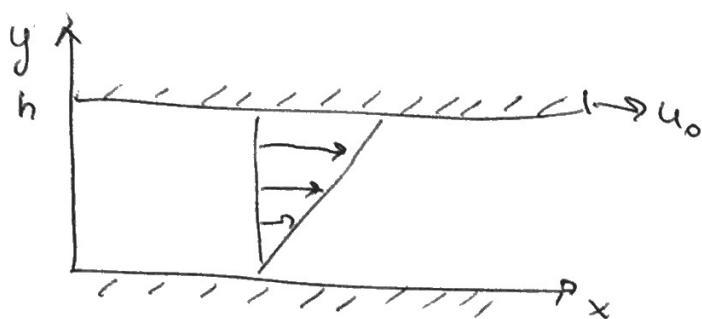


Planar Couette flow



$$\frac{\partial u}{\partial t} + u \cdot \nabla u = - \frac{\nabla p}{\rho} + \nu \Delta u, \quad \nabla \cdot u = 0$$

(a) Steady flow: $\frac{\partial u}{\partial t} = 0$, (b) all quantities depend only on y .

(c) velocity is parallel ~~x-axis~~
 (u, v) , $v = 0$

$$N.S \rightarrow \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial p}{\partial y} = 0$$

$$\Rightarrow p = \text{const}, \quad u = ay + b$$

Boundary conditions: $u = 0$ at $y = 0$, $u = u_0$ at $y = h$
 $u = y u_0 / h$.

$$\text{Mean velocity: } \langle u \rangle = \frac{1}{h} \int_0^h u dy = \frac{1}{2} u_0.$$

Shear stress:

$$F_{\text{shear}} = (-p \vec{n} + \sigma \vec{n}) A, \quad \sigma = 2\mu D \quad (\text{incompressible})$$

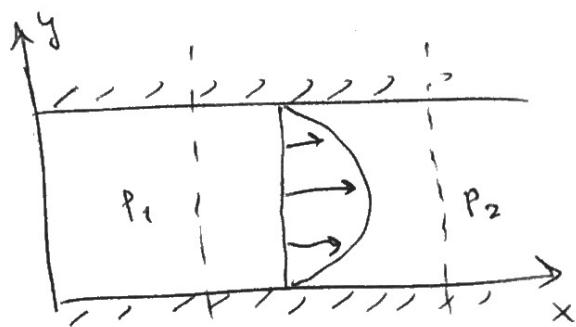
$$D = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} \end{pmatrix}$$

In our case:

$$D = \begin{pmatrix} 0 & \frac{1}{2} \frac{\partial u}{\partial y} \\ \frac{1}{2} \frac{\partial u}{\partial y} & 0 \end{pmatrix} = \frac{1}{2} \frac{u_0}{h} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

\Rightarrow Force on a plane: $\frac{\mu u_0}{h}$ (per unit length).

Poiseuille flow (pipe)



pressure gradient
in x-direction

- (a) stationary $\frac{\partial u}{\partial t} = 0$
- (b) velocity depends only on y
- (c) (u, v) , $v=0$ (only x-component is nonzero)

$$0 = -\frac{1}{g} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0 \quad \left(\nu = \frac{\mu}{g} \right) \quad (95)$$

$$\underbrace{\frac{\partial^2 u}{\partial y^2}}_{\text{dep. on } y.} = \underbrace{\frac{1}{\mu} \frac{\partial p}{\partial x}}_{\text{dep. on } x}, \quad \underbrace{\frac{\partial p}{\partial y}}_{\Rightarrow p = p(x)} = 0$$

\Rightarrow both sides are constants, $\frac{\partial p}{\partial x} = \text{const}$

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + a y + b$$

Boundary cond.: $u=0$ for $y=0, h$

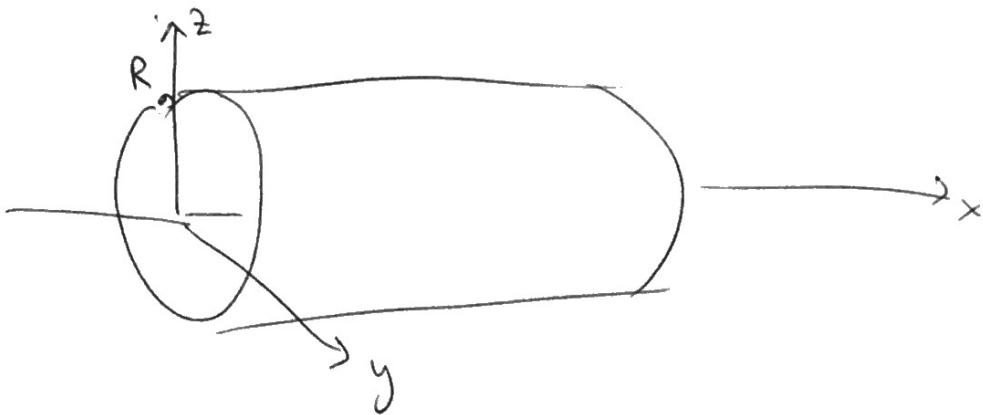
$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y(y-h)$$

Mean velocity: $\langle u \rangle = \frac{1}{h} \int_0^h u dy = + \frac{1}{h} \frac{1}{2\mu} \frac{\partial p}{\partial x} \int_0^h (y^2 - hy) dy$

$$= + \frac{1}{2\mu} \frac{\partial p}{\partial x} \left(\frac{y^3}{3} - \frac{hy^2}{2} \right) \Big|_0^h = - \frac{h^2}{12\mu} \frac{\partial p}{\partial x}.$$

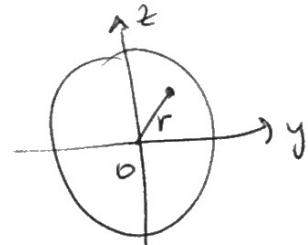
(96)

Pipe with circular cross-section:



Stationary, $(u, v, w) = (u, 0, 0)$, ~~assumption~~

$$u = u(y, z) = u(r), \quad r = \sqrt{y^2 + z^2}$$



$$\text{N.S. eqs: } 0 = -\frac{\rho \partial p}{\rho \partial x} + \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = 0 \quad \Rightarrow \quad p = p(x)$$

$$\text{Same argument} \rightarrow \frac{\partial p}{\partial x} = \text{const}$$

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu} \frac{\partial p}{\partial x} \quad (\text{const})$$

$$\Delta_{(y^2)} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

In polar coordinates (independent of angle)

$$\Delta_{(1,2)} = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = + \frac{1}{\mu} \frac{\partial p}{\partial x}$$

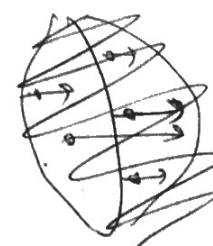
$$\frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \frac{r}{\mu} \frac{\partial p}{\partial x} \Rightarrow r \frac{\partial^2 u}{\partial r^2} = \frac{r^2}{2\mu} \frac{\partial p}{\partial x} + a$$

$$\frac{\partial u}{\partial r} = \frac{r}{2\mu} \frac{\partial p}{\partial x} + \frac{a}{r} \Rightarrow u = \frac{r^2}{4\mu} \frac{\partial p}{\partial x} + a \log r + b$$

