

- Course:
- classical mechanics
 - oscillations
 - waves

Hamilton's principle

Space $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

Time $t \in \mathbb{R}$

Material point (point particle, point mass);
small object fully described by its position
 $x = x(t)$.

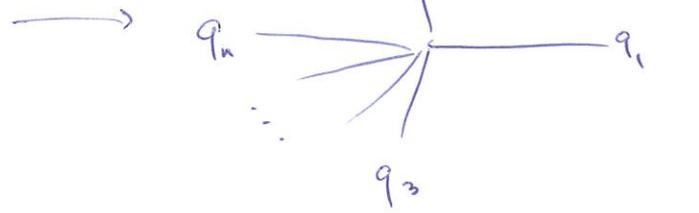
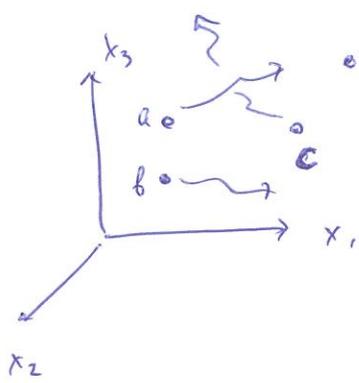
System of points $a, b, c, \dots \rightarrow x_a(t), x_b(t), \dots$

Vector of coordinates of the system:

$$q = \begin{pmatrix} x_{1a} \\ x_{2a} \\ x_{3a} \\ x_{1b} \\ x_{2b} \\ \vdots \end{pmatrix} \in \mathbb{R}^n, \quad n = 3 \cdot \text{number of points.}$$

Evolution of the system: $q(t) = (q_1(t), \dots, q_n(t))$.

How one derives equations of motion?



Hamilton's principle suggests that dynamics of a physical system can be determined as a minimum (extremum) of some functional (~~element~~) depending on a trajectory $q(t)$.

$q(t)$, $t \in [t_0, t_1]$ is a minimum of $S[q(t)]$

action $S: \begin{matrix} q(t) \\ \text{trajectory} \end{matrix} \rightarrow \mathbb{R}$ assigns a number to every trajectory.

Properties of S : should not depend on $t < t_0$ or $t > t_1$
 should work for any t_0, t_1

$$S = \int_{t_0}^{t_1} L(t, q, \dot{q}) dt$$

↑
Lagrangian function.

Obs Why only derivatives up to first order in L ?

$L(t, q)$ does not work (exercise)

$L(t, q, \dot{q}, \ddot{q})$ may work, ... But we take a simplest choice

Hamilton's principle (~~Principle of least action~~)
(Principle of least action)

The action attains ~~the~~ minimum

$$S \rightarrow \min$$

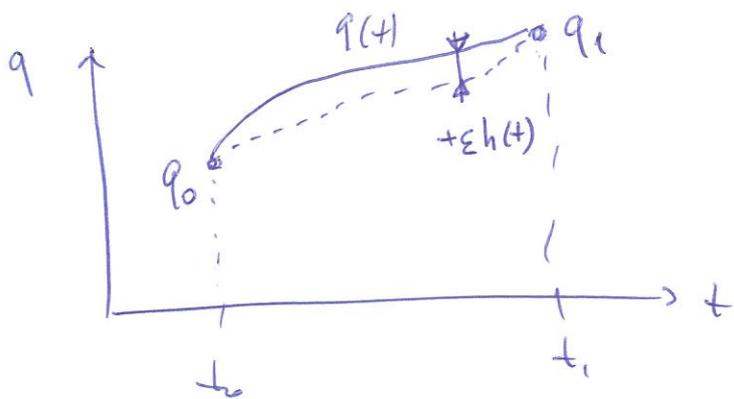
on a physical trajectory $q(t)$, $t \in [t_0, t_1]$

among all trajectories with the same end points

$$q_0 = q(t_0), \quad q_1 = q(t_1).$$

Euler-Lagrange equations

We consider the case $n=1$ first: $q \in \mathbb{R}$.



Consider a physical trajectory $q(t)$ and its small perturbation $q(t) + \epsilon h(t)$.

Since end points must be the same;

$$h(t_0) = h(t_1) = 0.$$

Least action principle :

$$S [q(t) + \epsilon h(t)] \geq S [q(t)]$$

$$\int_{t_0}^{t_1} L(t, q + \epsilon h, \dot{q} + \epsilon \dot{h}) dt \geq \int_{t_0}^{t_1} L(t, q, \dot{q}) dt$$

Taylor series in ϵ

$$L(t, q + \epsilon h, \dot{q} + \epsilon \dot{h}) = L(t, q, \dot{q}) + \epsilon \left(\frac{\partial L}{\partial q} h + \frac{\partial L}{\partial \dot{q}} \dot{h} \right) + o(\epsilon)$$

$$\epsilon \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} h + \frac{\partial L}{\partial \dot{q}} \dot{h} \right) dt + o(\epsilon) \geq 0 \quad \text{for any } \epsilon!$$

$$\Rightarrow \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} h + \frac{\partial L}{\partial \dot{q}} \dot{h} \right) dt = 0$$

Second term:

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}} \frac{\partial h}{\partial t} dt = \underbrace{\frac{\partial L}{\partial \dot{q}} h \Big|_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) h dt$$

$$= - \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) h dt$$

$$\Rightarrow \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) h dt = 0 \quad \text{for any } h(t).$$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0} \quad \text{Euler-Lagrange equation (for } n=1).$$

In a general case $n \geq 1$:

$$L(t, q, \dot{q}) = L(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n).$$

~~Take $q_i(t) = q_i(t) + \epsilon h(t)$~~ Exercise:

Perturb $q_i(t) \rightarrow q_i(t) + \epsilon h(t)$ for some fixed $i \in \{1, \dots, n\}$ keeping other components unchanged. Perform the same derivation as for $n=1$. This must yield

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n}$$

System of Euler-Lagrange equations.

Obs Why fixing end points $q(t_0)$ and $q(t_1)$?
 otherwise, the H. principle fails because of
 the term $\left. \frac{\partial L}{\partial \dot{q}} h \right|_{t_0}^{t_1} = \left(\frac{\partial L}{\partial \dot{q}} h \right)_{t_1} - \left(\frac{\partial L}{\partial \dot{q}} h \right)_{t_0}$

~~the term $\left. \frac{\partial L}{\partial \dot{q}} h \right|_{t_0}^{t_1}$ is not zero (Exercise).~~

Obs Is the Lagrangian function unique?

The Answer is no.

Consider any function $f(t, q)$ and define

$$\tilde{L}(t, q, \dot{q}) = L(t, q, \dot{q}) + \frac{d}{dt} f(t, q)$$

$$= L(t, q, \dot{q}) + \frac{\partial f}{\partial t} + \sum_{i=1}^n \frac{\partial f}{\partial q_i} \dot{q}_i$$

New action is

$$\tilde{S} = \int_{t_0}^{t_1} \tilde{L}(t, q, \dot{q}) dt$$

$$= \int_{t_0}^{t_1} L(t, q, \dot{q}) dt + \int_{t_0}^{t_1} f(t, q) dt$$

$$= S + \underbrace{f(t_1, q(t_1)) - f(t_0, q(t_0))}_{\text{fixed in the H. principle!}}$$

fixed in the H. principle!

Hence ~~we~~ $S \rightarrow \text{min} \Leftrightarrow \tilde{S} \rightarrow \text{min}$ \square

Equations of motion (Euler-Lagrange) for S and \tilde{S} must be equivalent.

Ex. check that E.L. eqs. are the same.

Obv We write $\frac{\partial}{\partial t}$ for a partial derivative

$$\frac{\partial f(t, q)}{\partial t}$$

and we write $\frac{d}{dt}$ for a material derivative assuming that q, \dot{q} are functions of time.

$$\frac{d}{dt} f(t, q) = \frac{\partial f}{\partial t} + \sum_{i=1}^n \frac{\partial f}{\partial q_i} \dot{q}_i$$

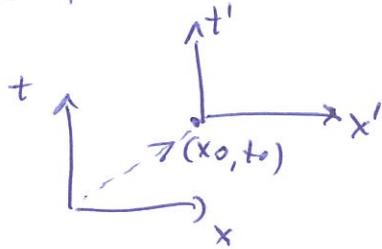
time derivative along a physical trajectory.

What is $L(t, q, \dot{q})$?

