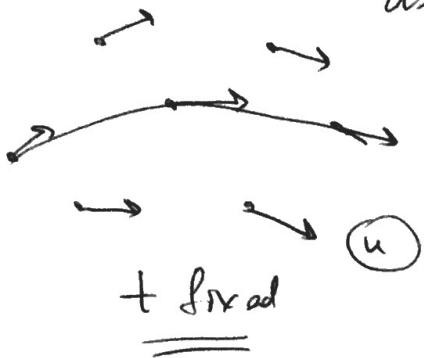


## Bernoulli's theorem

Given a fluid flow with  $u = u(r, t)$ ,  
a streamline at a fixed time is an integral  
 curve of  $u$ :  $r = r(s)$

$$\frac{dr}{ds} = u(r(s), t) , \quad t \text{ fixed.}$$



Stationary flow:  $\frac{\partial u}{\partial t} = 0 , \quad u = u(r)$ .

(streamline = trajectory)

Let  $f = 0$  (no body forces)

Theorem (Bernoulli)

In stationary isentropic flow,

$$\frac{1}{2} |u|^2 + \overset{\uparrow}{w}$$

enstrophy

is constant along streamlines (trajectories).

Proof

$$\frac{1}{2} \nabla |u|^2 = (u \cdot \nabla) u + u \times (\nabla \times u)$$

(Verify)

Euler eq. for steady isentr. flow :

$$\cancel{\frac{\partial u}{\partial t}} + (u \cdot \nabla) u = - \nabla w$$

$$\Rightarrow \nabla \left( \frac{1}{2} |u|^2 + w \right) = u \times (\nabla \times u)$$

Integrating this expression along the streamline between  $r_1 = r(s_1)$  and  $r_2 = r(s_2)$ , we have

$$\cancel{\int \left( \frac{1}{2} |u|^2 + w \right) ds} \Big|_{s_1}^{s_2} = \int_{r_1}^{r_2} \nabla \left( \frac{1}{2} |u|^2 + w \right) \cdot dr =$$

$$= \int_{s_1}^{s_2} [u \times (\nabla \times u)] \cdot \underbrace{r'(s)}_u ds = 0. \quad \square$$

In case of incompressible flow : ~~del~~

$$(u \cdot \nabla) u = - \frac{\nabla p}{\rho}, \quad p = \text{const}$$

$$\Rightarrow \left\{ \begin{array}{l} w \rightarrow \frac{\nabla p}{\rho} \\ \frac{1}{2} |u|^2 + \frac{p}{\rho} = \text{const} \end{array} \right. \text{ along a streamline}$$

Obs: increase of speed is related to decrease of pressure.

# Bernoulli's eq. theorem for flows with gravity

(31)

$$f = -g e_2 \quad \downarrow g$$

$$\text{Euler eq. : } (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla w - g e_2 = -\nabla(w + gz)$$

$$\Rightarrow \frac{1}{2} |\mathbf{u}|^2 + w + gz = \text{const} \quad (\text{along a streamline})$$

For incompressible flow

$$\frac{1}{2} |\mathbf{u}|^2 + \frac{P}{\rho} + gz = \text{const}$$

$$\underline{\hspace{10cm}} \sim \underline{\hspace{10cm}}$$

Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left( \partial_y w - \partial_z v, \partial_z u - \partial_x w, \partial_x v - \partial_y u \right)$$

(vorticity vector)  $\frac{\text{rot } \mathbf{u}}{\text{curl } \mathbf{u}}$

Local velocity field :

$$\mathbf{u}(\mathbf{r} + \Delta \mathbf{r}) = \mathbf{u}(\mathbf{r}) + \left( \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \right) \Delta \mathbf{r} + o(|\Delta \mathbf{r}|)$$

Jacobian matrix

$$\frac{\partial \mathbf{u}}{\partial \mathbf{r}} = D + S$$

symmetric matrix  $\nwarrow$  antisymmetric matrix

$$D = \frac{1}{2} \left[ \frac{\partial \mathbf{u}}{\partial \mathbf{r}} + \left( \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \right)^T \right]$$

$$S = \frac{1}{2} \left[ \frac{\partial \mathbf{u}}{\partial \mathbf{r}} - \left( \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \right)^T \right].$$

Using  $-\frac{\partial \mathbf{u}}{\partial r} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$   $\rightarrow S = \frac{1}{2} \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$

for  $\omega = (\omega_x, \omega_y, \omega_z)$

Hence  $S \Delta r = \frac{1}{2} \begin{pmatrix} \omega_y \Delta z - \omega_z \Delta y \\ \omega_z \Delta x - \omega_x \Delta z \\ \omega_x \Delta y - \omega_y \Delta x \end{pmatrix} = \frac{1}{2} \omega \times \Delta r$

As a result,

$$\mathbf{u}(r + \Delta r) = \mathbf{u}(r) + \frac{1}{2} \omega \times \Delta r + o(|\Delta r|).$$

Meaning of D and  $\omega$ :

(a) Let  $\mathbf{u} = Dr$  ( $D = \text{const}$ )  $\Rightarrow$

$\exists$  orthonormal basis with coordinates  $\tilde{r}$  such that

$$\tilde{\mathbf{u}} = \begin{pmatrix} D_x & 0 & 0 \\ 0 & D_y & 0 \\ 0 & 0 & D_z \end{pmatrix} \tilde{r} \Rightarrow \frac{dx}{dt} = D_1 x, \frac{dy}{dt} = D_2 y, \dots$$

$\Rightarrow x = x_0 e^{\frac{D_x t}{2}}$  expansion ( $D_x > 0$ ) or contraction ( $D_x < 0$ )  
in direction x.

same for y and z.

for a volume  $V = xyz$ , we have (33)

$$\frac{dV}{dt} = (D_x + D_y + D_z)xyz = (\text{tr } D)V$$

$\text{tr } D$  does not depend on a change of coordinates

$$\Rightarrow \text{tr } D = \frac{1}{2} \text{Tr} \left[ \frac{\partial u}{\partial r} + \left( \frac{\partial u}{\partial r} \right)^T \right] = \text{Tr} \left( \frac{\partial u}{\partial r} \right) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

or  $\text{tr } D = \nabla \cdot u$  ( $dV u$  is responsible for volume expansion/contraction)

(6) Let  $u = \frac{1}{2} \omega \times r$  ( $\omega = \text{const}$ )

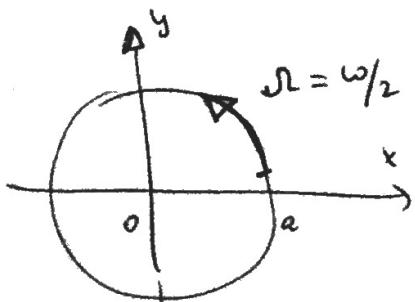
Let us consider coordinates with  $\omega = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$

$$u = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix}$$

\* Trajectories:

$$\frac{dx}{dt} = -\frac{\omega}{2}y, \quad \frac{dy}{dt} = \frac{\omega}{2}x \Rightarrow \frac{d^2x}{dt^2} = -\frac{\omega^2}{4}x$$

$$\Rightarrow x = a \cos\left(\frac{\omega t}{2} + \varphi\right), \quad y = -\frac{2}{\omega} \frac{dx}{dt} = a \sin\left(\frac{\omega t}{2} + \varphi\right)$$



rotation with angular velocity  $\omega_2 = \omega/2$ .

(with no change of volume)

(34)

There are no tangential forces in ~~ideal~~ ideal fluid.



$$\vec{p} = \rho \vec{u}.$$

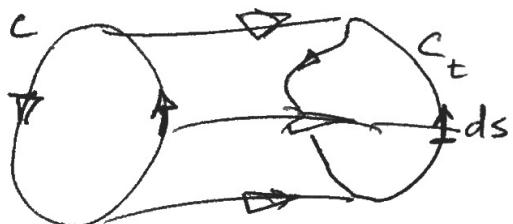
$\Rightarrow$  rotation cannot be induced or stopped.