Since $u \not\rightarrow \infty$ at $r=0 \Rightarrow a=0$

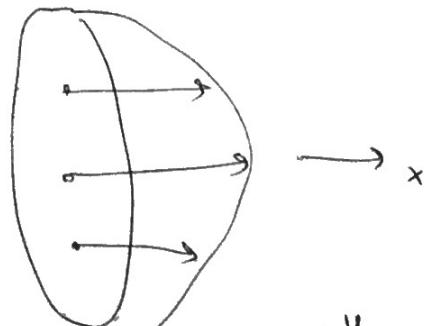
$$u=0 \text{ at } r=R \Rightarrow b = - \frac{R^2}{4\mu} \frac{\partial p}{\partial x}.$$

Solution : $u = \cancel{\frac{r^2 - R^2}{4\mu}} \frac{\partial p}{\partial x}$



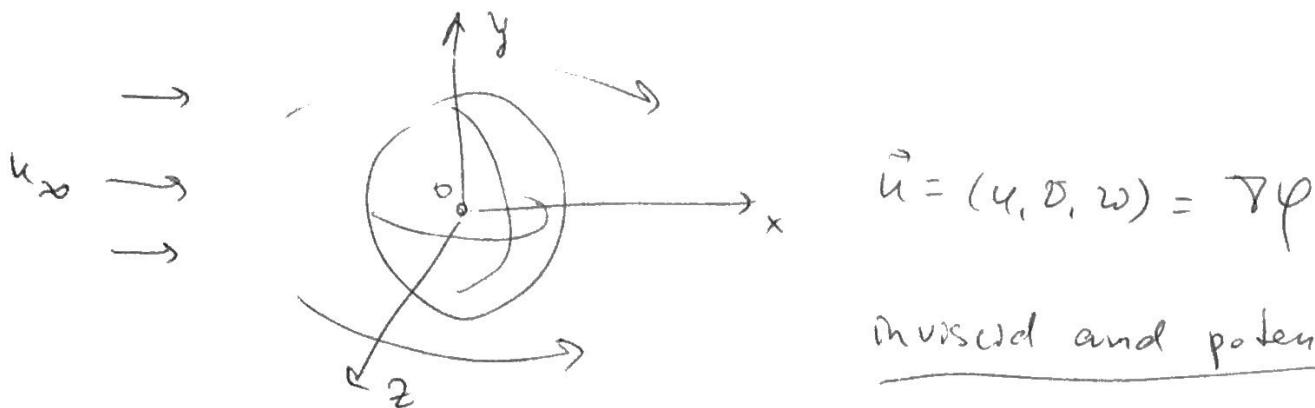
Fluid flux :

$$Q = g \int_{\text{cross-section}} u ds = 2\pi p \int_0^R u \cancel{r} dr =$$



$$= \frac{\pi p}{2\mu} \frac{\partial p}{\partial x} \int_0^R (r^3 - R^2 r) dr = \frac{\pi p}{2\mu} \frac{\partial p}{\partial x} \left(\frac{r^4}{4} - \frac{R^2 r^2}{2} \right)_0^R = - \frac{\pi p R^4}{8\mu} \frac{\partial p}{\partial x}.$$

Flow around a sphere (inviscid)



Inviscid and potential

$$\psi = u_{\infty} x \left(1 + \frac{R^3}{2r^3} \right), \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla r = \frac{(x, y, z)}{r} = \vec{n}$$

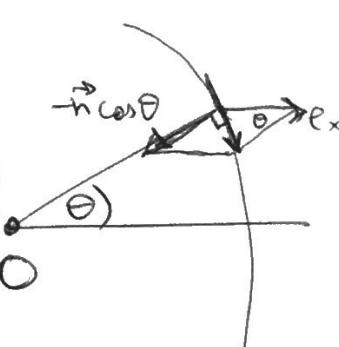
$$\vec{u} = \nabla \psi = u_{\infty} \vec{e}_x + \frac{u_{\infty} R^3}{2r^3} \left(\vec{e}_x - \frac{3x}{r} \vec{n} \right)$$

At the boundary : ($r = R$)

$$\vec{u} = \frac{3}{2} u_{\infty} \left(\vec{e}_x - \frac{x}{r} \vec{n} \right)$$

~~boundary~~

~~cos\theta~~



$\vec{u} \parallel \text{boundary}$.

Check : $\Delta \psi = 0$:

$$(\Delta x = 0; \Delta \frac{x}{r^3} = -\frac{\partial}{\partial x} \frac{1}{r^3} = -\frac{\partial}{\partial x} \Delta \frac{1}{r} = 0)$$

From symmetry & $|u|_{(x,y,z)} = |u|_{(-x,y,z)}$ } Total force

and Bernoulli eq.

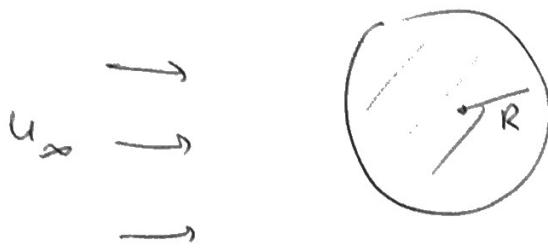
$$\frac{|u|^2}{2} + \frac{P}{\rho} = \text{const}$$

= 0.

d'Almber's paradox

Viscous flow around a sphere (small Re)

(99)



Stationary N.S. eqs: $\underbrace{(u \cdot \nabla) u = -\frac{\nabla p}{\rho} + \nu \Delta u}_{\sim \frac{u_\infty^2}{R}}, \quad \nabla \cdot u = 0$

$\sim \frac{u_\infty^2}{R}$ $\sim \frac{\nu u_\infty}{R^2}$

$$\left(\frac{\text{convection term}}{\text{viscous term}} \right) = \frac{u_\infty^2 / R}{\nu u_\infty / R^2} = \frac{u_\infty R}{\nu} \quad (= \text{Re Reynolds number})$$

When $\text{Re} \ll 1$ (slow motion) \Rightarrow we can neglect $(u \cdot \nabla) u$.

Stokes equations: $0 = -\frac{\nabla p}{\rho} + \nu \Delta u, \quad \nabla \cdot u = 0$

$$(\nabla \times) \Rightarrow \nabla \times \Delta u = \Delta (\nabla \times u) = \Delta \omega = 0$$

$$\nabla \cdot u = 0 \Rightarrow \cancel{u - u_\infty} = \nabla \times A \quad (\text{Helmholtz decomposition})$$

Clever guess for A:

- (1) A depends linearly on \vec{u}_∞ (boundary cond. is linear)
~~at ∞~~
- (2) A is axial (invariant under $\vec{r} \mapsto -\vec{r}$)

$$A = g(r) \vec{n} \times \vec{u}_\infty = f'(r) \vec{n} \times \vec{u}_\infty = \nabla f \times \vec{u}_\infty \quad \left(\text{for some } g(r) = f'(r) \right)$$

$$u = u_\infty + \nabla \times A = u_\infty + \nabla \times (\nabla f \times \vec{u}_\infty) = \\ = \vec{u}_\infty + \nabla \times \nabla \times (f \vec{u}_\infty).$$

check
(for $u_\infty = \text{const}$)

$$\omega = \nabla \times u = \nabla \times \nabla \times \nabla \times (f u_\infty) = \cancel{\nabla \times \nabla \times} \quad \left(\begin{array}{l} \nabla \times \nabla \times = \\ = \nabla \nabla \cdot - \Delta \\ = \text{grad div} - \Delta \end{array} \right)$$

$$= \cancel{\nabla \nabla \cdot (\nabla \times (f u_\infty))} - \Delta \nabla \times (f u_\infty),$$

$$\omega = -\Delta \nabla \times (f u_\infty).$$

$$\text{Since } \Delta \omega = 0 \Rightarrow \Delta^2 \nabla \times (f u_\infty) = \Delta^2 (\nabla f \times u_\infty) = \\ = (\Delta^2 \nabla f) \times u_\infty = 0.$$

(Since f is isotropic, $f = f(r) \Rightarrow \Delta^2 \nabla f = 0$).