LP

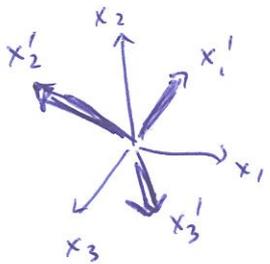
Space-time symmetries (Galilean group)

(1) Shifts: $t = t' + t_0$, $x = x' + x_0$



change of the origin in
Space-time

(2) Rotations in space ($x \in \mathbb{R}^3$)



$$x = \Theta x' \quad , \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad , \quad x' = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{21} & \dots & \vdots \\ \Theta_{31} & \dots & \Theta_{32} \end{pmatrix} \quad \begin{matrix} 3 \times 3 \\ \text{matrix} \end{matrix}$$

$$\|x\|^2 = (x, x) = (\Theta x', \Theta x') = (\Theta x')^T \Theta x' = x'^T \Theta^T \Theta x'$$

Demanding $\|x\|^2 = \|x'\|^2 = x'^T x'$, we have

the orthogonality condition $\Theta^T \Theta = I$ (identity matrix)

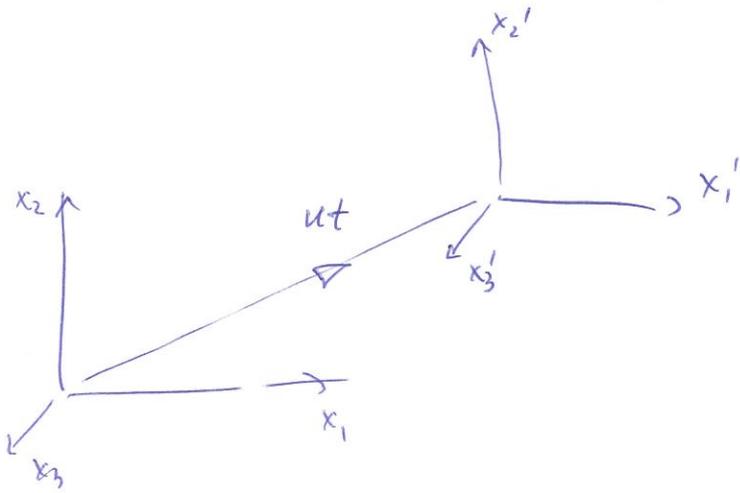
Hence, rotations are given by orthogonal matrices Θ .

Obs This includes reflections $x = -x'$.

(3) Galilean transformation

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$$x = x' + ut' \quad , \quad t = t' \quad \text{for a fixed velocity} \\ u \in \mathbb{R}^3$$



reference frame x'
moves with a constant
velocity u .

Obs It appears naturally from a Poincaré (space-time ~~relations~~ ^{isometry}) group in the limit of small velocities.

Obs Time reversal $t = -t'$ can also be considered.

Def ~~The~~ Galilean group \mathcal{G}_3 is a group of space-time transformations generated by (1) - (3):

$$x = x_0 + \Theta x' + ut' \quad , \quad t = t_0 + t'$$

An element of \mathcal{G}_3 can be represented as

$$a = (t_0, x_0, \Theta, u) \in \mathcal{G}_3.$$

Let us recall what is a group.

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Def A set G with a binary operation $*$

such that $a * b \in G$ for any $a, b \in G$ is

called a group if it satisfies the properties:

(a) Associativity: $(a * b) * c = a * (b * c), \forall a, b, c \in G$

(b) Existence of identity element: ~~$e \in G$~~

$\exists e \in G$ s.t. $e * a = a * e = a, \forall a \in G$

(c) Inverse element:

$\forall a \in G, \exists a^{-1} \in G$ s.t. $a * a^{-1} = a^{-1} * a = e$.

Ex: check that our definition of the Galilean group is really a group.

Obv Galilean group is not commutative:

$a * b \neq b * a$ in general.

Lesson 2

Summary : $q \in \mathbb{R}^n$ (coordinates of all mat. points)

Hamilton's principle :

$$S = \int_{t_0}^{t_1} L(t, q, \dot{q}) dt \longrightarrow \text{min with } \mathbb{R} q(t_0), q(t_1) \text{ fixed.}$$

Euler-Lagrange equations :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n$$

Freedom to choose a Lagrangian function :

$$L \longrightarrow \tilde{L} = L + \frac{d}{dt} f(t, q) = L + \frac{\partial f}{\partial t} + \sum_{i=1}^n \frac{\partial f}{\partial q_i} \dot{q}_i$$

Galilean group generators :

① Translations in space and time :

$$x = x' + x_0, \quad t = t' + t_0$$

② Rotations in space

$$x = O x' \quad (O^T O = I) \quad t = t'$$

③ Galilean transformations

$$x = x' + ut', \quad t = t'$$

Lagrangian function of a material point

We assume that ~~there~~ there exists a coordinate system (x, t) , called inertial reference frame, such that the laws of motion ~~do not~~ are invariant with respect to the action of the Galilean group.

This will give us ~~$L(t, x, \dot{x})$~~ $L(t, x, \dot{x})$

Let ~~on~~ a system consisting of a single material point ($q = x \in \mathbb{R}^3$).

1) ~~Shift~~ Shift in space-time: $t = t' + t_0, x = x' + x_0$

$$\Rightarrow L(t, x, \dot{x}) = L(t' + t_0, x' + x_0, \dot{x}') = \tilde{L}(t', x', \dot{x}')$$

If equations of motion are the same for x and x' $\Rightarrow L = \tilde{L} \Rightarrow L$ does not depend on t, x .

$$L = L(\dot{x}).$$

(homogeneity of space and time).

(2) Rotations in space:

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$$x = G x' \quad (\text{preserve } \|x\|)$$

$$\Rightarrow L = L(v^2), \quad v = \|x'\|.$$

(isotropy of space)

Obs we use $v^2 = \|x\|^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$ in order

to avoid ~~then~~ singularities, e.g., ~~the~~

$\|x\| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}$ is not ~~defined~~ a smooth function.

(3) Galilean transformation:

$$x = x' + \varepsilon u t \quad \text{for small } \varepsilon.$$

$$\dot{x} = \dot{x}' + \varepsilon u$$

Let us also consider $L(v^2)$ as a Taylor series

$$L = a_0 + a_2 v^2 + a_4 v^4 + \dots$$

We have

$$v^2 = \|x\|^2 = \|\dot{x}' + \varepsilon u\|^2 = (\dot{x}' + \varepsilon u, \dot{x}' + \varepsilon u)$$

$$= (\dot{x}', \dot{x}') + 2\varepsilon (\dot{x}', u) + o(\varepsilon) = v'^2 + 2\varepsilon (\dot{x}', u) + o(\varepsilon)$$

$$v^4 = (v^2)^2 = v'^4 + 4\varepsilon (\dot{x}', u) v'^2 + o(\varepsilon)$$

$$v^6 = (v^2)^3 = v'^6 + 6\varepsilon (\dot{x}', u) v'^4 + o(\varepsilon)$$

etc

$$\Rightarrow L(v^2) = a_0 + a_2 v^2 + a_4 v^4 + \dots$$

$$= a_0 + a_2 [v'^2 + 2\varepsilon (\dot{x}', u) + o(\varepsilon)] + a_4 [v'^4 + 4\varepsilon (\dot{x}', u) + o(\varepsilon)] + \dots$$

$$= a_0 + a_2 v'^2 + a_4 v'^4 + \dots$$

$$+ \varepsilon (\dot{x}', u) [2a_2 + 4a_4 v'^2 + \dots] + o(\varepsilon)$$

$$= L(v'^2) + \underbrace{\varepsilon (\dot{x}', u) [2a_2 + 4a_4 v'^2 + \dots]}_{\text{problematic term!}} + o(\varepsilon)$$

problematic term!

$$(L(v^2) \neq L(v'^2))$$

If we demand $L(v^2) = L(v'^2)$, then all

$$a_2 = a_4 = a_6 \dots = 0. \Rightarrow L(v^2) = a_0 \text{ is trivial (no good)}$$

Recall that the same laws (equations of motion) follow if L is changed as

$$\tilde{L} = L + \frac{d}{dt} f(x, t) = L + \underbrace{\frac{\partial f(x, t)}{\partial t} + \sum_{i=1}^3 \frac{\partial f}{\partial x_i} \dot{x}_i}_{\text{linear expression in speeds.}}$$

Hence, it is enough to set $a_4 = a_6 = \dots = 0$ of nonlinear terms: ~~of the~~ Then,

$$L(v^2) = L(v'^2) + \underbrace{2a_5 (x', t)}_{\frac{d}{dt} f(x, t)} + o(\epsilon)$$

Ex: find which f

Result: $L(v^2) = a_0 + a_2 v^2$

~~Obs: a_0 does not matter, as E.L. equations depend on derivatives of L only:~~

Coefficient a_0 does not matter, as E.L. equations depend on derivatives of L only:

$$L(v^2) = a_2 v^2.$$

Def We ~~set~~ set $a_2 = \frac{m}{2}$, where we call m the mass of a particle.

$$L = \frac{m\dot{v}^2}{2} = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

Ob The condition that $S \rightarrow m\dot{v}$ on a particle trajectory requires that $m > 0$.

If $m < 0 \Rightarrow$ infinitely fast motion (~~etc~~ ex.)

Newton's laws of motion

Consider a single (isolated) material point

$$L = \frac{m\dot{v}^2}{2} = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

E.L. eqs: $q = x = (x_1, x_2, x_3)$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, 3.$$

$$\frac{d}{dt} (m\dot{x}_i) = 0 \Rightarrow \overset{00}{\dot{x}_i} = 0 \Rightarrow \ddot{x}_i = 0$$

Solution: $x = x_0 + ut$ for any $x_0, u \in \mathbb{R}^3$.

1st Newton's law: Every body continues in its state of rest, or of uniform motion in a straight

line, unless, it is compelled to change 16
that state by forces impressed upon it.