Let  $C$  be a closed contour in  $\mathbb{R}^3$  at  $t_0$ .

$C_t = X(C, t)$  is a material contour (moving with) fluid

Def Circulation around  $C_t$  is defined as

$$F_{C_t} = \oint_C u \cdot ds \quad , \text{ where } ds \text{ is a counter length element along the counter}$$



Th (Kelvin circulation theorem)

For isentropic flow,  $F_{C_t}$  does not depend on time.  
(in the case  $b=0$ ).

Lemma

$$\frac{d}{dt} \oint_C u \cdot ds = \oint_C \frac{Du}{Dt} \cdot ds$$

## Proof (Lemme)

Let  $a(z)$  be parametrization of the loop  $C$  at  $t_0$ .

$$C = \{a(z) : 0 \leq z \leq 1\}.$$

$$\text{Then } C_t = \{r = X(a(z), t) : 0 \leq z \leq 1\}.$$

$$\begin{aligned} \frac{d}{dt} \int_C u \cdot ds &= \frac{d}{dt} \int_0^1 u(X(a(z), t), t) \cdot \underbrace{\frac{\partial}{\partial z} X(a(z), t)}_{ds} dz = \\ &= \int_0^1 \frac{Du}{Dt} \cdot \frac{\partial X}{\partial z} dz + \underbrace{\int_0^1 u(X, t) \cdot \frac{\partial}{\partial t} \frac{\partial}{\partial z} X(a(z), t) dz}_{\left( = \frac{\partial}{\partial z} \frac{\partial}{\partial t} X = \frac{\partial u}{\partial z} \right)} = \\ &= \cancel{\int_C \frac{Du}{Dt} \cdot ds} + \cancel{\int_0^1 \frac{\partial}{\partial z} \frac{|u|^2}{2} dz} \stackrel{o(\text{loop!})}{=} \int_C \frac{Du}{Dt} \cdot ds. \quad \blacksquare \end{aligned}$$

## Proof (theorem)

$$\text{Euler equation: } \frac{Du}{Dt} = -\nabla w \quad (\text{centropic flow})$$

$$\frac{d}{dt} \int_{C_t} u \cdot ds = \frac{d}{dt} \int_{C_t} u \cdot ds \stackrel{\text{Lemme}}{=} \int_{C_t} \frac{Du}{Dt} \cdot ds = - \int_{C_t} \nabla w \cdot ds = 0 \quad \text{Loop!} \quad \blacksquare$$

Obs. Decrease of loop  $\Rightarrow$  increase of speed  $\rightarrow \omega$ .

Obs. Theorem is valid for potential body forces, e.g.

$$\text{gravity: } \frac{Du}{Dt} = -\nabla w - g e_z = -\nabla(w + gz).$$

Ob: Incompressible flow with  $\rho = \text{const}$ :  $\nabla \cdot \mathbf{v} = 0 \Rightarrow \nabla \cdot (\frac{\rho}{\rho} \mathbf{v}) = \nabla \cdot \mathbf{v} = 0$  (36)

Stokes theorem:  $\int_C \mathbf{v} \cdot d\mathbf{s} = \iint_{\Sigma} (\nabla \times \mathbf{v}) \cdot d\mathbf{A} =$

$$= \iint_{\Sigma} (\nabla \times \mathbf{v}) \cdot \mathbf{n} dA = \iint_{\Sigma} \omega \cdot \mathbf{n} dA$$

flux of vorticity across  $\Sigma$ .

Corollary Flux of vorticity across a material surface  $\Sigma_t = X(\Sigma, t)$  is constant in time

Def Vortex line (vortex sheet) is a curve (surface) tangent at every point to a vorticity vector.



Th If  $L(S)$  is the vortex curve (sheet) at  $t_0 \Rightarrow$   
 $\Rightarrow L_t = X(L, t)$  and  $(S_t = X(S, t))$  is the vortex curve (sheet) for isentropic fluid. at any  $t$ .

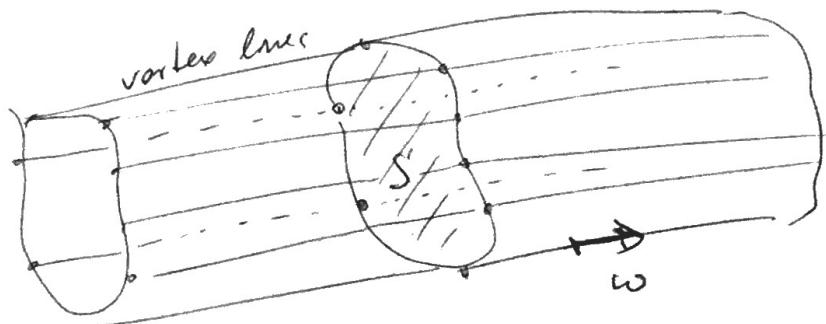
Proof For a ~~non~~ vortex sheet  $S$ . let  $\tilde{S} \subset S$  be any small portion of  $S$ . We have

$$\iint_{\tilde{S}_t} \omega \cdot \mathbf{n} dA = \iint_{\tilde{S}} \omega \cdot \mathbf{n} dA \stackrel{\text{(corollary)}}{=} 0 \stackrel{\text{vortex sheet}}{=} \Rightarrow \omega \cdot \mathbf{n} = 0 \text{ for any point of } S_t.$$

For a vortex line consider  $L = S_1 \cap S_2$  (intersection of two vortex sheets)

Let  $S$  be a surface transversal (not tangent) to vorticity field  $\omega$ . Assume that  $S$  is diffeomorphic to a disc.

Def Vortex tube: a union of vortex lines passing through all points of  $S$ :



### Helmholtz's theorem (isentropic fluid)

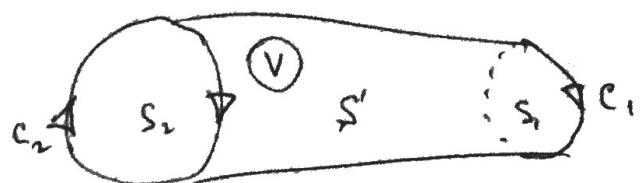
(a) For any circles  $C_1$  and  $C_2$  encircling the vortex tube,  $\int_{C_1} u \cdot ds = \int_{C_2} u \cdot ds$ ,

called the strength of the vortex tube.

(b) The strength is constant in time as the vortex tube moves with fluid.

### Proof

Consider the volume  $V$  in the tube between  $C_1$  and  $C_2$ .



$$\text{Surface: } \cancel{\partial V} = S_1 S, V S_2$$

Since  $\omega = \nabla \times u \Rightarrow \nabla \cdot \omega = 0$ .

$$0 = \int_V \nabla \cdot \omega \, dV = \int_V \omega \cdot n \, dA = \left( \int_{S_1} + \int_{S_2} + \int_S \right) \omega \cdot n \, dA$$

(Gauss' theorem)

In  $S$ :  ~~$\omega \cdot n = 0$~~  (~~Def of the vortex line~~)  
 $\Rightarrow \int_S \omega \cdot n \, dA = 0$ .

By Stokes' theorem:

$$\int_{S_1} \omega \cdot n \, dA = \int_{S_1} (\nabla \times u) \cdot n \, dA = \int_{C_1} u \cdot \vec{ds}$$

$$\int_{S_2} \omega \cdot n \, dA = - \int_{C_2} u \cdot \vec{ds} \quad (\text{orientation is opposite!})$$

$$\Rightarrow \int_{C_1} u \cdot ds = \int_{C_2} u \cdot ds = 0. \Rightarrow (a).$$

Kelvin's circulation theorem  $\Rightarrow (b)$ .