We have $\nabla \Delta^2 f = 0 \Rightarrow \Delta^2 f = \text{const}$

Since $u - u_\infty = \nabla \times \nabla \times (f u_\infty) \rightarrow 0 \text{ at } \infty \Rightarrow \text{const} = 0$.

$\Delta^2 f = 0$ biharmonic ~~eq~~ function.

In 3D : $\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \text{part with angular derivatives}$
 $(\Delta_m \text{ spherical coordinates})$

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \Delta f = 0 \Rightarrow \Delta f = \frac{2a_1}{r} + c, \quad a_1, c.$$

Wirksanzahligkeit

$$u - u_\infty = \underbrace{\nabla \times \nabla \times (\vec{f} \cdot \vec{n})}_{\text{2 d derivatives}} \rightarrow 0 \quad \text{et } \infty \Rightarrow C = 0.$$

$$\Delta f = \frac{a}{r} \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} = \frac{a}{r} \Rightarrow$$

$$\frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} = 2ar \Rightarrow$$

$$\frac{\partial f}{\partial r} = a \Leftrightarrow \frac{f}{r^2} =$$

$$f = ar + \frac{b}{r}.$$

$$u = u_\infty - a \frac{u_\infty + n(u_\infty \cdot n)}{r} + b \frac{3n(u_\infty \cdot n) - u_\infty}{r^3}.$$

At $r=R$; $u=0$ (boundary conditions):

$$-u_\infty \left(\frac{a}{R} + \frac{b}{R^3} - 1 \right) + n(u_\infty \cdot n) \left(-\frac{a}{R} + \frac{3b}{R^3} \right) = 0$$

$$\Rightarrow a = \frac{3}{4} R, \quad b = \frac{1}{4} R^3.$$

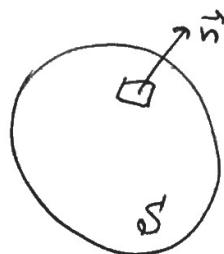
Pressure: ~~Eq 2~~

$$\nabla p = \rho \nabla \Delta u = \mu \Delta u = \mu \Delta \underbrace{\nabla \times \nabla \times (\delta u_\infty)}_{(\nabla \nabla \cdot - \Delta)} = \\ = \mu \Delta (\nabla \nabla \cdot (\delta u_\infty) - u_\infty \Delta f)$$

$$\text{But } \Delta^2 f = 0 \Rightarrow \nabla p = \mu \Delta (\nabla \nabla \cdot (\delta u_\infty)) = \\ = \nabla (\mu \Delta \nabla \cdot (\delta u_\infty)) \\ = \nabla (\mu u_\infty \cdot \nabla \Delta f)$$

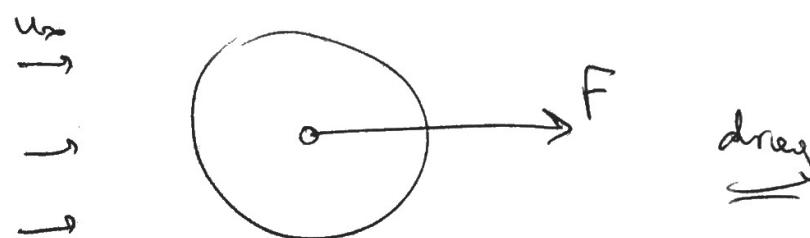
$$\Rightarrow p = p_0 + \mu u_\infty \cdot \nabla \Delta f.$$

Using $f = \dots \Rightarrow$ $p = p_0 - \frac{3}{2} \mu \frac{u_\infty^2 h}{r^2} R.$



$$\text{Force} = \int_{S'} \left(-\rho \vec{n} + \sigma_{visc} \vec{n} \right) dA \quad \left. \right\} \Rightarrow F = 6\pi \mu R \vec{u}_\infty$$

$$\sigma_{visc} = 2\mu \frac{\partial u}{\partial r}$$



Obs The term $\frac{1}{r} \frac{\partial u_0}{\partial r}$ is first-order term in expansion

$$\text{in } Re = \frac{R u_0}{V} \ll 1.$$

Obs In 2D the Stokes eq. does not have a solution satisfying bound. conditions (Stokes paradox)

This is because Stokes eq. is not valid far from sphere. \rightarrow Oseen eq. use

$$(u \cdot \nabla) u \approx (u_\infty \cdot \nabla) u.$$

Stability of steady solutions (linear stability)

$$u = u_0(r) \quad \left(\frac{\partial u_0}{\partial t} = 0 \right)$$

$$\text{Linearization} \quad u = u_0 + u_1, \quad p = p_0 + p_1,$$

where $u_1(r, t)$ and $p_1(r, t)$ are small perturbations

Linearized eqs (neglecting $u_1 \cdot \nabla u_1$):

$$\left\{ \begin{array}{l} \frac{\partial u_1}{\partial t} + u_0 \cdot \nabla u_1 + u_1 \cdot \nabla u_0 = - \frac{\nabla p_1}{\rho} + \nu \Delta u_1 \\ \nabla \cdot u_1 = 0 \end{array} \right. + \text{B.C.: } u_1 = 0 \text{ at boundary.}$$

Solutions of linearized system (coeff. do not depend on time)

have the form $u_1 = u_1(r) e^{\lambda t}$.

Stable if $\operatorname{Re} \lambda \leq 0$. Unstable if $\operatorname{Re} \lambda > 0$

Many ~~other~~ stationary solutions become unstable for large Re (transition to turbulence for $\operatorname{Re} \gg 1$)

Example von Karman vortex street



periodic separation of vortices from above/below.

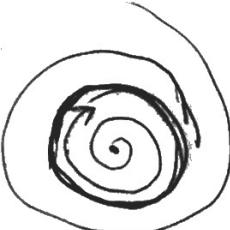
Analogous behavior of ODEs (dynamical systems)



stable



Parit



$P < P_{\text{crit}}$

Hopf bifurcation

$P > P_{\text{crit}}$: fixed point unstable
periodic solution stable

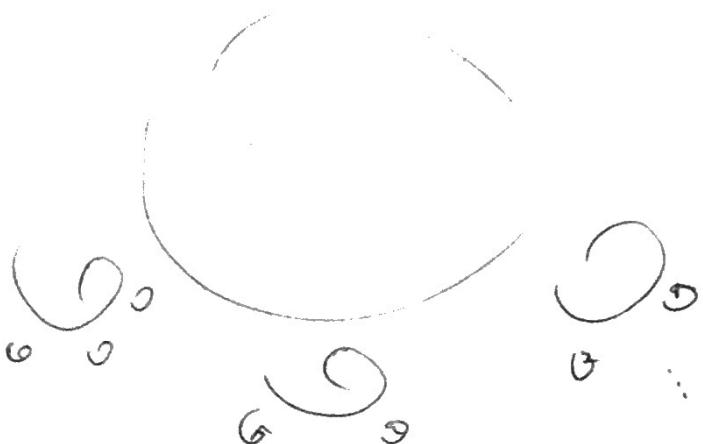
Turbulence (brief introduction)

		Re $\rightarrow \infty$
Further increase of Re	(developed turbulence) $Re \rightarrow \infty$	→
		→
		→
Mixing coffee	10^4	
Ball in flight	10^5	$Re \gg 1 \Rightarrow$
Fish swimming	10^6	viscous term in N.S.
Car / air	10^7	
Ship	10^8	$\frac{1}{Re} \Delta u$ becomes small.
atmosphere	10^9	
astrophysics	$10^{12} \dots$	N.S. \rightarrow Euler ?