Two material points $x_a = (x_{1a}, x_{2a}, x_{3a})$

$$x_b = (x_{1b}, x_{2b}, x_{3b})$$

When points are far away (isolated) we have

$$L_a = \frac{m_a v_a^2}{2}, \quad L_b = \frac{m_b v_b^2}{2} \quad \left(\begin{array}{l} \text{different} \\ \text{masses } m_a, m_b \end{array} \right)$$

and the joint Lagrangian can be taken as

$$L := T = \frac{m_a v_a^2}{2} + \frac{m_b v_b^2}{2} \quad (\text{kinetic energy})$$

Note that $S = S_a + S_b \rightarrow \text{min} \Rightarrow S_a \rightarrow \text{min}, S_b \rightarrow \text{min}$.

In classical mechanics, the interaction is modelled by a function of ~~coordinates~~ ^{distance} $U(\frac{r}{|x_a - x_b|})$ called the potential energy, subtracted in the Lagrangian:

$$L = T - U = \frac{m_a v_a^2}{2} + \frac{m_b v_b^2}{2} - U(r).$$

where $r = \|x_a - x_b\|$.

Obs $U(r)$ is the only form of a function of

x_a and x_b that is invariant w.r.t Galilean group.

Now $q = \begin{pmatrix} x_a \\ x_b \end{pmatrix} \in \mathbb{R}^6$. E.L. equations are:

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$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{ia}} - \frac{\partial L}{\partial x_{ia}} = 0 \quad i = 1, 2, 3$$

$$\frac{d}{dt} (m \dot{x}_{ia}) - \left(- \frac{\partial U}{\partial x_{ia}} \right) = m \ddot{x}_{ia} + \frac{\partial U}{\partial x_{ia}} = 0.$$

$$\Rightarrow \boxed{m \ddot{x}_a = F_a}, \quad \text{where we defined a force}$$

$$F_a = (F_{1a}, F_{2a}, F_{3a}) = \left(- \frac{\partial U}{\partial x_{1a}}, - \frac{\partial U}{\partial x_{2a}}, - \frac{\partial U}{\partial x_{3a}} \right).$$

2nd Newton's law: The change of motion of an object is proportional to the force impressed, and it is made in the direction of the straight line in which the force is impressed.

Second material point: $m \ddot{x}_b = F_b$

$$F_b = (F_{1b}, F_{2b}, F_{3b}) = \left(- \frac{\partial U}{\partial x_{1b}}, - \frac{\partial U}{\partial x_{2b}}, - \frac{\partial U}{\partial x_{3b}} \right)$$

But $U = U(r)$, $r = \sqrt{(x_{1a} - x_{1b})^2 + (x_{2a} - x_{2b})^2 + (x_{3a} - x_{3b})^2}$

$$\begin{aligned} \text{Hence, } F_{1a} &= - \frac{\partial U}{\partial x_{1a}} = - \frac{\partial U}{\partial r} \frac{\partial r}{\partial x_{1a}} = \\ &= - \frac{\partial U}{\partial r} \frac{x_{1a} - x_{1b}}{\sqrt{\dots}} = - \frac{\partial U}{\partial r} \frac{x_{1a} - x_{1b}}{r}. \end{aligned}$$

$$O_r \quad F_a = - \frac{\partial U}{\partial r} \frac{x_a - x_b}{r}$$

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$$\text{Similarly } F_b = - \frac{\partial U}{\partial r} \frac{x_b - x_a}{r} = -F_a$$

3^d Newton's law:

To every action, there is always opposite an equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

System of many material points: x_a, x_b, x_c, \dots

$$L = T - U$$

$$T = \sum_{\alpha=a,b,\dots} \frac{m_\alpha v_\alpha^2}{2} \quad \text{kinetic energy}$$

$U(x_a, x_b, \dots)$ potential energy depending only on the distances $\|x_\alpha - x_\beta\|$ among points.

Generalized coordinates

We can use any vector $q \in \mathbb{R}^n$ that uniquely represents positions of all material points

$$x_a = x_a(q), \quad x_b = x_b(q), \quad \dots$$

For example, angles and particle distances.

Speeds are determined as

$$\dot{x}_\alpha(q, \dot{q}) = \frac{dx_\alpha(q)}{dt} = \sum_{i=1}^n \underbrace{\frac{\partial x_\alpha}{\partial q_i}}_{\text{function of } q} \dot{q}_i$$

We define a new Lagrangian as

$$\begin{aligned} \tilde{L}(q, \dot{q}) &= L(x_a(q), x_b(q), \dots, \dot{x}_a(q, \dot{q}), \dot{x}_b(q, \dot{q}), \dots) \\ &= \tilde{T}(q, \dot{q}) - \tilde{U}(q) \end{aligned}$$

where

$$\tilde{T}(q, \dot{q}) = \sum_{\alpha=a, b, \dots} \frac{m_\alpha}{2} (\dot{x}_\alpha)^2 = \frac{1}{2} \sum_{i,j=1}^n m_{ij}(q) \dot{q}_i \dot{q}_j$$

$$m_{ij}(q) = \sum_{\alpha=a, b, \dots} m_\alpha \left(\frac{\partial x_\alpha}{\partial q_i}, \frac{\partial x_\alpha}{\partial q_j} \right)$$

$$\tilde{U}(q) = U(x_a(q), x_b(q), \dots)$$

Since $\tilde{L}(q, \dot{q}) = L(x_a, x_b, \dots, \dot{x}_a, \dot{x}_b, \dots)$ ~~and $\frac{\partial L}{\partial x_\alpha} = \frac{\partial L}{\partial x_\alpha}$~~

$\Rightarrow \tilde{S} = \int_{t_0}^{t_1} \tilde{L} dt \rightarrow$ min on a physical trajectory

\Rightarrow one can use E.L equations written directly for $\tilde{L}(q, \dot{q})$.

Gravity :

Interaction between particles a and b given by

$$U = - \frac{k}{\|x_a - x_b\|}, \quad \text{~~potential energy~~$$

where $k = G m_a m_b$, $G = 6.67 \dots \times 10^{-11} \frac{Nm^2}{kg^2}$ ~~universal~~ gravitation constant.

Consider generalized coordinates

$$R = \frac{m_a x_a + m_b x_b}{m_a + m_b} \quad (\text{center of mass})$$

$$x = x_a - x_b \quad (\text{relative position})$$

The inverse is

$$x_a = R + \frac{m_b}{m_a + m_b} x, \quad x_b = R - \frac{m_a}{m_a + m_b} x$$

$$\dot{x}_a = \dot{R} + \frac{m_b}{m_a + m_b} \dot{x}, \quad \dot{x}_b = \dot{R} - \frac{m_a}{m_a + m_b} \dot{x}$$

Kinetic energy :

$$T = \frac{m_a}{2} \|\dot{x}_a\|^2 + \frac{m_b}{2} \|\dot{x}_b\|^2 = \frac{m_a}{2} (\dot{x}_a, \dot{x}_a) + \frac{m_b}{2} (\dot{x}_b, \dot{x}_b) \\ = \dots (\text{exercise}) = \frac{M}{2} \|\dot{R}\|^2 + \frac{m}{2} \|\dot{x}\|^2$$

with total mass $M = m_a + m_b$ and reduced mass $m = \frac{m_a m_b}{m_a + m_b}$

$$U = -\frac{k}{\|x_a - x_b\|} = -\frac{k}{\|r\|}$$

Lagrangian function: $L = \frac{M\|\dot{R}\|^2}{2} + \frac{m\|\dot{r}\|^2}{2} + \frac{k}{\|x\|}$

E.L. eq. for R: $M\ddot{R} = 0 \Rightarrow R = R_0 + Ut$
 uniform motion of the center of mass.

E.L. eq. for r:

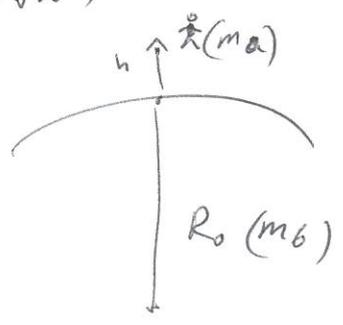
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \frac{d}{dt} (m\dot{x}_i) + \frac{kx_i}{\|x\|^3} = 0$$

$$\Rightarrow \left[\ddot{x} = -\frac{k}{m} \frac{x}{\|x\|^3} \right]$$

equation for the relative motion in 2-body problem.

When $m_a \ll m_b$ (m_b is the Earth)

$$m = \frac{m_a m_b}{m_a + m_b} \approx \frac{m_a m_b}{m_b} = m_a$$



$$V = -\frac{k}{R} \approx -\frac{k}{R_0} + \frac{k}{R_0^2} h + o(h)$$

$$\frac{k}{R_0^2} = \frac{Gm_a m_b}{R_0^2} = m_a g, \quad g = \frac{Gm_b}{R_0^2} \Rightarrow V \approx m_a g h$$

Lecture 3

Summary: ~~coordinates~~ $q \in \mathbb{R}^n$ generalized coordinates
(describe the system uniquely)

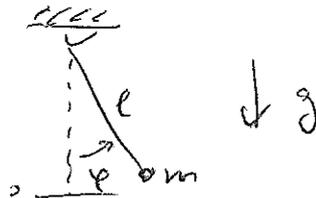
E. L. eqs: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n$

$L = T - U$. Kinetic energy $T = \sum_{\alpha} \frac{m_{\alpha} v_{\alpha}^2}{2}$
potential energy $U = U(q)$. - interaction.