Typical shape of vortex tubes:



ends at a boundary

(39)

Obs ~~that~~ Above statements are valid for incompressible ideal fluid. ( $\rho = \text{const}$ )

### Euler equation for vorticity

(isentropic flow or incompressible fluid with  $\rho = \text{const}$ )

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla w \quad \text{or} \quad \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{\nabla p}{\rho}$$

Since:  $\frac{1}{2} \nabla |u|^2 = (u \cdot \nabla) u + u \times (\nabla \times u)$ , we have

$$(u \cdot \nabla) u = -u \times (\nabla \times u) + \frac{1}{2} \nabla |u|^2$$

$$\Rightarrow \frac{\partial u}{\partial t} \underset{\nabla \times u = \omega}{=} u \times (\nabla \times u) \underset{\text{or } -\nabla \left( \frac{p}{\rho} + \frac{|u|^2}{2} \right)}{=} -\nabla \left( w + \frac{|u|^2}{2} \right) \quad \left( \begin{array}{l} \text{or } -\nabla \left( \frac{p}{\rho} + \frac{|u|^2}{2} \right) \\ \text{for } \rho = \text{const.} \end{array} \right)$$

$$(\nabla \times) \Rightarrow \boxed{\frac{\partial \omega}{\partial t} = \nabla \times (u \times \omega)} \quad \frac{\partial \omega}{\partial t} = \text{rot}(u \times \omega).$$

$$\nabla \times (u \times \omega) \stackrel{\text{verify}}{=} u \cancel{(\nabla \omega)}^{\circ} - \omega (\nabla \cdot u) \cancel{-} (u \cdot \nabla) \omega + (\omega \cdot \nabla) u$$

$(\omega = \nabla \times u)$

(a) ~~incompressible~~ (isentropic) flow

~~$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + u \times (\nabla \times u) + \omega (\nabla \cdot u) \cancel{-} (u \cdot \nabla) \omega + (\omega \cdot \nabla) u$$~~

~~$$\frac{\partial \omega}{\partial t} = -\omega (\nabla \cdot u) - (u \cdot \nabla) \omega + (\omega \cdot \nabla) u.$$~~

Continuity:  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$  or

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho + \rho (\nabla \cdot \mathbf{u}) = 0$$

Using both eqs for  $\frac{\partial}{\partial t} \left( \frac{\omega}{\rho} \right)$  yields

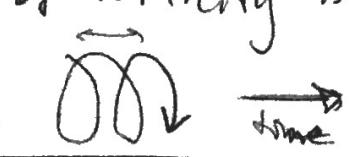
$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\omega}{\rho} \right) &= \frac{1}{\rho} \frac{\partial \omega}{\partial t} - \frac{\omega}{\rho^2} \frac{\partial \rho}{\partial t} = \frac{1}{\rho} [ -\omega (\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla) \omega + \\ &+ (\omega \cdot \nabla) \mathbf{u} ] + \frac{\omega}{\rho^2} [ (\mathbf{u} \cdot \nabla) \rho + \rho (\nabla \cdot \mathbf{u}) ] = \\ &= -\frac{1}{\rho} (\mathbf{u} \cdot \nabla) \omega + \frac{\omega}{\rho^2} (\mathbf{u} \cdot \nabla) \rho + \left( \frac{\omega}{\rho} \cdot \nabla \right) \mathbf{u} = \\ &= -(\mathbf{u} \cdot \nabla) \left( \frac{\omega}{\rho} \right) + \left( \frac{\omega}{\rho} \cdot \nabla \right) \mathbf{u}. \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial t} \left( \frac{\omega}{\rho} \right) + (\mathbf{u} \cdot \nabla) \left( \frac{\omega}{\rho} \right) \stackrel{\text{def}}{=} \boxed{\frac{D}{Dt} \left( \frac{\omega}{\rho} \right) = \left( \frac{\omega}{\rho} \cdot \nabla \right) \mathbf{u}}$$

another version of vorticity eq.

(b) InCompressible flow:  $\nabla \cdot \mathbf{u} = 0$ ,  $\rho = \text{const.}$

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = \boxed{\frac{D \omega}{Dt} = (\omega \cdot \nabla) \mathbf{u}}$$

Vortex stretching: a change of vorticity is related to diverging of velocity in direction of  $\omega$ : 

# Irrational flow (isentropic) ~~(irrotational)~~

$$\omega \equiv 0 \text{ at } t=t_0 \Rightarrow \frac{D\omega}{Dt} \equiv 0 \Rightarrow \omega \equiv 0 \text{ at } t > t_0.$$

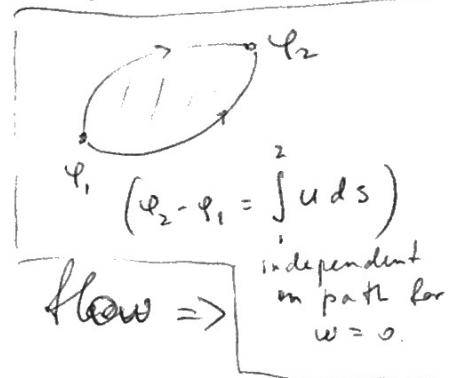
~~As vorticity~~

zero vorticity at  $t_0 \rightarrow$  zero vorticity at ~~every~~ all times.

Calculus (Helmholtz' theorem): A vector field can

be decomposed into a curl-free and divergence-free

components :  $\mathbf{u} = \underbrace{\nabla \varphi}_{\nabla \times \mathbf{u} \rightarrow 0} + \underbrace{\nabla \times \mathbf{A}}_{\nabla \cdot \mathbf{A} \rightarrow 0}$



Since  $\omega = \nabla \times \mathbf{u} = 0$  for irrational flow  $\Rightarrow$

$\Rightarrow \mathbf{u} = \nabla \varphi$  (no  $\mathbf{A} \equiv 0$ ) with the potential function  $\varphi$ .

Def Potential flow is an irrational flow of ideal (isentropic) fluid.

Ob There cannot be closed streamlines, which can be shrunked to a point (otherwise  $\exists$  nonzero circulation  $\Rightarrow \omega \neq 0$ ).



Euler eq:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla w$$

$$\underbrace{\frac{1}{2} \nabla |u|^2}_{\text{kinetic energy}} - \mathbf{u} \times (\nabla \times \mathbf{u})^0$$

$$\Rightarrow (\text{for } \mathbf{u} = \nabla \varphi)$$

$$\nabla \left( \frac{\partial \varphi}{\partial t} + \frac{|u|^2}{2} + w \right) = 0$$

$$\Rightarrow \frac{\partial \varphi}{\partial t} + \frac{|u|^2}{2} + w = f(t) \quad (*)$$

constant independent of r.

Redefining  $\varphi \rightarrow (\varphi - \int f dt)$ , which does not affect  $\mathbf{u} = \nabla \varphi$ ,

we have

$$\frac{\partial \varphi}{\partial t} + \frac{|u|^2}{2} + w = 0$$

Obs For stationary flow, we have from (\*) the Bernoulli's equation  $\frac{|u|^2}{2} + w = \text{const.}$

But in this (irrotational) case const does not depend on a streamline (the same in whole space).

Irrational incompressible flow:

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \times \mathbf{u} = 0,$$

$$\nabla \times \mathbf{u} = 0 \Rightarrow \mathbf{u} = \nabla \varphi; \quad \nabla \cdot \mathbf{u} = 0 \Rightarrow \Delta \varphi = 0$$

Potential  $\varphi$  satisfies the Laplace eq.