Richardson (1922)

description

Big whirls have little whirls
that feed on their velocity,
and little whirls have
lesser whirls and so on
to viscosity



until viscous term

$$\frac{1}{Re} \Delta u \sim \frac{\Delta u}{Re \cdot \delta L^2} \sim 1.$$

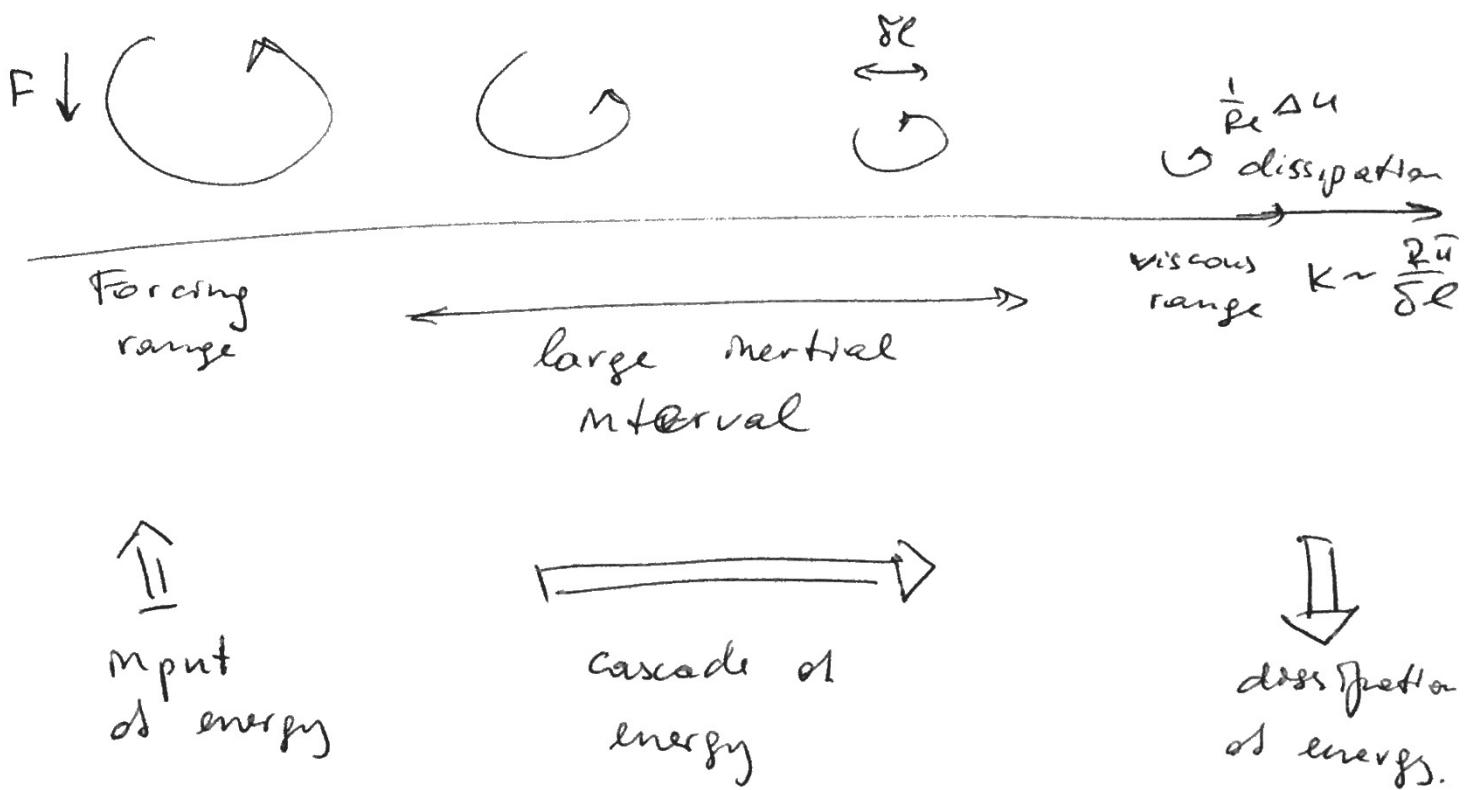
becomes important.

Kolmogorov theory (1941).

Forcing at large scale (mixing) }
 Dissipation at small scales (viscosity) } \Rightarrow

there is a range of scales in between, where

- (1) no forcing (2) no dissipation \rightarrow Euler eqs.



Kolmogorov hypothesis:

- u_1 In the limit $Re \rightarrow \infty$ ($v \rightarrow 0$) symmetries of N.S. eqs are restored in statistical sense:

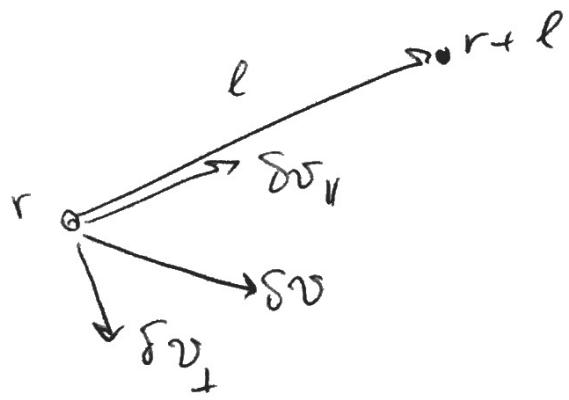
$$u_1 \rightarrow \textcircled{1} \quad \dots \quad \dots$$

stationary, symmetric
statistics!

At small scales:

homogeneous, isotropic dynamics described by velocity increments.

$$\delta_l v = \tilde{v}(r+l) - v(r) \quad \text{for "small" distance } l.$$



$$\delta v = \delta v_{||} + \delta v_{\perp}$$

(parallel and orthogonal)
components

H2

Self-similarity

~~Fractal signature~~

$$\delta_h v = \lambda^h \delta_l v \quad \text{for some } h.$$

H3

Dissipation rate $\varepsilon \rightarrow \varepsilon_m > 0$ has nonzero limit as $Re \rightarrow \infty$. (also known as Onsager dissipation anomaly).

Small-scale statistics is described only by scale l and dissipation rate ε .