T and U must have the same value for whatever choice of coordinates q .

Examples:

① Pendulum



$$T = \frac{mv^2}{2} = \frac{ml^2 \dot{\varphi}^2}{2}$$

$$U = mgh = mgl(1 - \cos \varphi)$$

$$L = \frac{ml^2 \dot{\varphi}^2}{2} - mgl(1 - \cos \varphi)$$

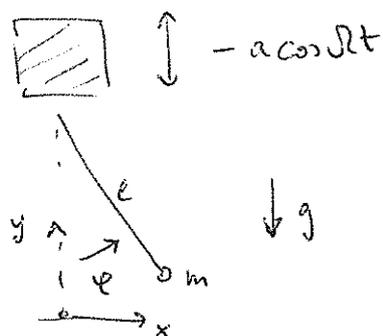
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{d}{dt} (ml^2 \dot{\varphi}) + mgl \sin \varphi = 0$$

$$\ddot{\varphi} + \frac{g}{l} \sin \varphi = 0$$

② Pendulum with periodic support

$$x = l \sin \varphi, \quad y = l(1 - \cos \varphi) - a \cos \Omega t$$

$$v_x = l \dot{\varphi} \cos \varphi, \quad v_y = l \dot{\varphi} \sin \varphi + a \Omega \sin \Omega t$$



$$L = m \frac{v_x^2 + v_y^2}{2} - mgy$$

$$= \frac{m}{2} (l^2 \dot{\varphi}^2 \cos^2 \varphi + l^2 \dot{\varphi}^2 \sin^2 \varphi + 2l \dot{\varphi} a \Omega \sin \varphi \sin \Omega t + a^2 \Omega^2 \sin^2 \Omega t) - mg(l(1 - \cos \varphi) - a \cos \Omega t)$$

We only keep terms with φ and/or $\dot{\varphi}$. (Other terms ~~disappear~~ are not important in E.L. eqs.)

$$L = \cancel{\frac{m l^2}{2} \dot{\varphi}^2} \quad m l^2 \left(\frac{\dot{\varphi}^2}{2} + \frac{a \Omega}{l} \dot{\varphi} \sin \varphi \sin \Omega t + \frac{g}{l} \cos \varphi \right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \Rightarrow$$

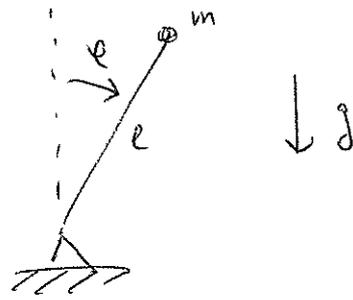
$$\frac{d}{dt} \left[m l^2 \left(\dot{\varphi} + \frac{a \Omega}{l} \sin \varphi \sin \Omega t \right) \right] - m l^2 \left[\frac{a \Omega}{l} \dot{\varphi} \cos \varphi \sin \Omega t - \frac{g}{l} \sin \varphi \right] = 0$$

$$\ddot{\varphi} + \frac{a \Omega}{l} \dot{\varphi} \cos \varphi \sin \Omega t + \frac{a \Omega^2}{l} \sin \varphi \cos \Omega t - \frac{a \Omega}{l} \dot{\varphi} \cos \varphi \sin \Omega t + \frac{g}{l} \sin \varphi = 0$$

$$\ddot{\varphi} + \frac{g}{l} \left(1 + \frac{a \Omega^2}{g} \cos \Omega t \right) = 0$$

Exercises (write L and E.L. eqs).

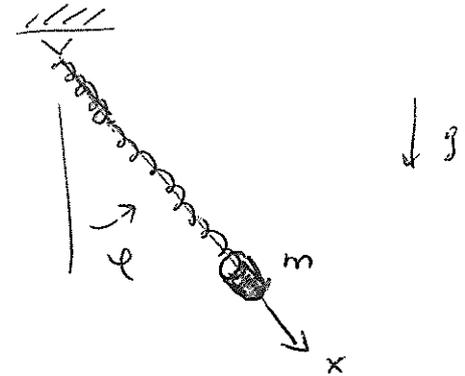
(a) Inverted pendulum



(b) Pendulum with a spring

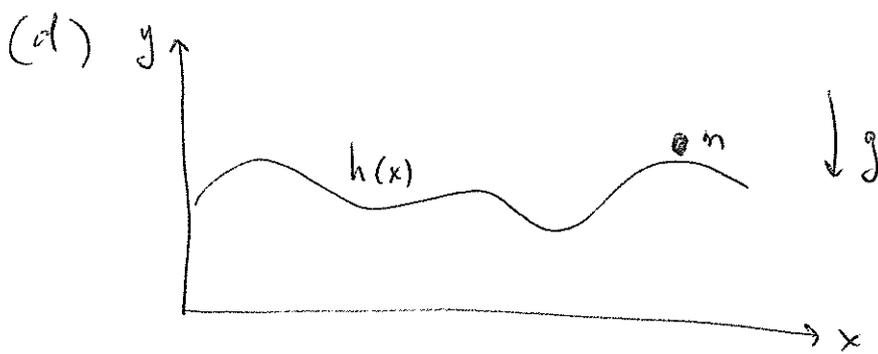
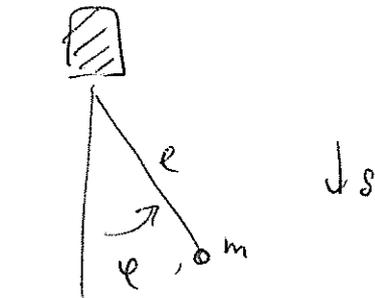
Potential energy

$$U = \underbrace{mgh}_{\text{gravity}} + \underbrace{\frac{k}{2}(x-x_0)^2}_{\text{harmonic oscillator}}$$



(c) Pendulum with a horizontally moving support

$x = x_s(t)$ given function of time.



(assuming no jumps).

Conservation laws

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Generalized coordinates $q \in M$ (n -dim smooth manifold)
~~locally \mathbb{R}^n~~

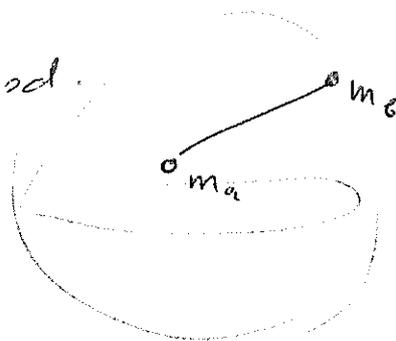
Pendulum: $\varphi \in S^1 \pmod{2\pi}$ $M = S^1$

Pendulum with a spring $q = (x, \varphi) \in M = \mathbb{R}_+ \times S^1$

Two masses connected with a rigid rod:

~~$q = (x, \varphi)$~~

$q \in M = \underbrace{\mathbb{R}^3}_{\text{position of } m_a} \times \underbrace{S^2}_{\text{orientation of the rod}}$



We can always define a local coordinate system

$q \in \mathbb{R}^n$ at each point of M . $q = (q_1, \dots, q_n)$.

Change of coordinates:

$h: M \rightarrow M$ diffeomorphism (invertible, h and h^{-1} are smooth)

$q' = h(q) = (h_1(q), \dots, h_n(q))$

Examp: Standard pendulum \rightarrow inverted pendulum

$h: S^1 \rightarrow S^1$ $h(\varphi) = \varphi + \pi \pmod{2\pi}$

$$\dot{q}' = \frac{dh}{dt} = \frac{\partial h}{\partial q} \dot{q} = \begin{pmatrix} \frac{\partial h_1}{\partial q_1} & \dots & \frac{\partial h_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial q_1} & \dots & \frac{\partial h_n}{\partial q_n} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \quad \boxed{26}$$

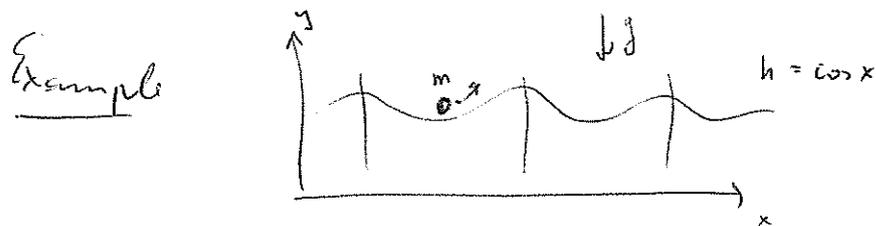
Jacobian matrix

New Lagrangian: $L'(t, q', \dot{q}') = L(t, q, \dot{q})$ for $q' = h(q)$
 $\dot{q}' = \frac{\partial h}{\partial q} \dot{q}$.