Boundary condition on rigid surface:  $\mathbf{u} \cdot \mathbf{n} = 0 \Rightarrow$

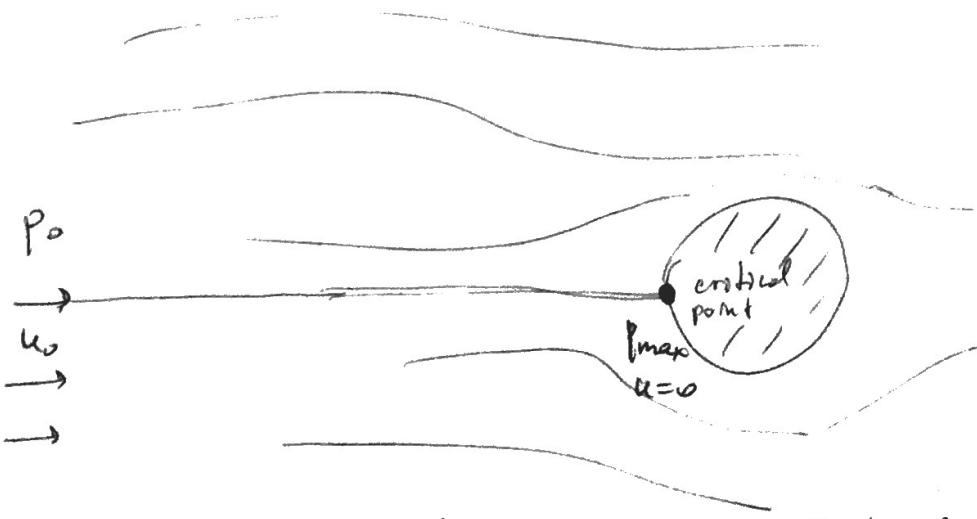
$$\Rightarrow \frac{\partial \varphi}{\partial n} = 0 \quad (\text{Neumann boundary condition}).$$

When surface is moving:  $\frac{\partial \varphi}{\partial n} = \frac{u_n}{\rho}$  (normal velocity) at a surface.

As for isentropic flow  $\nabla w \rightarrow \frac{\nabla p}{\rho} = \nabla \left( \frac{P}{\rho} \right)$  for  $\rho = \text{const.}$

$$\Rightarrow \frac{\partial \varphi}{\partial t} + \frac{|u|^2}{2} + \frac{P}{\rho} = f(\varphi).$$

Obs: In potential incompressible flow around a moving rigid body, the ~~absolute~~ fluid velocity depends only on the velocity of the body (not acceleration etc.), because  $u$  is given by the Laplace eq. + Neumann b.c.



For stationary flow:

$$\frac{|u|^2}{2} + \frac{P}{\rho} = \text{const}$$

(same in all space)

Critical point:  $|u| = 0 \rightarrow P_{\max}$ .

$$P_{\max} = P_0 + \underbrace{\frac{\rho u_0^2}{2}}_{\text{conditions at } \infty}$$

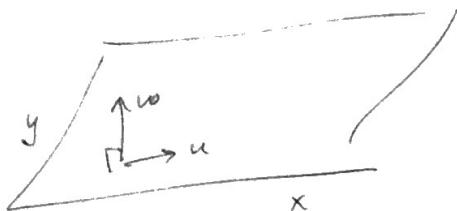
# Two-dimensional fluid dynamics (ideal fluid) (44)

$$u = u(x, y, t)$$

no dependence of  $z$

or motion in thin regions

(example: atmosphere, ~~thin~~, soap film, etc.)



$$\text{Vorticity: } \omega = \nabla \times u = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) e_z$$

(one can consider  $\omega$  as a scalar).

$$\text{Isentropic flow: } \frac{D}{Dt} \left( \frac{\omega}{\rho} \right) = \left( \frac{\omega}{\rho} \cdot \nabla \right) u = 0 \Rightarrow \frac{\omega}{\rho} = \text{const}$$

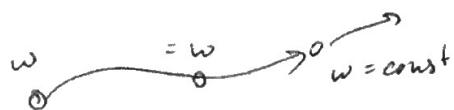
in in  
(\$\parallel e\_z\$) (only  $\partial_{x,y}$ )

along a flow trajectory.

$$\text{Incompressible flow: } \frac{D\omega}{Dt} = (\omega \cdot \nabla) u = 0 \Rightarrow \omega = \text{const}$$

along a flow trajectory

( $\omega$  is the scalar converted by the flow.).



Obs Math. fluid dynamics proves that if the strong solution of Euler equation for incompressible fluid cannot be continued for  $t > t_*$  (blowup), then ~~the flow~~

$$\lim_{t \rightarrow t_*} \sup \omega = \infty \quad (\text{unbounded growth of maximum vorticity}).$$

(45)

This implies ~~that~~ the global (in time) regularity  
 of 2D incompressible Euler equation (no blowup),  
 because  $\max \omega = \text{const.} < \infty$ .

## Potential 2D flow for incompressible fluid

$$\omega \equiv 0 \Rightarrow \mathbf{u} = \nabla \varphi, \quad \text{or} \quad u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}.$$

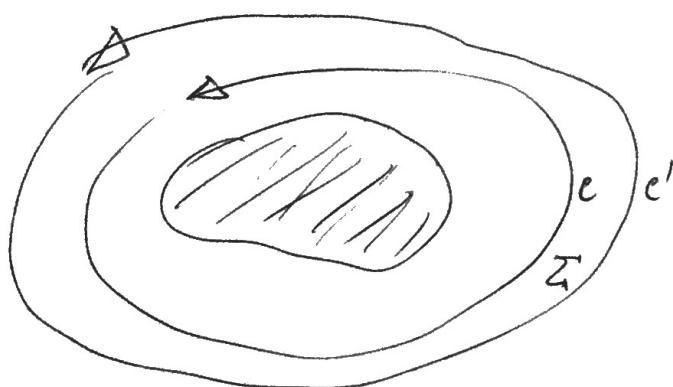
Incompressibility  $\Rightarrow \nabla \cdot \mathbf{u} = 0 \quad \text{in } D \quad \left. \right\}$

Boundary conditions  $\Rightarrow \frac{\partial \varphi}{\partial n} = b_{t_n}(+) \quad \text{in } \partial D \quad \left. \right\}$

Given  $\varphi \rightarrow$  velocity  $\mathbf{u} = \nabla \varphi$ , pressure  $P = -\frac{\rho |\mathbf{u}|^2}{2} \cancel{-}$   
 $- \rho \frac{\partial P}{\partial t} + f(t).$

Theory

flow near a body (nonsimply connected domain)



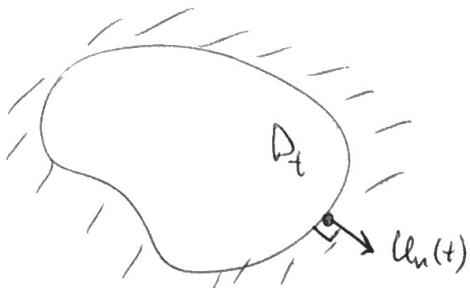
Stokes theorem:

$$\int_{\Sigma} \omega dA = \int_{C} \mathbf{u} \cdot d\mathbf{s} - \int_{C'} \mathbf{u}' \cdot d\mathbf{s} = \\ = \Gamma_c - \Gamma_{c'}$$

For irrotational flow  $\omega \equiv 0 \Rightarrow \Gamma_c = \Gamma_{c'}$  but in general  $\Gamma_c \neq 0$ .

$(\Gamma_c = 0 \text{ if there is no body inside})$

Theorem Consider a simply connected bounded region  $D_t$  with prescribed motion of boundary  $\partial D_t$ , such that its volume = const. (2D or 3D)



Then, (a)  $\exists!$  potential incompressible flow in  $D_t$  with B.C.  $u \cdot n = u_n(+)$  at  $\partial D_t$

(b) This flow minimizes the kinetic energy

$$E = \frac{\rho}{2} \int_D |u|^2 dV \rightarrow \min$$

among all divergence-free vector fields,

$$\nabla \cdot u = 0 \text{ in } D_t, \quad u \cdot n = u_n(+) \text{ at } \partial D_t.$$

Proof (a) Volume = const  $\Rightarrow \int_{\partial D_t} u_n dA = 0 \Rightarrow \exists$  solution. (analysis)

Uniqueness: let  $u$  and  $u'$  be solutions with potentials  $\varphi$  and  $\varphi'$ . Let  $\tilde{u} = u - u'$ ,  $\tilde{\varphi} = \varphi - \varphi'$ . Then

$$\Delta \tilde{\varphi} = 0, \quad \frac{\partial \tilde{\varphi}}{\partial n} = 0, \quad \tilde{u} = \nabla \tilde{\varphi}. \quad \text{Hence,}$$

$$\Rightarrow \int_D \nabla \cdot (\tilde{\varphi} \tilde{u}) dV = \int_D \tilde{u} \cdot \nabla \tilde{\varphi} dV + \int_D \tilde{\varphi} \nabla \cdot \tilde{u} dV = \int_D |\tilde{u}|^2 dV = 0$$

$$\text{But } \int_D \nabla \cdot (\tilde{\varphi} \tilde{u}) dV = \int_D \tilde{\varphi} \tilde{u} \cdot n dA = 0 \Rightarrow \int_D |\tilde{u}|^2 dV = 0$$

$$\Rightarrow \tilde{u} = 0.$$

$$\Rightarrow u = u'.$$

(6) Let  $u$  be a solution, and  
 $u'$  be any ~~vector field~~ vector field with  $\nabla \cdot u' \in D$   
and  $u \cdot n = u' \cdot n$  on  $\partial D$ .