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Inertial interval: Euler eqs: $\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + u \cdot \nabla u = - \frac{\nabla p}{\rho} \\ \nabla \cdot u = 0 \end{array} \right.$

it has scaling symmetry

(arbitrary way to choose
units m, s, kg.)

$$\text{e.g.: } x \mapsto x/L, \quad t \mapsto t$$

$$u \mapsto u/L, \quad p \mapsto Lp.$$

$$\rho \mapsto L^3 \rho$$

L [m], ~~length~~

$$E = \frac{\rho}{2} \int v^2 dV \quad \left[\frac{\text{kg}}{\text{m}^3} \left(\frac{\text{m}}{\text{s}} \right)^2 \text{m}^3 \right] \rightarrow \left[\frac{\text{kg m}^2}{\text{s}^2} \right]$$

$$\frac{dE}{dt} \quad \left[\frac{\text{kg m}^2}{\text{s}^3} \right]$$

$$\varepsilon = \underbrace{\frac{1}{\rho V} \frac{dE}{dt}}_{(\text{volume})} \quad (\text{dissipation per unit volume mass})$$

$$\text{g} \left[\frac{\text{kg}}{\text{m}^3} \right], \text{V} [\text{m}^3] \quad \Rightarrow \quad \varepsilon \quad \left[\frac{\text{m}^2}{\text{s}^3} \right]$$

$$v \left[\frac{\text{m}}{\text{s}} \right] \sim \varepsilon^{1/3} \ell^{1/3} \quad \left[\frac{\text{m}^{2/3}}{\text{s}} \text{ m}^{1/3} \right] = \left[\frac{\text{m}}{\text{s}} \right]$$

$$\text{or } \cancel{\langle \delta v \rangle} \quad \langle |\delta_e v| \rangle = C_1 \varepsilon^{1/3} \ell^{1/3}$$

mean value.

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Similarly $\langle |\delta_e v|^p \rangle = C_p \varepsilon^{p/3} l^{p/3}$

Obs 1 This implies that velocity function is not smooth in the limit $l \rightarrow \infty$.

It must be $C^{0,1/3}$ ~~smooth~~

(Hölder continuous function.)

$$|f|_{C^{0,1/3}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{1/3}}.$$

Obs 2 Kolmogorov theory is not true!

~~$$\langle \delta_e v \rangle \sim l^{\beta_p} \cdot \langle |\delta_e v|^p \rangle \sim l^{\beta_p},$$~~

where $\beta_p \neq p/3$ (~~nonlinear function of p~~)

Anomaly \rightarrow intermittency. (open problem)
 (strong bursts in ~~calm~~ regions)
 (calm regions)

~~ex~~

Obs 3 β_p is not far from $p/3$ for p not too large

and $\beta_3 = 1$ (exact result): $\langle |\delta_e v|^3 \rangle = -\frac{4\varepsilon l}{5}$
 Kolmogorov $4/5$ -th law.

Dissipation

viscous term :

$$\nabla \Delta u \sim \nu \frac{\delta_e u}{\ell^2} \sim \nu \varepsilon^{1/3} \ell^{-5/3}$$

(at scale ℓ)

$$\delta_e u \sim \varepsilon^{1/3} \ell^{1/3} \quad (\text{Kolmogorov})$$

↓

~~fluid acceleration~~ ~~$\propto \nu \Delta u \propto \varepsilon^{1/3} \ell^{-5/3}$~~

turnover time : $t_e \sim \frac{\ell}{\delta_e u} \sim \nu \varepsilon^{-1/3} \ell^{2/3}$

Q

fluid acceleration : $a_e \sim \frac{\delta_e u}{t_e} \sim \varepsilon^{2/3} \ell^{-1/3}$

$\left(\frac{du}{dt} \right)$

dissipation starts when $\frac{Du}{Dt} \sim \nu \Delta u$
 (acceleration at scale ℓ) \sim (viscous term at scale ℓ)

$$\nu \varepsilon^{1/3} \ell^{-5/3} \sim \varepsilon^{2/3} \ell^{-1/3}$$

$$\ell^{4/3} \sim \nu \varepsilon^{-1/3}$$

$$\ell^{(\text{def})} = \eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$$

(Kolmogorov scale) Scale influenced by viscosity

Implications for numerical simulations:

one has to resolve scales between L and η .

to take into account forcing (L) and viscosity (η).

Required number of grid points in 3D:

$$N \approx \left(\frac{L}{\eta}\right)^3 = L^3 \varepsilon^{3/4} \nu^{-9/4}.$$

Now ε = dissipation rate = ~~energy input~~
into system
at large scale L

$$\Rightarrow \varepsilon \sim \left[\frac{\text{m}^2}{\text{s}^3}\right] \sim \frac{U^3}{L}, \quad U \text{ is a large-scale velocity.}$$

$$N \approx L^3 U^{9/4} L^{-3/4} \nu^{-9/4} = \left(\frac{UL}{\nu}\right)^{9/4} = \underline{\underline{\text{Re}}}^{9/4}$$

For coffee $\text{Re} \sim 10^4 \Rightarrow N \approx \text{Re}^{9/4} = 10^9 !$

For fish $\text{Re} \sim 10^6 \Rightarrow N \approx 10^{13.5} !$
not accessible for modern computers.

Boundary layer

Flow at large Re.

Navier - Stokes (dimensionless)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = - \nabla p + \frac{1}{Re} \Delta u \xrightarrow{Re \rightarrow \infty}$$

Euler eq.

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = - \nabla p$$

Boundary conditions

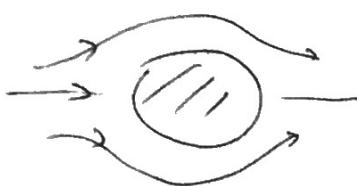
$$u=0 \text{ at } \partial D$$

$\xrightarrow[\text{problem}]{?}$

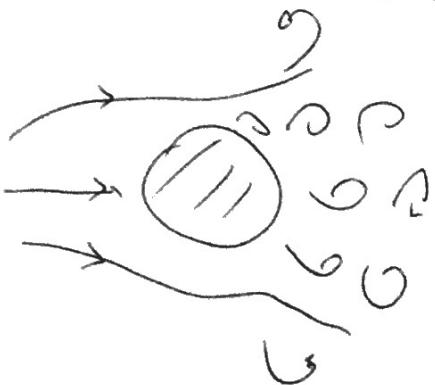
$$u \cdot n = 0 \text{ at } \partial D$$

Laminar vs. turbulent flow:

Laminar flow is the flow in "parallel" layers without lateral mixing



Turbulent flow is characterized by eddies of various sizes resulting in lateral mixing

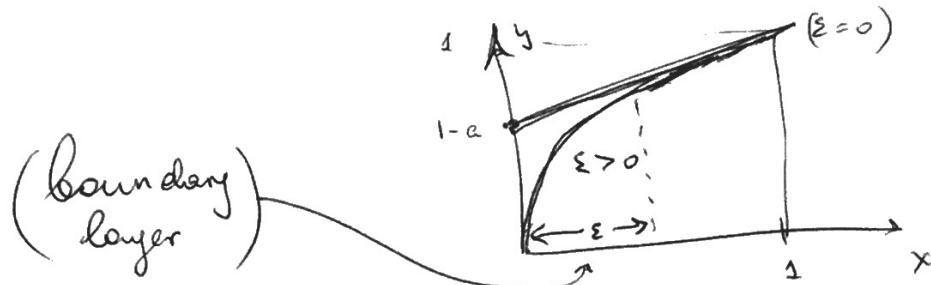


The Prandtl theory considers a laminar flow near the boundary

Obs In the limit $Re \rightarrow \infty$, the flow is typically turbulent.