We call h a symmetry if it does not change the Lagrangian, i.e.,

$$L(t, q, \dot{q}) = L(t, q', \dot{q}') \text{ for } q' = h(q), \dot{q}' = \frac{\partial h}{\partial q} \dot{q}.$$

Obs We do not consider here symmetries that change L or $L \rightarrow L + \frac{d}{dt} f(q, t)$. Hence, Galilean transformations is ~~not~~ not a symmetry in this sense.



$$q = x$$

$$h(x) = x + 2\pi.$$

(check for L)

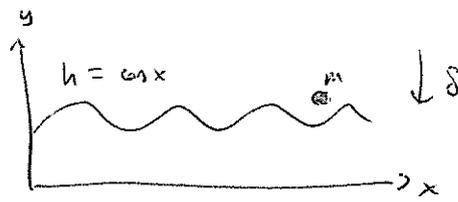
Def h^s is a (uniparametric) continuous symmetry if

$$\textcircled{1} h^{s_1} \circ h^{s_2} = h^{s_1 + s_2}, \quad q'' = h^{s_1}(h^{s_2}(q)) = h^{s_1 + s_2}(q).$$

$\textcircled{2}$ h^s is a symmetry for any s

Obs: ~~is~~ uniparametric means $s \in \mathbb{R}$ ~~all~~ (locally near $s=0$).

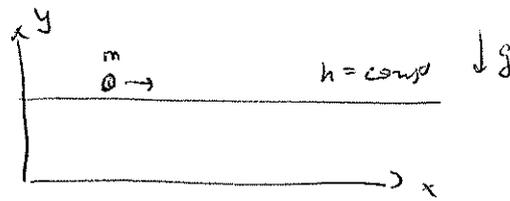
Previous example



~~h(x) = x + 2\pi n~~

$$h(x) = x + 2\pi n,$$

$n \in \mathbb{Z} \Rightarrow$ not a contin
sym. group.



$$h(x) = x + x_0$$

$x_0 \in \mathbb{R} \Rightarrow$ ok!

Noether's theorem

If a system has a continuous symmetry h^s . Then there is a ~~conservation~~ corresponding conservation law.

Namely, the conserved quantity defined as

$$C(t, q, \dot{q}) = \sum_{j=1}^n \frac{\partial L}{\partial \dot{q}_j} \frac{\partial h^s}{\partial s} \Big|_{s=0}$$

is constant along any solution $q(t)$ of the E.L. equations.

Proof Let us consider the \mathbb{R} core $q \in \mathbb{R}$ (ex: do this for \mathbb{R}^n).

$$C(t, q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{\partial h^s}{\partial s} \Big|_{s=0}$$

Let us compute

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \frac{\partial h^s}{\partial s} \right) = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \frac{\partial h^s}{\partial s} + \frac{\partial}{\partial \dot{q}}$$

(for any s)

Let us take $q' = h^s(q)$, $\dot{q}' = \frac{dh^s}{dt}$.

[28]

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}'} \frac{\partial h^s}{\partial s} \right) = \underbrace{\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}'} \right)}_{= \frac{\partial L}{\partial q} \text{ (E.L.)}} \frac{\partial h^s}{\partial s} + \frac{\partial L}{\partial \dot{q}'} \frac{d}{dt} \frac{\partial h^s}{\partial s}$$

(for any s)

$$= \frac{\partial L}{\partial q} \text{ (E.L.)}$$

$$= \frac{\partial L}{\partial q} \frac{\partial h^s}{\partial s} + \frac{\partial L}{\partial \dot{q}'} \frac{\partial}{\partial s} \frac{dh^s}{dt} = \frac{\partial}{\partial s} L(t, \underbrace{q'}_{h^s}, \underbrace{\dot{q}'}_{\dot{h}^s}) =$$

$$\stackrel{\text{(symm)}}{=} \frac{\partial}{\partial s} L(t, q, \dot{q}) = 0.$$

Taking $s=0 \Rightarrow q' = q, \dot{q}' = \dot{q} \Rightarrow$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \frac{\partial h^s}{\partial s} \Big|_{s=0} \right) = \frac{dC(t, q, \dot{q})}{dt} = 0 \quad \square$$

Examples

Linear momentum

Consider a system of ~~point~~ material points

x_a, x_b, \dots

$\alpha = a, b, \dots$

Homogeneity of space in direction x_i means that

$$h^s : x_{1\alpha} \rightarrow x_{1\alpha} + s, \quad s \in \mathbb{R}$$

$$(x_{2\alpha}, x_{3\alpha} \rightarrow x_{2\alpha}, x_{3\alpha})$$

is a symmetry (does not change L).

$$\Rightarrow C_1 = \sum_{j=1}^n \frac{\partial L}{\partial \dot{q}_j} \frac{\partial h_n^s}{\partial s} \Big|_{s=0}$$

$$= \sum_{\alpha=a,b,\dots} \frac{\partial L}{\partial \dot{x}_{1\alpha}} \underbrace{\frac{\partial h_{1\alpha}^s}{\partial s}}_{m_\alpha \dot{x}_{1\alpha} \cdot 1} \Big|_{s=0} = \sum_{\alpha=a,b,\dots} m_\alpha \dot{x}_{1\alpha}$$

Similarly $C_i = \sum_{\alpha=a,b,\dots} m_\alpha \dot{x}_{i\alpha}, \quad i = 1, 2, 3.$

\Rightarrow we have the conservation of (linear) momentum defined as

$$P = \sum_{\alpha=a,b,\dots} m_\alpha \dot{x}_\alpha$$

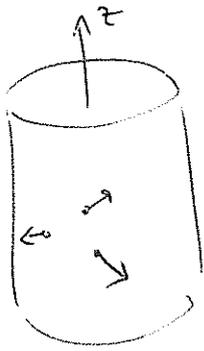
We define the center of mass as $R = \frac{m_a x_a + m_b x_b + \dots}{m_a + m_b + \dots}$

Then, $\dot{R} = \frac{P}{m_a + m_b + \dots} = \text{const.}$ for the isolated system (Galilean group!) includes space translations

Center of mass of an isolated system

moves uniformly along a straight line.

Breaking translational symmetries:



particles inside a tube with rigid walls $\langle \mathcal{H} \rangle$

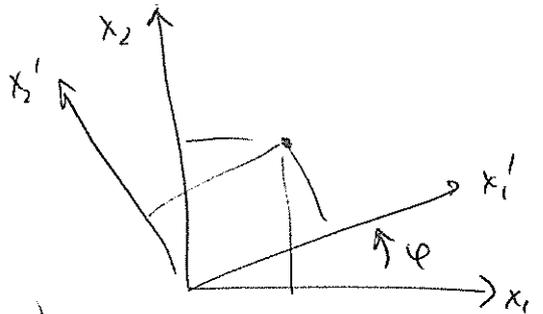
$$U_{\text{well}} = \begin{cases} 0 & \text{inside} \\ \infty & \text{outside.} \end{cases}$$

We still have the translation invariance of L along $axM z \Rightarrow P_z$ is conserved,

(P_x and P_y are not).

Angular momentum

Rotation (consider a plane x, y).



~~$\mathcal{H}(\mathbf{x}, \mathbf{p})$~~ $\varphi \in S^1$ parameter

$$h^\varphi: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos \varphi + x_2 \sin \varphi \\ -x_1 \sin \varphi + x_2 \cos \varphi \end{pmatrix}$$

more labels $\alpha = a, b, \dots$

If h^φ is a symmetry for all $\varphi \Rightarrow$ the conservation of

$$M_3 = - \sum_{\alpha = a, b, \dots} \left(\frac{\partial L}{\partial x_{1\alpha}} \frac{\partial h_{1\alpha}^\varphi}{\partial \varphi} + \frac{\partial L}{\partial x_{2\alpha}} \frac{\partial h_{2\alpha}^\varphi}{\partial \varphi} \right)_{\varphi=0}$$

$$= \sum_{\alpha=a,b,\dots} \left[m_{\alpha} \dot{x}_{1\alpha} (-x_{1\alpha} \sin \varphi + x_{2\alpha} \cos \varphi) + m_{\alpha} \dot{x}_{2\alpha} (-x_{1\alpha} \cos \varphi - x_{2\alpha} \sin \varphi) \right]_{\varphi=0}$$

$$= \sum_{\alpha=a,b,\dots} m_{\alpha} (x_{1\alpha} \dot{x}_{2\alpha} - x_{2\alpha} \dot{x}_{1\alpha})$$

Similar expressions follow from rotations in

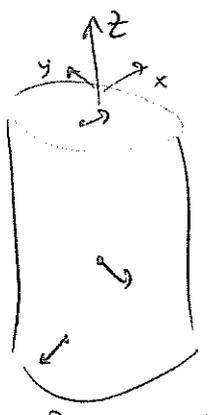
$$(x_2, x_3) \rightarrow M_1, \quad (x_3, x_1) \rightarrow M_2.$$

$$M = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \sum_{\alpha=a,b,\dots} m_{\alpha} \begin{pmatrix} x_{2\alpha} \dot{x}_{3\alpha} - x_{3\alpha} \dot{x}_{2\alpha} \\ x_{3\alpha} \dot{x}_{1\alpha} - x_{1\alpha} \dot{x}_{3\alpha} \\ x_{1\alpha} \dot{x}_{2\alpha} - x_{2\alpha} \dot{x}_{1\alpha} \end{pmatrix} = \sum_{\alpha=a,b,\dots} m_{\alpha} (\mathbf{x}_{\alpha} \times \dot{\mathbf{x}}_{\alpha})$$

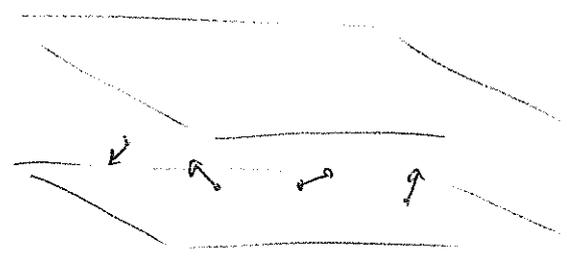
Angular momentum of an isolated system is conserved

in time \mathbb{E} as a consequence of space isotropy (rotations in \mathbb{E}) (Galil. group)

Breaking of rotational symmetry by rigid walls



$P_z = \text{const.}$
 $M_z = \text{const.}$



$P_x, P_y = \text{const.}$
 $M_z = \text{const.}$

Continuous symmetry:

~~Consider~~ one-parameter family of diffeomorphisms

$$h^s: M \rightarrow M$$

where $h^s(q)$ is smooth w.r.t. q and s ,

$$h^{s_1} \circ h^{s_2} = h^{s_1 + s_2}$$

$$L(t, q, \dot{q}) \equiv L(t, q', \dot{q}') \quad \text{for } q' = h(q), \quad \dot{q}' = \frac{\partial h}{\partial q} \dot{q}.$$

Noether's Theorem: continuous symmetry implies the

conservation law $C(t, q, \dot{q}) = \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \frac{\partial h_i^s}{\partial s} \Big|_{s=0} = \text{const}$

along any solution q of the E.L. equations.