Then, for  $\tilde{u} = u - u'$ , we have  $\nabla \cdot \tilde{u} = 0$  in  $D$  and  
 $\tilde{u} \cdot n = 0$  on  $\partial D$ .

$$\Rightarrow \frac{1}{2} \int_D (|u|^2 - |u'|^2) dV = \int_D (u - u') \cdot u dV - \frac{1}{2} \int_D |u - u'|^2 dV \leq$$

$$(\text{use } |u| = \sqrt{u \cdot u})$$

$$\leq \int_D (u - u') \cdot u dV = \int_D \tilde{u} \cdot \nabla \varphi dV$$

But  $\tilde{u} \cdot \nabla \varphi = \nabla \cdot (\varphi \tilde{u}) - \varphi \nabla \cdot \tilde{u} = \nabla \cdot (\varphi \tilde{u}) \Rightarrow$

$$\Rightarrow \frac{1}{2} \int_D (|u|^2 - |u'|^2) dV \leq \int_D \nabla \cdot (\varphi \tilde{u}) dV = \int_{\partial D} \varphi \tilde{u} \cdot n dA = 0.$$

$$\Rightarrow \frac{1}{2} \int_D |u|^2 dV \leq \frac{1}{2} \int_D |u'|^2 dV.$$

Obs It was important to have a bounded domain!

# Exercises

① Prove  $\frac{1}{2} \nabla |u|^2 = (u \cdot \nabla) u + ux(\nabla x u)$ .

② Consider 1D isentropic flow,  $u \in \mathbb{R}$ ,  $x \in \mathbb{R}$

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad p = p(\rho), \\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \end{array} \right.$$

Write equations ~~in~~ in Lagrangian formulation:

~~$x(a, t), u(a, t), \rho(a, t),$~~

where  $a = x(a, t=0)$ .

③ prove ~~the~~ existence and uniqueness (up to a constant) of a potential ~~function~~  $\varphi$ , such that  $u = \nabla \varphi$ , for irrotational flow,  $\nabla \times u = 0$  in whole space

# (49)

## Complex potential for incompressible irrotational 2D flow.

Speed  $(u, v) \in \mathbb{R}^2$

Incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1) \quad (\nabla \cdot \mathbf{u} = 0)$$

Zero vorticity

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \quad (2) \quad (\nabla \times \mathbf{u} = 0)$$

Note that (1-2) are Cauchy-Riemann eqs.

For the function  $F = u - i v$  ~~on~~ on complex plane  $z = x + iy \Rightarrow F(z)$  is an analytic function.  
(holomorphic)

Complex potential (Def):  $W(z)$  such that  $F(z) = \frac{dW}{dz}$ .

(it can be a multivalued function)

$$W = \varphi + i\psi : \quad u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Analytic (holomorphic)  
function of complex  
variable  $z \in D$

$\Leftrightarrow$

2D incompressible  
irrotational ideal  
flow

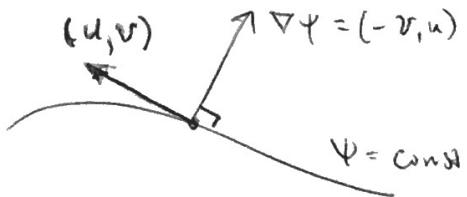
$\varphi$  is a (real) potential :  $(u, v) = \nabla \varphi \quad \left| \Delta \varphi = 0 \right.$

$\psi$  is called a stream function:  $(-v, u) = \nabla \psi \quad \left| \Delta \psi = 0 \right.$

Level curves of  $\psi$  are streamlines:

$\psi = \text{const}$  is orthogonal to  $\nabla \psi = (-v, u) \perp (u, v)$

$\Rightarrow$  level line is tangent to the speed vector at all points



Flux of fluid  $Q$  across any curve equals with  
end points  $z_1$  and  $z_2$  is equal to

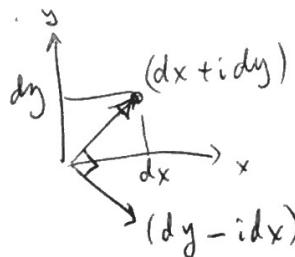
$$Q = \oint_C u_n dl = \oint_C (-v dx + u dy) =$$

↑  
normal  
speed  $u_n = u \cdot n$



$$= \int_1^2 \nabla \psi \cdot ds = \psi(z_2) - \psi(z_1).$$

(difference of values of  
the stream function)



Rigid boundary: boundary ~~is streamline~~ coincides  
with a stream line  $\Rightarrow \psi = \text{const}$   
at the boundary.

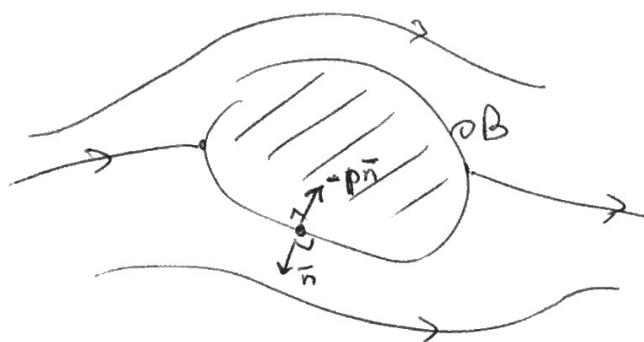


Obs: for a single boundary one can take  $\text{const} = 0$ .



But! different constants at diff. boundaries

# Force on a rigid body



$$\text{Force} = (F_x, F_y) =$$

$$= - \oint_{\partial B} p \vec{n} dl$$

For stationary flow :  $p = -\frac{1}{2} \rho |u|^2 + \text{const}$

(Bernoulli for potential flow)

$$\Rightarrow \text{Force} = \frac{\rho}{2} \int_{\partial B} |u|^2 \vec{n} dl, \text{ because } \oint_{\partial B} \vec{n} dl = \int_{\partial B} (dy, -dx) = 0$$

$$\vec{n} dl = (dy, -dx).$$

Th (Blastius) for incompres. irrotational stationary ideal flow:

$$\overline{\text{Force}} = F_x - iF_y = \frac{i\rho}{2} \oint_{\partial B} \left( \frac{dw}{dz} \right)^2 dz, \text{ where } W(z) \text{ is a complex potential.}$$

Proof

~~$$\vec{n} dl = (dy, -dx) \Rightarrow \text{Force} = - \oint_{\partial B} p (dy, -dx)$$~~

~~$$\Rightarrow F_x - iF_y = - \oint_{\partial B} p (dy + idx) = \frac{\rho}{2} \int_{\partial B} |u|^2 (dy + idx) =$$~~

~~$$= \frac{i\rho}{2} \int_{\partial B} |u|^2 \bar{dz}, \quad \cancel{\text{del}}$$~~

$$\text{Now, } |u|^2 = u^2 + v^2 = |u + iv|^2 = |\frac{dw}{dz}|^2 = \frac{dw}{dz} \left( \frac{\bar{dw}}{\bar{dz}} \right).$$

$$\Rightarrow F_x - iF_y = \frac{i}{2} \int_{\partial B} \frac{dw}{dz} \overline{\left( \frac{dw}{dz} \right)} dz$$

Here  $\overline{\left( \frac{dw}{dz} \right)} = u + iv$  is tangent to the boundary  
and  $dz$  is also tangent to the boundary.