Example 1

$$\frac{dy}{dx} = a, \quad y(1) = 1 \Rightarrow y_0 = a(x-1) + 1$$

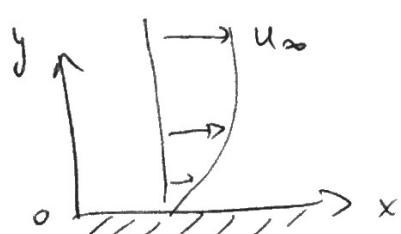


$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = a, \quad y(0) = 0, \quad y(1) = 1 \Rightarrow$$

$$y_\epsilon = \left(\frac{1-a}{1-e^{-1/\epsilon}} \right) \left(1 - e^{-x/\epsilon} \right) + ax$$

$\epsilon \rightarrow 0$: solution $y_\epsilon \rightarrow y_0$.

Example 2 * Flow (unsteady!) over a flat bottom



$$v=0 \text{ at } y=0$$

Assumptions: $v=0$, $u=u(y, t)$, $P=0$

Boundary conditions: $u=0$ at $y=0$

$$u = u_\infty \text{ at } y \rightarrow \infty$$

$$N.S. \text{ eqs: } \frac{\partial u}{\partial t} = v \frac{\partial^2 u}{\partial y^2}, \quad (\text{heat eq.})$$

in dimensionless form
($y \rightarrow y_L$, $u \rightarrow u/u_\infty$ etc.)

$$\frac{\partial u}{\partial t} = \frac{1}{Re} \frac{\partial^2 u}{\partial y^2}, \quad Re = \frac{U_L L}{v}$$

There is no spatial/temporal scales in the problem

$u(y, t)$ is a solution $\Rightarrow u\left(\frac{y}{L}, \frac{t}{T}\right)$ is also a solution
for $L^2 = T$.

If the solution is unique, $u(y, t) = u\left(\frac{y}{L}, \frac{t}{T}\right) \Rightarrow$

$$u(y, t) = u\left(\frac{y}{\sqrt{t}}, 1\right), \quad (L = \sqrt{t}, T = t)$$

(self-similar solution)

$$u = u_\infty f(y), \quad y = \frac{\sqrt{Re}}{2} \frac{y}{\sqrt{t}} \quad \left(\text{in dimensionless coordinates } u_\infty = 1 \right)$$

$$\text{N.S.} \Rightarrow -\underbrace{\frac{\sqrt{Re}}{2} \frac{y}{t^{3/2}} f'}_{\frac{\partial u}{\partial t}} = \underbrace{\frac{1}{Re} \left(\frac{Re}{4t} \right) f''}_{\frac{\partial^2 u}{\partial y^2}}$$

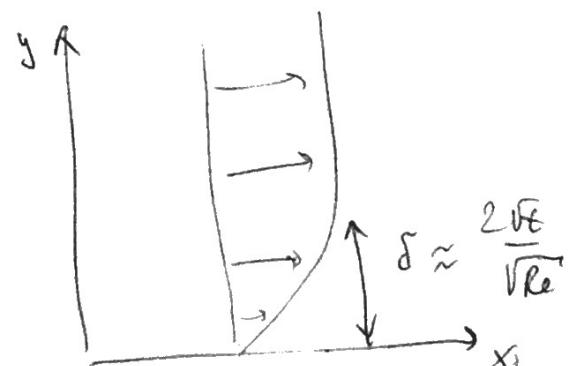
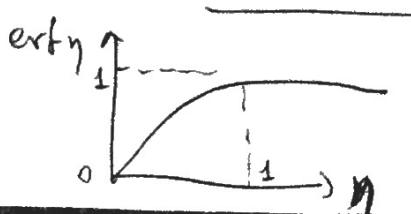
$$\Rightarrow f'' + 2y f' = 0, \quad f(0) = 0, \quad f(\infty) = 1$$

$$f' = c e^{-y^2}, \quad f = \cancel{\operatorname{erf}}(y)$$

(Dimensionless boundary conditions)

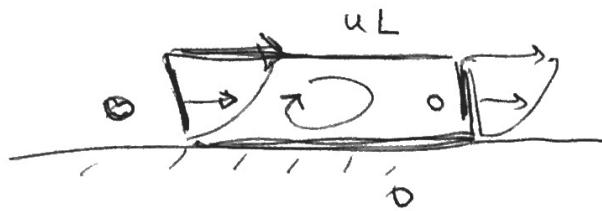
$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-s^2} ds, \quad \operatorname{erf}(\infty) = 1.$$

$$\Rightarrow u = u_\infty \operatorname{erf}\left(\frac{\sqrt{Re}}{2} \frac{y}{\sqrt{t}}\right).$$



Obs Boundary layer decreases as $\underline{\underline{Re}}^{1/2}$.

Obs Boundary layer produces vorticity



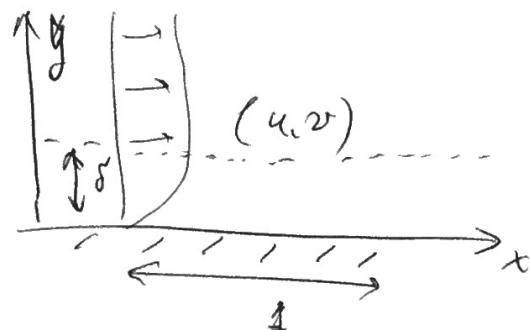
circulation is

$$u_L = \frac{u_\infty L}{\sqrt{Re}} \operatorname{erf}\left(\frac{\sqrt{Re}y}{2\sqrt{E}}\right)$$

General case

Consider 2D incompressible ~~homogeneous~~ slow over a rigid bottom $y=0$.

N.S. eqs (dimensionless)



$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{1}{Re} \Delta u$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} + \frac{1}{Re} \Delta v. \quad \text{④} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = v = 0 \text{ at } y = 0.$$

Assume that a thin boundary layer $\delta \sim \frac{1}{\sqrt{Re}}$ is developed. ($Re \gg 1$).

$$\frac{\partial v}{\partial y} = - \frac{\partial u}{\partial x} \sim 1 \quad (\text{dimensionless large-scale flow}) \Rightarrow v \sim \delta \text{ in boundary layer}$$

with $v=0$ at $y=0$.

Change of variables :

$$x' = x, \quad y' = \frac{y}{\delta}, \quad t' = t, \quad u' = u, \quad v' = \frac{v}{\delta}, \quad p' = p$$

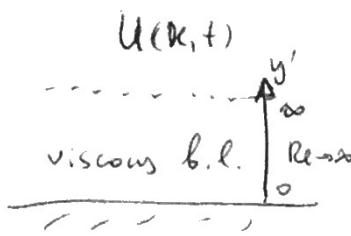
(all new variables ~ 1 inside b.l.)

$$\left\{ \begin{array}{l} \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = - \frac{\partial p'}{\partial x'} + \frac{1}{Re} \left(\frac{\partial^2 u'}{\partial x'^2} + \frac{1}{\delta^2} \frac{\partial^2 u'}{\partial y'^2} \right) \\ \delta \frac{\partial v'}{\partial t'} + \delta u' \frac{\partial v'}{\partial x'} + \delta v' \frac{\partial v'}{\partial y'} = - \frac{1}{\delta} \frac{\partial p'}{\partial y'} + \frac{1}{Re} \left(\delta \frac{\partial^2 v'}{\partial x'^2} + \frac{1}{\delta} \frac{\partial^2 v'}{\partial y'^2} \right) \\ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \quad u' = v' = 0 \text{ at } y' = 0 \end{array} \right.$$

