.$$

(mass conservation).