Generalized Noether's Theorem

Space-Time transformations, $(q, t) \rightarrow (q', t')$

$$(q, t) \in M \times \mathbb{R}$$

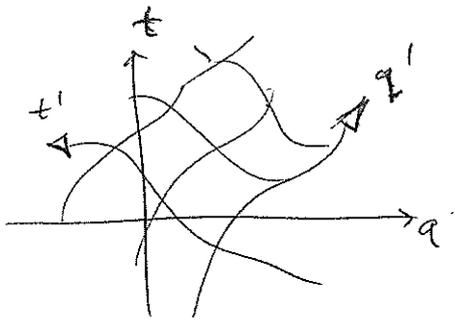
$$q' = h^s(q, t) = (h_1^s(q, t), \dots, h_n^s(q, t)); \quad h^s: M \times \mathbb{R} \rightarrow M.$$

$$t' = h_0^s(q, t); \quad h_0^s: M \times \mathbb{R} \rightarrow \mathbb{R}$$

~~The~~ $(h^s, h_0^s): M \times \mathbb{R} \rightarrow M \times \mathbb{R}$ is a uniparametric

group of diffeomorphisms:

$$(h^{s_1}, h_0^{s_1}) \circ (h^{s_2}, h_0^{s_2}) = (h^{s_1 + s_2}, h_0^{s_1 + s_2})$$



Can we define a symmetry and conservation laws in this case?

Action: $S = \int_{t_0}^t L(t, q, \dot{q}) dt \rightarrow min$

let us define new variables ~~Q, Q_0, \dot{Q}~~

$Q = q, Q_0 = t$ and new time τ

considering the dependences $Q(\tau), Q_0(\tau)$.

$$dt = \frac{dt}{d\tau} d\tau = \frac{dQ_0}{d\tau} d\tau = \dot{Q}_0 d\tau$$

$$\frac{dQ}{d\tau} = \frac{dq}{dt} \frac{dt}{d\tau} = \dot{q} \frac{dQ_0}{d\tau} \Rightarrow \dot{Q} = \frac{\dot{Q}_0}{Q_0} \cdot Q$$

Also $S = \int_{\tau_0}^{\tau_1} \underbrace{L(Q_0, Q, \frac{\dot{Q}}{\dot{Q}_0})}_{L_1} \dot{Q}_0 d\tau \rightarrow min$

New Lagrangian

$$L_1(Q_0, Q, \dot{Q}_0, \dot{Q}) = L(Q_0, Q, \frac{\dot{Q}}{\dot{Q}_0}) \dot{Q}_0$$

Change of variables

$$(h^s, h_0^s) : (q, q_0) \rightarrow (q', q_0') \quad M' \rightarrow M'$$

$$M' = M \times \mathbb{R}$$

Symmetry condition:

$$L_1(Q_0, Q, \dot{Q}_0, \dot{Q}) = L_1(Q'_0, Q', \dot{Q}'_0, \dot{Q}')$$

$$\text{with } \dot{Q}'_i = \sum_{j=0}^n \frac{\partial h_i^s}{\partial \dot{Q}_j} \dot{Q}_j, \quad Q'_0 = h_0^s(Q, Q_0) \\ Q' = h^s(Q, Q_0).$$

Noether's theorem: for any continuous symmetry

there is a conservation law ~~of a physical quantity~~

$$C(Q_0, Q, \dot{Q}_0, \dot{Q}) = \sum_{i=0}^n \frac{\partial L_1}{\partial \dot{Q}_i} \frac{\partial h_i^s}{\partial s} \Big|_{s=0}$$

$$= \sum_{i=0}^n \frac{\partial L_1}{\partial \dot{Q}_i} \frac{\partial h_i^s}{\partial s} \Big|_{s=0} = \text{const} \\ \text{(along solution)}$$

Let us write these relations in original variables.

$$L(t, q, \dot{q}) \frac{dt}{dt} = L(t', q', \dot{q}') \frac{dt'}{dt}$$

$$\Rightarrow \boxed{L(t, q, \dot{q}) = L(t', q', \dot{q}') \frac{dt'}{dt} \\ \text{with } t' = h_0^s(q, t), \quad q' = h^s(q, t) \\ \dot{q}' = \frac{\partial h^s}{\partial q} \dot{q} + \frac{\partial h^s}{\partial t}, \quad \frac{dt'}{dt} = \frac{\partial h_0^s}{\partial q} \dot{q} + \frac{\partial h_0^s}{\partial t}}$$

symmetry condition

for space-time transformations.

$$\frac{\partial L_1}{\partial \dot{q}_0} = \frac{\partial}{\partial \dot{q}_0} \left(L(q_0, q, \underbrace{\frac{\dot{q}}{\dot{q}_0}}_q) \dot{q}_0 \right) = \cancel{L}$$

$$= L + \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \left(-\frac{\dot{q}_i}{\dot{q}_0^2} \right) \dot{q}_0$$

$$= L - \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \underbrace{\frac{\dot{q}_i}{\dot{q}_0}}_{\dot{q}_i} = L - \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i$$

$$\frac{\partial L_1}{\partial \dot{q}_i} = \cancel{L} \frac{\partial}{\partial \dot{q}_i} \left(L(q_0, q, \frac{\dot{q}}{\dot{q}_0}) \dot{q}_0 \right) \quad i = 1, \dots, n$$

$$= \frac{\partial L}{\partial \dot{q}_i} \frac{1}{\dot{q}_0} \dot{q}_0 = \frac{\partial L}{\partial \dot{q}_i}$$

$$C = \frac{\partial L_1}{\partial \dot{q}_0} \frac{\partial h_0^s}{\partial s} \Big|_{s=0} + \sum_{i=1}^n \frac{\partial L_1}{\partial \dot{q}_i} \frac{\partial h_i^s}{\partial s} \Big|_{s=0}$$

$$C = \underbrace{\left(L = \sum_{i=0}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)}_{\text{new term.}} \frac{\partial h_0^s}{\partial s} \Big|_{s=0} + \underbrace{\sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \frac{\partial h_i^s}{\partial s} \Big|_{s=0}}_{\text{previous N. P.}}$$

Conservation of energy

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Homogeneity of time: $t \rightarrow t + t_0$ ($x \rightarrow x$)

$$t' = h_0^s(q, t) = t + s, \quad q' = h^s(q, t) = q, \quad s \in \mathbb{R}$$

Symmetry condition:

$$\begin{aligned} L(t, q, \dot{q}) &= L(t + s, q, \dot{q}) \\ &= L(t + s, q, \dot{q}). \end{aligned}$$

i.e. $L = L(q, \dot{q})$ independent of time.

Conserved quantity: ($\partial h_0^s / \partial s = 1, \partial h_i^s / \partial s = 0$)

$$\boxed{E = -C = \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L} \quad \text{energy}$$

For a system of material points:

$$L = T(q, \dot{q}) - U(q)$$

$$T(q, \dot{q}) = \frac{1}{2} \sum_{j, k=1}^n m_{jk}(q) \dot{q}_j \dot{q}_k, \quad m_{jk}(q) = m_{kj}(q).$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2} \sum_{k=1}^n m_{ik}(q) \dot{q}_k + \frac{1}{2} \sum_{j=1}^n m_{ji}(q) \dot{q}_j =$$

($j=i$) ($k=i$)

$$= \frac{1}{2} \sum_{j=1}^n m_{ij}(q) \dot{q}_j + \frac{1}{2} \sum_{j=1}^n m_{ij}(q) \dot{q}_j =$$

$$= \sum_{j=1}^n m_{ij} \dot{q}_j$$

$$\Rightarrow \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum_{i,j=1}^n m_{ij}(q) \dot{q}_i \dot{q}_j = 2T.$$

$$E = 2T - L = 2T - (T - U) = T + U$$

$$\boxed{E = T + U}$$

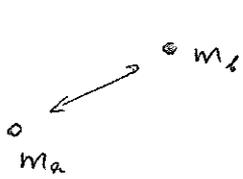
is conserved along

solutions of the e.l. eqs

if $L = L(q, \dot{q}) \leftarrow$ ~~also~~ homogeneity of time

Kepler problem (2-body problem)