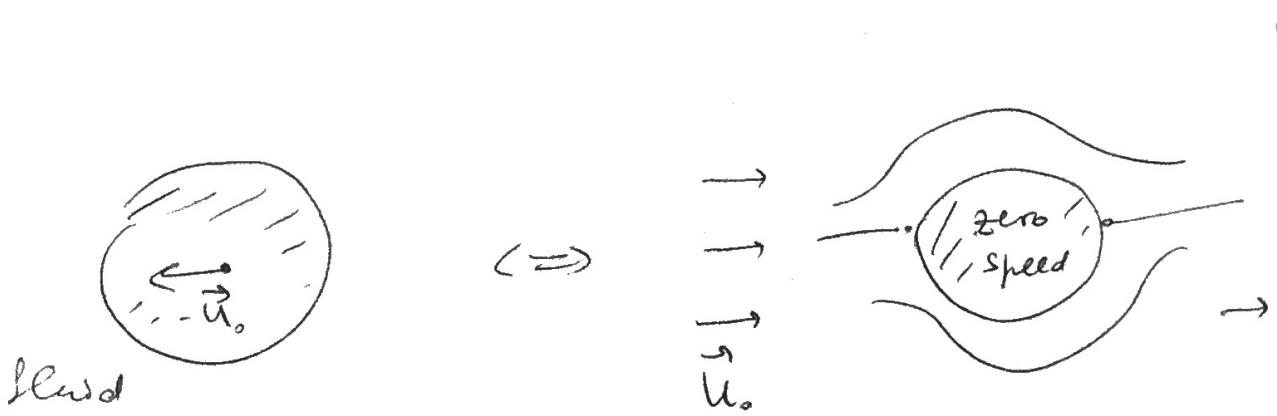
~~Also~~ Hence  $\overline{\left( \frac{dw}{dz} \right)} \in \mathbb{R} \Rightarrow \overline{\left( \frac{dw}{dz} \right)} dz = \frac{dw}{dz} dz$ .

$$\Rightarrow F_x - iF_y = \frac{i}{2} \int_{\partial B} \left( \frac{dw}{dz} \right)^2 dz. \quad \blacksquare$$

Force on a body moving with const speed  $\vec{U}$

~~stationary~~ in fluid stationary at  $\infty \Leftrightarrow$

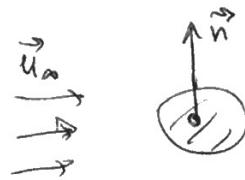
$\Leftrightarrow$  Stationary body in fluid moving with speed  $+\vec{U}$  at  $\infty$



Galilean Invariance :  $\tilde{r} = r + \vec{U}_0 t$ ,  $\tilde{u} = \vec{u} + \vec{U}_0$   
of fluid eqs.

Th Consider incompres. potential stationary ideal flow around body B (fixed in space), such that the flow speed  $\vec{U} = (U, V) \rightarrow \vec{U}_\infty = \text{const at } \infty$ . Then the force on the body is equal to

$$\text{Force} = - \oint_{\partial B} |U_\infty| \vec{n},$$



where  $\vec{n} \perp \vec{U}_\infty$  ( $\vec{n} = \frac{(U_\infty + iV_\infty)}{|U_\infty|}$ ) and  $\Gamma_{\partial B}$  is the circulation around  $\partial B$ .

Proof

Expand  $F(z) = \frac{dW}{dz}$  in Laurent series.

$$F = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

(since  $F \rightarrow U_\infty - iV_\infty$  at  $\infty$ , there are no positive powers)

$$a_0 = U_\infty - iV_\infty.$$

By Cauchy's theorem :  $\oint_{\partial B} F dz = 2\pi a_1$

$$\begin{aligned} \oint_{\partial B} F dz &= \oint_{\partial B} (u - iv)(dx + idy) = \oint_{\partial B} [(u dx + v dy) + i(u dy - v dx)] = \\ &= (u, v) \cdot (dx, dy) \text{ at } \partial B \end{aligned}$$

$$= \oint_{\partial B} \vec{u} \cdot d\vec{s} = \Gamma_{\partial B}$$

Hence  $a_1 = \frac{\Gamma_{\partial B}}{2\pi i} \quad (*)$

$$\left(\frac{dw}{dz}\right)^2 = F^2 = A_0^2 + \frac{2a_0a_1}{z} + \frac{2a_0a_2 + a_1^2}{z^2} + \dots = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots$$

~~Blasius~~

$$\frac{\text{Force}}{\text{Force}} = (F_x - iF_y) = \underbrace{\frac{i\rho}{2} \int_B F^2 dz}_{\text{Blasius th.}} = \cancel{\frac{i\rho}{2} \int_B (2a_0 + 2a_1 z + \dots)^2 dz} =$$

$$= \frac{i\rho}{2} 2\pi i \underbrace{A_1}_{2a_0 a_1} = \frac{i\rho}{2} 2\pi i \underbrace{2(4a_0 - iV_\infty)}_{a_0} \frac{\int_B}{\underbrace{2\pi i}_{a_1}} =$$

$$= + \oint_B (iU_\infty + V_\infty) \Rightarrow \text{Force} = - \oint_B \vec{n} \cdot \vec{U}_\infty.$$

Obs Force is orthogonal to  $\vec{U}_\infty$ ! (lift force).

$\Rightarrow$  No drag force opposing the flow.

Is  $\int_B = 0 \Rightarrow \text{Force} = 0$  d'Alembert's paradox

Obs As we will see below:

(1) Model resolving d'Alembert's paradox

takes into account viscosity (dissipation)

(2) Solution corresponding to vanishing viscosity  
is not smooth!

Open problems: vanishing visc. limit, turbulence.

## Examples:

(1) Flow around a cylinder (disc in  $\mathbb{R}^2$ )

$$\text{Let } W(z) = U \left( z + \frac{a^2}{z} \right)$$

$$u - iv = \frac{dw}{dz} = U \left( 1 - \frac{a^2}{z^2} \right) \rightarrow U \text{ as } |z| \rightarrow \infty$$

$(u \rightarrow U, v \rightarrow 0)$ .

$$\text{When } |z|=a \Rightarrow \frac{a^2}{z} = \bar{z} \Rightarrow w = U(z + \bar{z}) \in \mathbb{R}$$

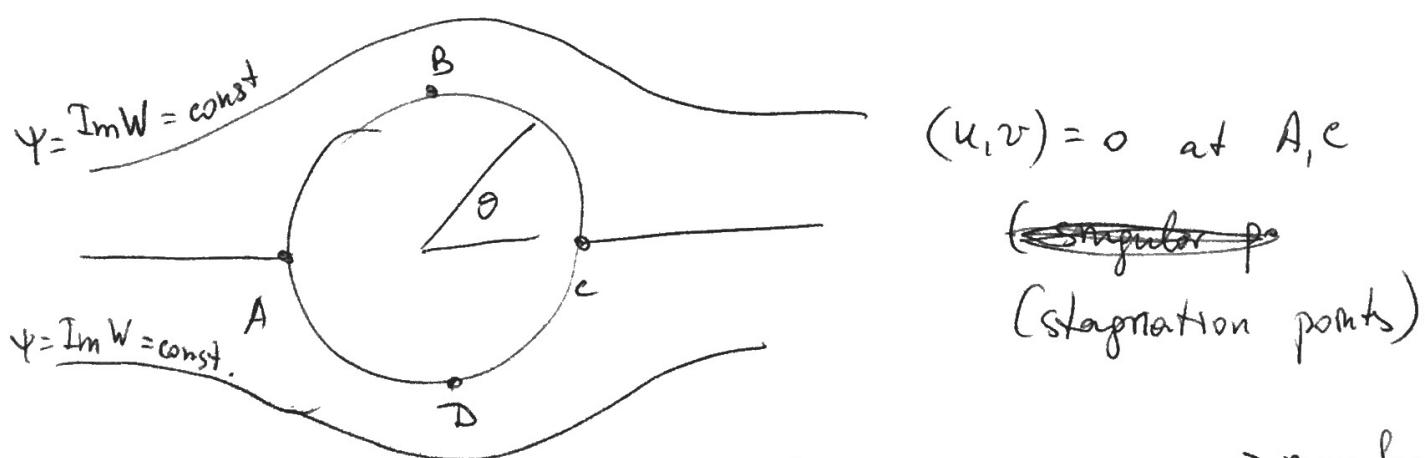
$$\Rightarrow \Psi = \phi \text{ (stream function)}$$

at the circle  $|z|=a$ .