We choose $\delta = \frac{1}{\sqrt{Re}}$ and consider leading terms
for $\frac{1}{Re} \ll 1$ (neglecting higher-order terms)

$$\left\{ \begin{array}{l} \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = - \frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2} \\ \frac{\partial p'}{\partial y'} = 0 \quad (\cancel{\text{at } y' = 0}) \\ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \quad u' = v' = 0 \text{ at } y' = 0 \end{array} \right.$$

AND: $u \rightarrow U(x, t)$ at $y' \rightarrow +\infty$

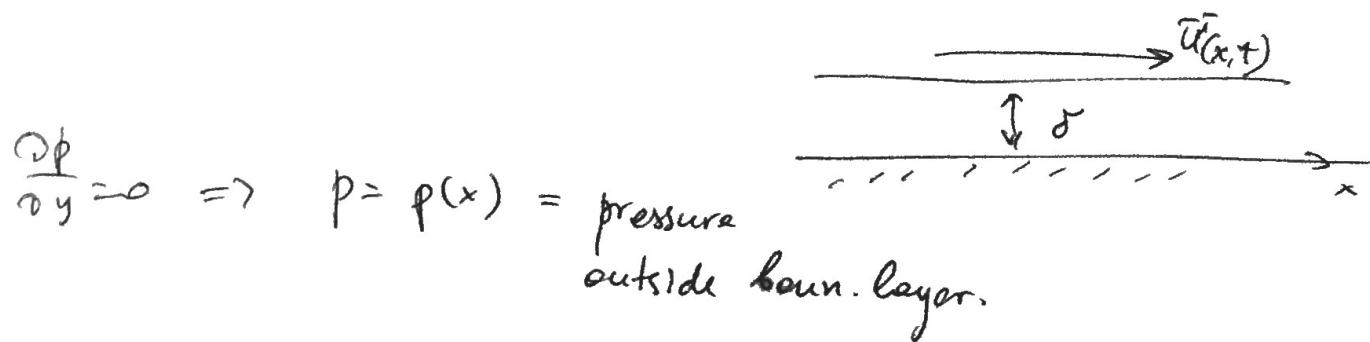


In dimensional form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad u, v = 0 \text{ at } y = 0$$

$$u \rightarrow U(x, t) \text{ at } y \gg \delta$$



When Bernoulli theorem is applicable (steady flow)

$$\text{we have } \frac{p}{\rho} + U^2(x) = \text{const} \Rightarrow \frac{\partial p}{\rho \partial x} = -U \frac{\partial U}{\partial x}$$

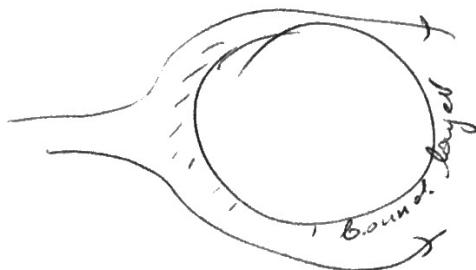
Prandtl's equations:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{\partial U}{\partial x} + v \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad + \text{B.C.}$$

These eqs approximate the flow near a rigid boundary (no math proof, turbulence!)

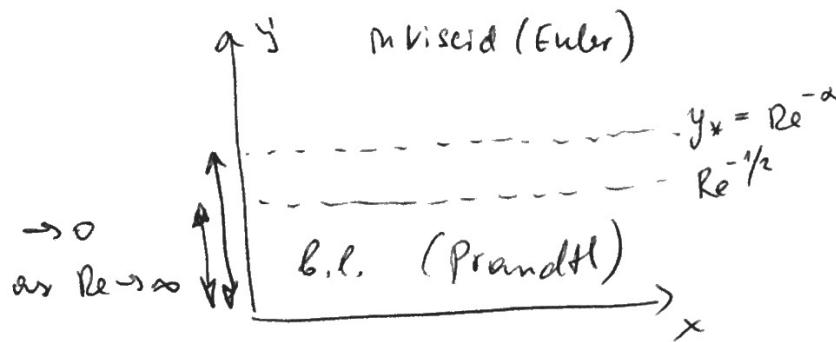
Obs In general, ~~Prandtl~~ Prandtl's eqs can be written for any smooth boundary



Obs Possible strategies:

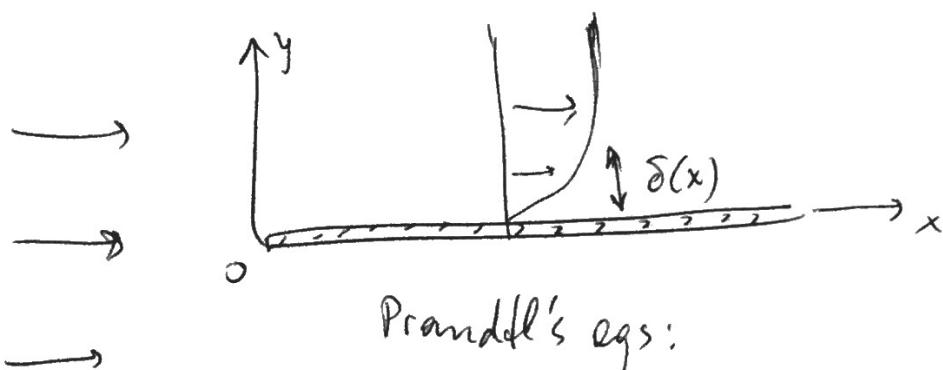
Match Euler solution with Prandtl solution,

e.g. at $y_* = Re^{-\alpha}$, $0 \leq \alpha \leq \frac{1}{2}$



or match solution w.r.t. to some norm.

Example 3 (semi-infinite plate)



Prandtl's eqs:

$$u = \text{const}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

There is no scale in the problem:

$$u(x,y), v(x,y) \text{ is a solution} \Rightarrow u\left(\frac{x}{L}, \frac{y}{\sqrt{L}}\right), v\left(\frac{x}{L}, \frac{y}{\sqrt{L}}\right)$$

is also a solution.

Self-similar solution ($L = \cancel{\pi} x$)

$$u = \tilde{u}\left(\frac{y}{\sqrt{x}}\right), v = \cancel{\frac{1}{\sqrt{x}}} \tilde{v}\left(\frac{y}{\sqrt{x}}\right)$$

$$\begin{cases} \Rightarrow \delta \propto \sqrt{x} \\ \left(\frac{y}{\sqrt{x}} = \text{const} \right) \\ \rightarrow \begin{array}{c} \tilde{u} \\ \downarrow y \end{array} \end{cases}$$

or (incompressibility)

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$$

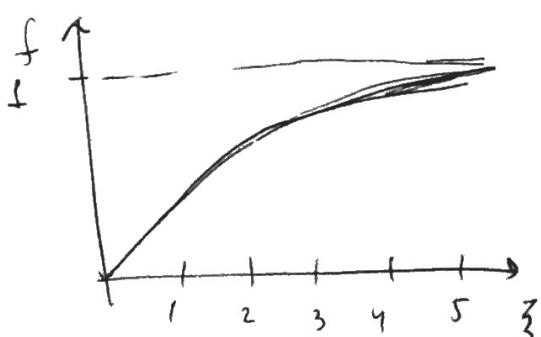
for a stream function $\psi = \sqrt{vxU} f(\xi)$,

$$\xi = y \sqrt{\frac{U}{vx}}$$

$$\Rightarrow u = \frac{\partial \psi}{\partial y} = U f'(\xi), v = -\frac{\partial \psi}{\partial x} = \frac{1}{2} \sqrt{\frac{U}{vx}} (\xi f' - f)$$