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$$U(r) = -\frac{k}{r}$$

$$(k = G m_a m_b)$$

$$r = \|x_a - x_b\|$$

$$\Rightarrow \boxed{\ddot{x} = -\frac{k}{m} \frac{x}{\|x\|^3}}$$

eq. for relative motion

$$x = x_a - x_b$$

~~Relative~~

It can be written as S.L. eq. for the Lagrangian

$$\boxed{L = \frac{m \| \dot{x} \|^2}{2} + \frac{k}{\|x\|}}$$

(full Lagrangian is obtained adding $\frac{M}{2} \| \dot{R} \|^2$)

Con Continuous symmetries:

space translations \rightarrow ~~no~~ $\|x\| \neq \|x+x_0\|$ no!

space rotations of x - yes! $\Rightarrow M = m \times x \times \dot{x}$
is conserved
(angular momentum)

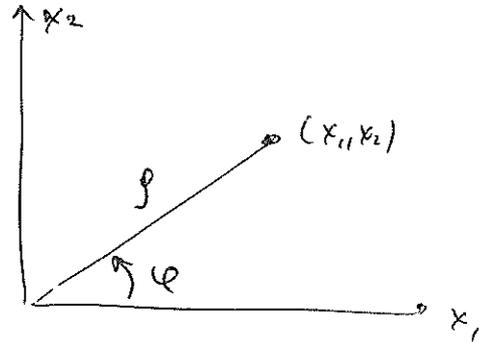
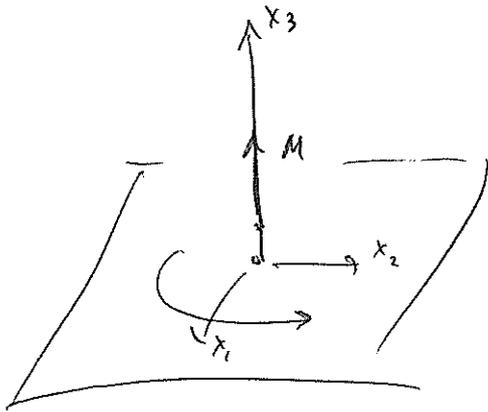
time translations - yes! $\Rightarrow E = \frac{m \|\dot{x}\|^2}{2} - \frac{k}{\|x\|}$

is conserved

(energy)

Since $\dot{z} \perp M = \text{const} \Rightarrow$ motion happens in a plane.

We set x_1, x_2 in the plane of motion and $Ox_3 \parallel M$.



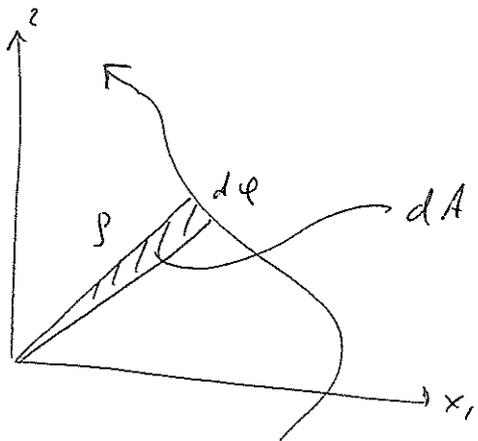
Polar coordinates : $x_1 = \rho \cos \varphi$, $x_2 = \rho \sin \varphi$,
(new - generalized coordinates).

$$\dot{x}_1 = \dot{\rho} \cos \varphi - \rho \dot{\varphi} \sin \varphi, \quad \dot{x}_2 = \dot{\rho} \sin \varphi + \rho \dot{\varphi} \cos \varphi.$$

$$M_3 = m(x_1 \dot{x}_2 - x_2 \dot{x}_1) = \text{const.}$$

$$\begin{aligned} \frac{M_3}{m} &= x_1 \dot{x}_2 - x_2 \dot{x}_1 = \rho \cos \varphi (\dot{\rho} \sin \varphi + \rho \dot{\varphi} \cos \varphi) - \\ &\quad - \rho \sin \varphi (\dot{\rho} \cos \varphi - \rho \dot{\varphi} \sin \varphi) \\ &= \rho \dot{\varphi} \cos^2 \varphi + \rho \dot{\varphi}^2 \sin^2 \varphi = \rho^2 \dot{\varphi} \end{aligned}$$

or $\boxed{\frac{d\varphi}{dt} = \frac{M_3}{m\rho^2}}$



$$\frac{dA}{dt} = \frac{\rho^2}{2} \frac{d\varphi}{dt} = \frac{\rho^2}{2} \frac{M_3}{m\rho^2} = \frac{M_3}{2m} = \text{const}$$

Kepler's law of equal areas: areas swept are equal for equal time intervals

Energy: $E = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{\rho} =$

$$= \frac{m}{2} (\dot{\rho}^2 \cos^2 \varphi - 2\dot{\rho}\rho\dot{\varphi} \cos \varphi \sin \varphi + \rho^2 \dot{\varphi}^2 \sin^2 \varphi + \dot{\rho}^2 \sin^2 \varphi + 2\dot{\rho}\rho\dot{\varphi} \sin \varphi \cos \varphi + \rho^2 \dot{\varphi}^2 \cos^2 \varphi) - \frac{k}{\rho}$$

$$= \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2) - \frac{k}{\rho} =$$

$$= \frac{m\dot{\rho}^2}{2} + \frac{m\rho^2}{2} \left(\frac{M_3}{m\rho^2} \right)^2 - \frac{k}{\rho}$$

$$= \frac{m\dot{\rho}^2}{2} + \frac{M_3^2}{2m\rho^2} - \frac{k}{\rho}$$

$$E = \frac{m\dot{\rho}^2}{2} + U_{\text{eff}}(\rho), \quad U_{\text{eff}} = \frac{M_3^2}{2m\rho^2} - \frac{k}{\rho}$$

effective potential energy.

$$\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2}{m} (E - U_{\text{eff}}(\rho))}$$

~~$$dt = \frac{d\rho}{\sqrt{\frac{2}{m} (E - U_{\text{eff}}(\rho))}} = dt$$~~

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{\frac{2}{m} (E - U_{\text{eff}}(\rho))}} = t - t_0.$$

implicit solution.

Form of trajectories

$\rho(\varphi)$:

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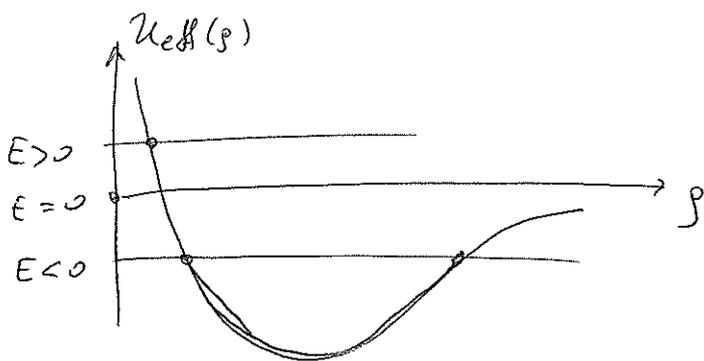
$$d\varphi = \frac{M_3}{m \rho^2} dt = \frac{M_3}{m \rho^2} \frac{d\rho}{\sqrt{\frac{2}{m}(E - U_{\text{eff}}(\rho))}}$$

$$\frac{d\rho}{d\varphi} = \frac{m \rho^2 \sqrt{\frac{2}{m}(E - U_{\text{eff}}(\rho))}}{M_3} = \frac{\rho^2}{P} \sqrt{e^2 - 1 - \frac{P^2}{\rho^2} - \frac{2P}{\rho}}$$

(ex)

$$P = \frac{M_3^2}{mk}, \quad e = \sqrt{1 + \frac{2EM_3^2}{mk^2}}$$

Solution: $\rho = \frac{P}{1 + e \cos \varphi}$ (ex).



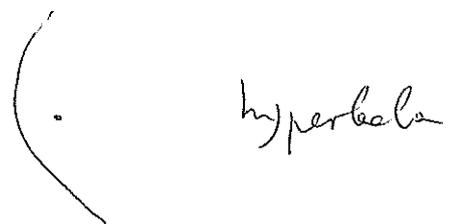
$$E < 0 \Rightarrow e < 1$$



$$E = 0 \Rightarrow e = 0$$



$$E > 0 \Rightarrow e > 0$$



e - eccentricity.

Exercises 2Consider 1D space $x \in \mathbb{R}$.Space-time distance between points (x_a, t_a) and (x_b, t_b)

$$\Delta S^2 = c^2 \Delta t^2 - \Delta x^2, \quad \Delta t = t_b - t_a$$

$$\uparrow \quad \Delta x = x_b - x_a$$

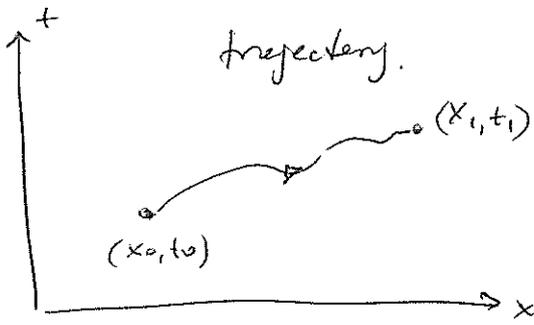
speed of light

This is the only form of metric (up to a change of coordinates), which is given by a quadratic form, (not positive definite), not degenerate and distinguishes between the space and time variables. It is called the Minkowski metric.