We see that the flow satisfied boundary condition for a disk of radius  $a$ .

$$\text{For } z = ae^{i\theta}, \quad \frac{dw}{dz} = U(1 - \cos 2\theta + i \sin 2\theta)$$

$$\Rightarrow u = U(1 - \cos 2\theta), \quad v = -U \sin 2\theta.$$



$$p = -\frac{\rho}{2} |u|^2 + \text{const} = -\frac{\rho U^2}{2} (1 - \cos 2\theta)$$

max for  $\theta = 0, \pi$   
(A, C)

min for  $\theta = \pm \pi/2$   
(B, D)

(2) Potential vortex:

$$W(z) = \frac{\Gamma}{2\pi i} \log z. \quad (\log z = \log|z| + i\arg z)$$

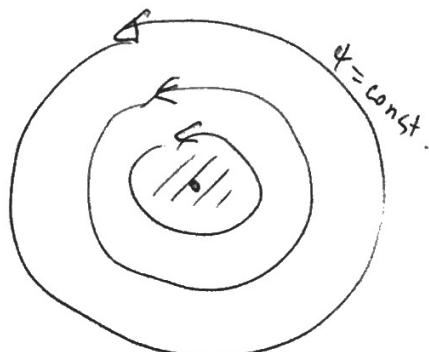
Not single-valued!

$$u - iv = \frac{dw}{dz} = \frac{\Gamma}{2\pi iz} \quad (\text{single-valued}).$$

Circulation is  $\Gamma$  (Laurent series + Cauchy's th.)

Stream function  $\psi = \text{Im } W = -\frac{\Gamma}{2\pi} \log|z|$

$$\psi = \text{const} \iff |z| = \text{const.}$$



Nonzero circulation,  
but flow is irrotational,  
with a singular point  
at the origin.

(3) combination of (1) and (2): flow around a disc with circulation

$$W(z) = U \left( z + \frac{a^2}{z} \right) + \frac{\Gamma}{2\pi i} \log z. \quad |z| \geq a$$

As  $|z| \rightarrow \infty$ ,  $u - iv = \frac{dw}{dz} \rightarrow U$ .

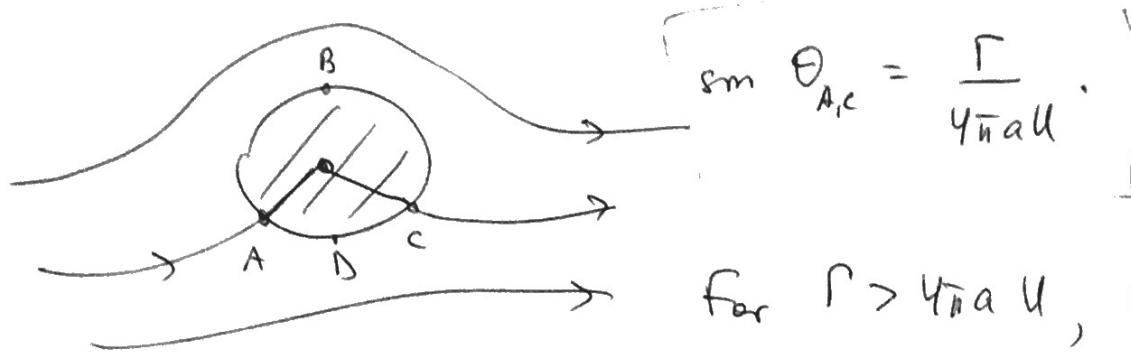
$$\text{At } |z|=a, \quad \psi = \operatorname{Im} W = \text{const} \quad (\text{as in (1) and (2)})$$

Velocity at disc boundary

$$(u, v) = \nabla \varphi, \quad |u| = a \frac{\partial \varphi}{\partial \theta}$$

$$\varphi = Re W = 2au \cos \theta + \frac{\Gamma \theta}{2\pi} \Rightarrow |u| = -2U \sin \theta + \frac{\Gamma}{2\pi a}$$

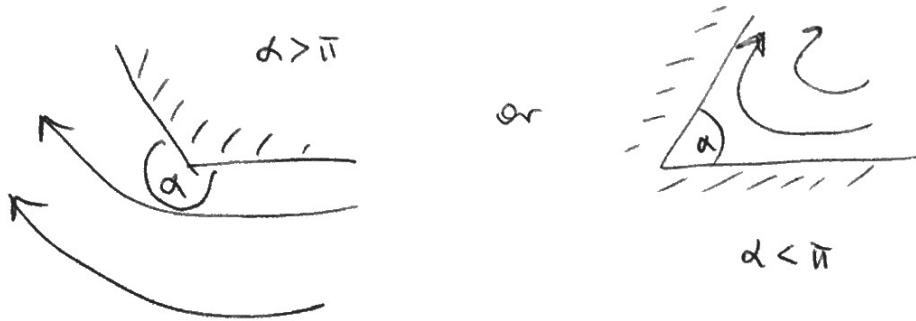
If  $\Gamma < 4\pi a U \Rightarrow \exists$  stagnation points A, B:



$\sin \theta_{Ac} = \frac{\Gamma}{4\pi a U}$

for  $\Gamma > 4\pi a U$ ,  $|u| \neq 0$  at the boundary.

(ii) Flow around an angle:



$$W = A z^n \Rightarrow \varphi = A \rho^n \cos n\theta, \quad \psi = A \rho^n \sin n\theta$$

( $A \in \mathbb{R}$ )

$(z = \rho e^{i\theta})$

If  $\alpha < \pi$ : flow inside angle, polar. coord.

$$n = \pi/\alpha < 1 \Rightarrow \psi = 0 \text{ for } \theta = 0, \theta = \alpha.$$

$(0 \leq \theta \leq \alpha)$

Streamlines:  $\psi = \text{const} \Rightarrow \rho = \frac{\text{const}}{\sin^{1/n}(n\theta)}$ .

~~BB 2021:~~

Velocity:  ~~$\psi = A\rho^n \cos n\theta$~~ ,  $(u, v) = \nabla \psi$

In polar coordinates,  $u_p = \frac{\partial \psi}{\partial \rho} = nA\rho^{n-1} \cos n\theta$

$$u_\theta = \frac{1}{\rho} \frac{\partial \psi}{\partial \theta} = -nA\rho^{n-1} \sin n\theta$$

$$|u| = n|A|\rho^{n-1} \rightarrow \infty \text{ as } \rho \rightarrow 0 \quad (n < 1)$$

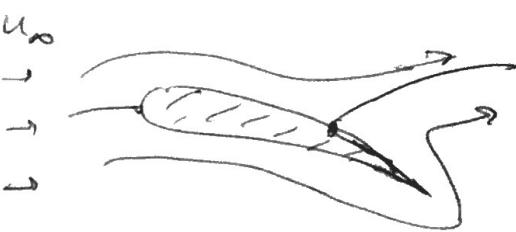
If  $\alpha \ll \pi$ : flow inside an angle

$n = \pi/\alpha > 1$ . All relations are the same,  
but  $|u| \rightarrow 0$  as  $\rho \rightarrow 0$ .