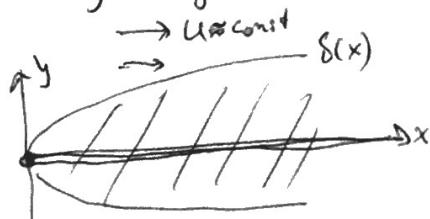
From Prandtl's eq.: ~~f'''~~ $u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2}$

$$\Rightarrow ff'' + 2f''' = 0, \quad \xi=0 : f = f' = 0 \quad (u, v=0)$$



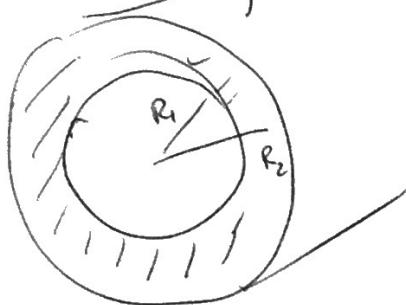
$$\xi = \infty : f' = 1 \quad (u \rightarrow U)$$

$$\text{Boundary layer : } \xi = 1 \Rightarrow \delta(x) \approx \sqrt{\frac{vx}{U}}$$



Exercises

- ① ~~Reynolds~~ Find viscous flow between cylinders



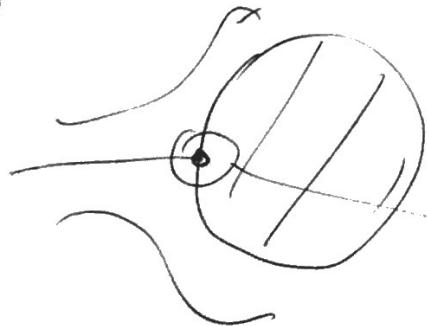
(generalize a solution for a pipe).

- ② Follow our derivation for 3D viscous flow (Stokes eq.) around a sphere, but in 2D case (around a disk).

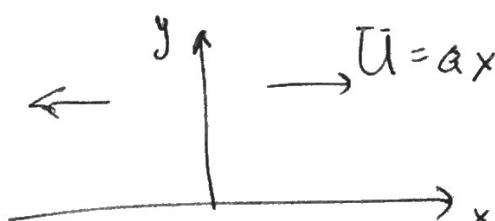
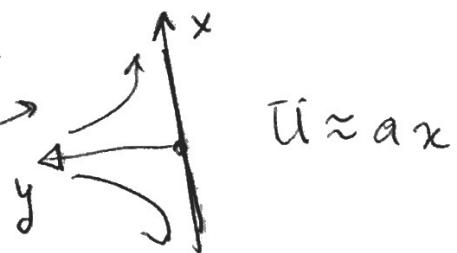
Show that no solution can be obtained in this way.

(Stokes paradox)

③



In ideal flow



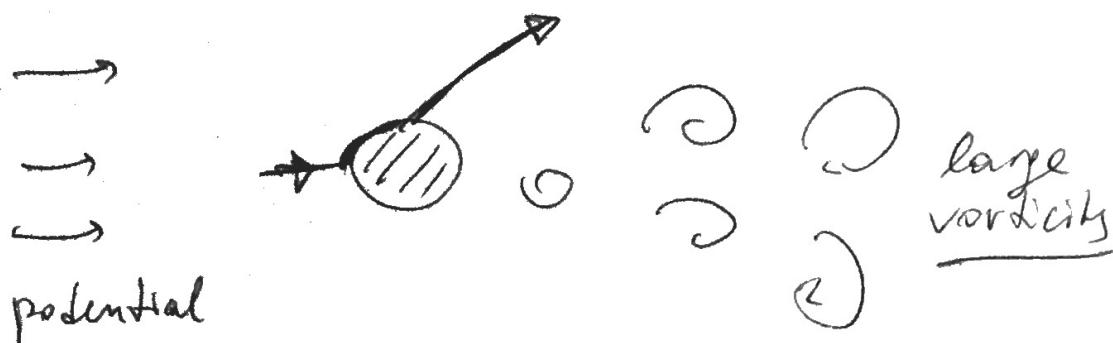
~~Reynolds~~ (1) Write Prandtl eqs
for boundary layer

(2) Find a form of
self-similar solution

- (3) What is the dependence $\delta(x)$ (4) Try to solve eqs.
for b.l. width

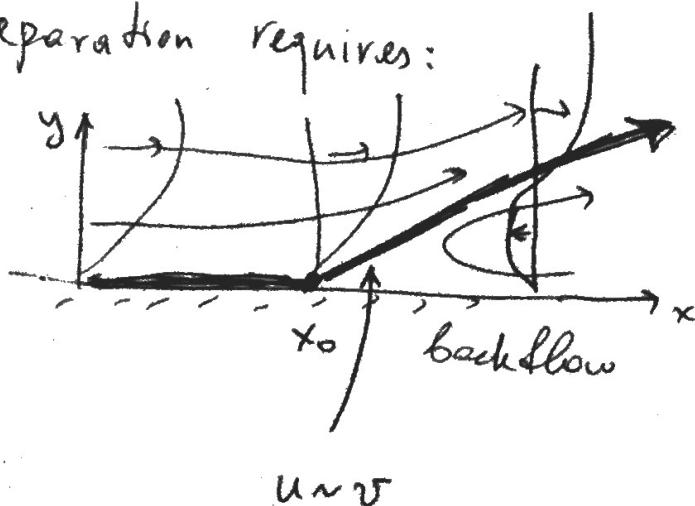
Separation of boundary layer

120

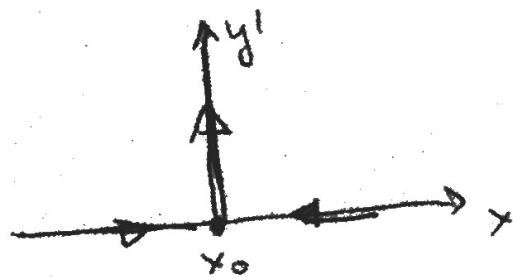


Vorticity generation requires separation of boundary layer from boundary (otherwise, the flow will remain potential).

Separation requires:



For coordinates $y' = \frac{y}{\delta}$, $v' = \frac{v}{\delta}$, $\delta = \frac{1}{\sqrt{Re}} \rightarrow 0$



$v' = \infty$ at $y' = 0$ $x = x_0$
(except $y' = 0$)

$$\frac{\partial u}{\partial x} + \frac{\partial v'}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial x}{\partial u} = 0$$

(121)

$$x = x(u, y)$$

$$\Rightarrow x = x_0 + f(y)(u - u_0(y))^2, \text{ where } u = u_0(y) \\ \text{at } x = x_0$$

$$x = u_0(y) + \alpha(y) \sqrt{x_0 - x}, \quad \alpha = f^{-1/2}$$

$$\frac{\partial v'}{\partial y} = - \frac{\partial u}{\partial x} = \frac{\alpha(y)}{2\sqrt{x_0 - x}} \Rightarrow v' = \frac{\beta(y)}{\sqrt{x_0 - x}}$$

If assumptions are correct, we cannot extend a solution behind point x_0 :

$\sqrt{x_0 - x}$ becomes complex.

What happens at separation point (as $R \rightarrow \infty$) is unknown. Turbulent flow is expected