### Flow around airfoils (streamlined bodies)

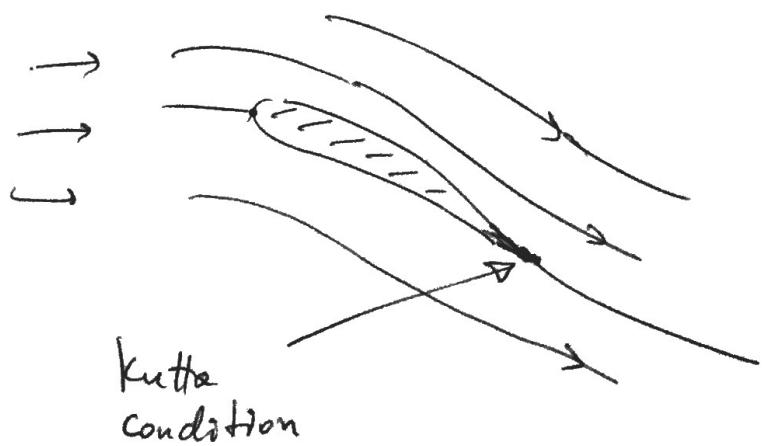
incompressible when speeds  $\ll$  sound speed (343 m/s)

$$\Gamma = 0$$



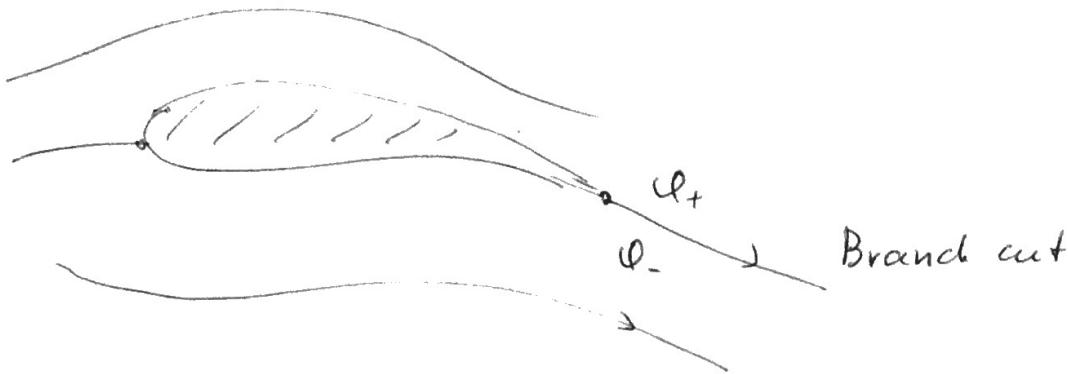
$$\text{Lift Force} = 0$$

$$\Gamma \neq 0$$



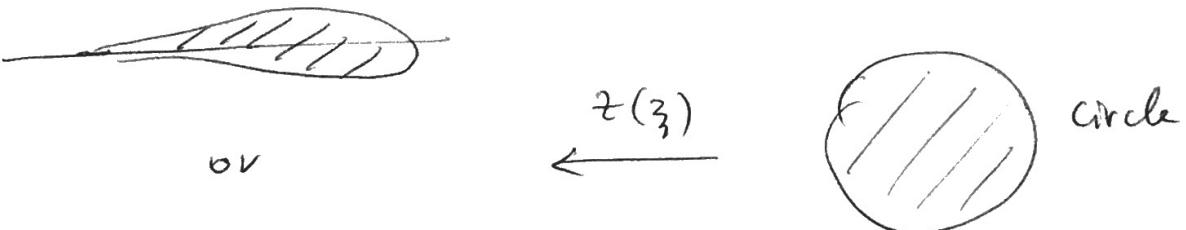
$$\Gamma = \int u \cdot ds = \int \nabla \varphi \cdot ds = \varphi_- - \varphi_+$$

59



$$\text{Lift force} = -\rho \Gamma u_\infty = \rho u_\infty (\varphi_+ - \varphi_-)$$

$$\text{Joukowski airfoil: } z = \xi + \frac{1}{\xi}$$



( $z$ -plane)

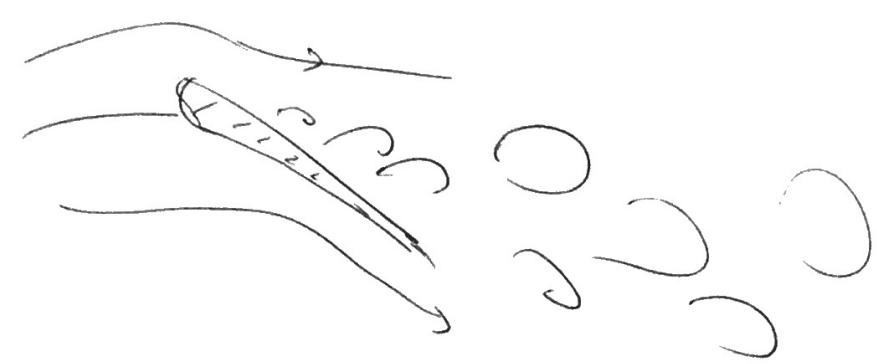
(depending on  
position of a  
circle)

$w(\xi)$  is a flow around a disc (with circulation)

$\Rightarrow w(\xi(z))$  describes the flow around airfoil

( $\varphi = \text{const}$  at the boundary)  $\Rightarrow u - i v = \frac{dw}{dz} = \frac{dw/d\xi}{d\xi/dz}$

# Real-life flows :



10-20% difference  
for lift force  
at small  
attack angles

large deviation  
from potential theory

## Incompressible flow with vorticity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \exists \text{ stream function } \psi(x, y) :$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$\left. \begin{array}{l} \text{(same argument as for potential)} \\ \text{flow } \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \Rightarrow \varphi \end{array} \right)$$

$$\Delta \psi = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\omega \quad \cancel{\text{vortex}}$$

Given  $\omega(r)$ ,  $r = (x, y) \in \mathbb{R}^2$  we have

(61)

$$\Psi = - \int \omega(r') G(r, r') dr'$$

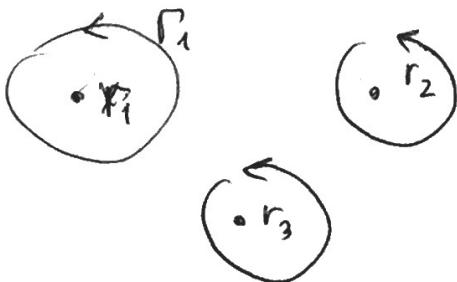
with the Green function

$$G(r, r') = \frac{1}{2\pi} \log |r - r'| \quad (2D \text{ Laplace eq.})$$

Special case (point vortices) :

$$\omega = \sum_j \Gamma_j \delta(r - r_j(+))$$

vortices of strength  $\Gamma_j$  (constant in time) concentrated at points  $r_j(+)$ .



Point-vortex solution :  $\Psi_j = -\frac{\Gamma_j}{2\pi} \log |r - r_j|$

$$\Rightarrow \Psi = \sum_j \Psi_j$$

Speeds :  $(u_j, v_j) = \left( \frac{\partial \Psi_j}{\partial y}, - \frac{\partial \Psi_j}{\partial x} \right) = \frac{\Gamma_j}{2\pi |r - r_j|^2} (-y_j, x_j)$

$$(u, v) = \sum_j (u_j, v_j).$$

Each vortex is advected by the velocity field generated by other vortices:

$$\frac{dx_j}{dt} = \frac{1}{2\pi} \sum_{i \neq j} \frac{\Gamma_i (y_j - y_i)}{r_{ij}^2},$$

$$\frac{dy_j}{dt} = \frac{1}{2\pi} \sum_{i \neq j} \frac{\Gamma_i (x_i - x_j)}{r_{ij}^2}, \quad r_{ij} = |r_i - r_j|.$$

Obs.

It can be shown that these eqs are related to smooth ~~other~~ solutions of Euler equations:

Initial cond for  $\{r_j, \Gamma_j\} \rightarrow$  smooth initial cond

$\Rightarrow$  solution  $r_j(t) \rightarrow$  smooth solution of Euler eqs.

(where  $\rightarrow$  is understood in some "weak" sense)

When all  $\Gamma_j = \Gamma$ , we can write

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}, \quad \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j}, \quad j = 1, \dots, N$$

(Hamiltonian system)

$$\text{for } H = -\frac{\Gamma}{4\pi} \sum_{i \neq j} \log |x_i - x_j|.$$