All linear transformations of space-time that preserve the distance ΔS^2 (isometries) form a Poincaré group. For example: space-time translations.

We want to define a theory which is invariant with respect to the Poincaré group (instead of the Galilean group).

Variational principle For one material point:



Invariant quantity along the trajectory is

$$ds^2 = c^2 dt^2 - dx^2$$

Action: (one material point)

$$S = a \int_{\text{initial point}}^{\text{final point}} ds =$$

$$= a \int_{t_0}^{t_1} \sqrt{c^2 dt^2 - dx^2}$$

(unknown prefactor)

$$= a \int_{t_0}^{t_1} \sqrt{c^2 - \left(\frac{dx}{dt}\right)^2} dt$$

$$= a \int_{t_0}^{t_1} \sqrt{c^2 - v^2} dt$$

$$\Rightarrow S = \int_{t_0}^{t_1} L dt, \quad L = a \sqrt{c^2 - v^2}, \quad v = \frac{dx}{dt}$$

Obs ~~L vs~~ L vs real iff $|v| \leq c$ (no motion with the speed above light speed).

Problem (a): Consider all linear transformations of space-time:

$$(x, t) \rightarrow (x', t')$$

$$\begin{cases} x' = ax + bt + c \\ t' = dx + et + f \end{cases}$$

$$a, b, c, d, e, f \in \mathbb{R}.$$

Find all transformations that preserve the distance

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 = c^2 (\Delta t')^2 - (\Delta x')^2.$$

This is the Poincaré group.

Problem (b) Define the transformation of velocity

$$v = \frac{dx}{dt} \rightarrow v' = \frac{dx'}{dt'}$$

under the action of transformations from the Poincaré group. Show that

if $|v| = c$, then $|v'| = c$ (the light speed is invariant).

Problem (c) Consider the case ~~for~~ $|v| \ll c$

(small speeds of motion). Show that the Poincaré group is approximately the Galilean group.

(Obs: in $x \in \mathbb{R}$ there are no rotations).

Problem (d) For small speeds $|v| \ll c$ compare

$$L = a \sqrt{c^2 - v^2} \approx L_{\text{classical}} = \frac{mv^2}{2} + \text{const.}$$

Find the unknown constant a .

Problem (e) Using ~~the~~ Noether's theorem define the linear momentum P (space translations)

and the energy E (time translations)

Show that $E = mc^2$ for the material point at rest, $v = 0$.



Hamiltonian mechanics

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Motivation : asymmetry in q coordinates vs. speeds.

Can we use a more "symmetric" formulation?

Consider the Lagrangian $L(t, q, \dot{q})$, $q \in \mathbb{R}^n$.

We define the generalized momenta as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n.$$

Recall (last lecture) that for the isolated system of material points

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^n m_{ij} \dot{q}_j$$

Hence
$$p_i = \sum_{j=1}^n m_{ij} \dot{q}_j \quad \text{or}$$

$$\begin{array}{ccc} \mathbb{R}^n & \begin{array}{c} \mathbb{R}^n \\ [m_{ij}] \\ n \times n \text{ matrix} \end{array} & \mathbb{R}^n \\ p & = & M(q) \dot{q} \Rightarrow \dot{q} = M^{-1}(q) p. \end{array}$$

So, we can express (in general)

$$\dot{q} = \dot{q}(q, p, t).$$

We define a Hamiltonian function H as

$$H(q, p, t) = \left(\sum_{i=1}^n \dot{q}_i p_i - L \right)_{\dot{q} = \dot{q}(q, p, t)}$$

where we substitute speeds as $\dot{q}(q, p, t)$.

Obs This has the same form as the energy but expressed in terms of (q, p, t) .

Let us compute derivatives of H :

$$\begin{aligned} \frac{\partial H}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\sum_{i=1}^n \dot{q}_i p_i - L(t, q, \dot{q}) \right)_{\dot{q}(q, p, t)} \\ &= \sum_{i=1}^n \frac{\partial \dot{q}_i}{\partial q_j} p_i - \frac{\partial L}{\partial q_j} - \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial q_j} = - \frac{\partial L}{\partial q_j} = \end{aligned}$$

E.L. eq. $= - \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{p_j} = - \dot{p}_j$

E.L. eqs $\Rightarrow \dot{p}_j = - \frac{\partial H}{\partial q_j}$

$$\begin{aligned} \frac{\partial H}{\partial p_j} &= \frac{\partial}{\partial p_j} \left(\sum_{i=1}^n \dot{q}_i p_i - L(t, q, \dot{q}) \right)_{\dot{q}(q, p, t)} \\ &= \dot{q}_j + \sum_{i=1}^n \frac{\partial \dot{q}_i}{\partial p_j} p_i - \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial p_j} = \dot{q}_j \end{aligned}$$

Hamilton's equations:

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$$\dot{p}_i = - \frac{\partial H}{\partial q_i}, \quad \dot{q}_i = + \frac{\partial H}{\partial p_i}, \quad i = 1, \dots, n$$

Let the Hamiltonian $H(t, q, p)$.

Now the ~~descriptions~~ equations of motion are symmetric w.r.t. ~~the~~ coordinates:

the change $(q, p, H) \rightarrow (p, q, -H)$ does not change the equations of motion. \Rightarrow

Symplectic geometry etc

When $H = H(q, p)$ does not depend on time:

$$H(q, p) = \sum_{i=1}^n \dot{q}_i p_i - L = \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = E = \text{const.}$$

i.e. the Hamiltonian has the value of energy and it is conserved along the solutions.

Indeed:

$$\begin{aligned} \frac{dH(q, p)}{dt} &= \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) = \\ &= \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial \dot{q}_i} + \frac{\partial H}{\partial p_i} \left(- \frac{\partial H}{\partial q_i} \right) \right) = 0. \end{aligned}$$

"Real-world" models

We distinguish the coordinates ~~of the system~~ $q \in \mathbb{R}^n$ (or M) used to describe the system, and coordinates $Q \in \mathbb{R}^N$ for the "rest of the world".

$$L = T - U$$

$$T = T_S(q, \dot{q}) + ~~T_R(Q, \dot{Q})~~ T_R(Q, \dot{Q})$$

$$U = U_S(q) + U_R(Q) + U_I(q, Q).$$

Obs Additivity of T ~~is~~ follows ~~for~~ when q and Q correspond to different objects (sets of material points) or to the center of mass (macroscopic variable) vs. relative motion (microscopic variables).

$U_I(q, Q)$ is the interaction term, which is usually non zero.

E.L. equation for our system:

$$\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$L = \underbrace{T_s(q, \dot{q}) - U_s(q)}_{L_s} - \underbrace{U_I(q, Q)}_{\text{interaction}} + \underbrace{T_R(Q, \dot{Q}) - U_R(Q)}_{\text{not important}}$$

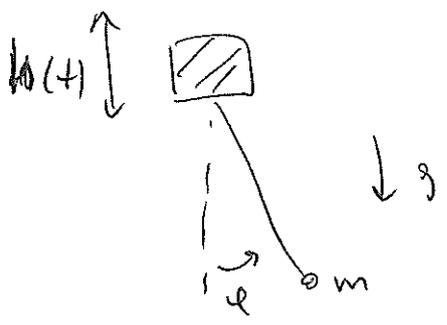
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L_s}{\partial \dot{q}_i} - \frac{\partial L_s}{\partial q_i} = 0.$$

~~$$\frac{d}{dt} \frac{\partial T_s}{\partial \dot{q}_i} - \frac{\partial L_s}{\partial q_i} = 0$$~~

$$\underbrace{\frac{d}{dt} \frac{\partial L_s}{\partial \dot{q}_i} - \frac{\partial L_s}{\partial q_i}}_{\text{function of } q, \dot{q}.} = - \frac{\partial U_I(q, Q)}{\partial q} = \underbrace{F_{\text{ext}}(q, Q)}_{\text{external forces.}}$$

The model is defined if we know $F_{\text{ext}}(q, Q)$.

For example, if we know $Q(t)$ exactly:



$q = \varphi$, ~~$Q = h(t)$~~ A position of ~~the~~ ~~center~~

$Q \rightarrow h(t)$ and position relative to the center of the Earth (g).

Work of a force (e.g. external force)

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$$dA = F_{\text{ext}} \cdot dq \quad (\text{scalar product.}) \quad dq = \dot{q} dt.$$

$$A = \int_{t_0}^{t_1} F_{\text{ext}} \cdot dq = \int_{t_0}^{t_1} (F_{\text{ext}} \cdot \dot{q}) dt \quad \text{work at time interval } [t_0, t_1].$$

Virtual work (arbitrary ~~var~~ small variation δq)

$$\begin{aligned} \delta A &= f_{\text{ext}} \cdot \delta q = - \frac{\partial \mathcal{U}_{\text{I}}(q, Q)}{\partial q} \cdot \delta q \approx \\ &\approx - (\mathcal{U}_{\text{I}}(q + \delta q, Q) - \mathcal{U}_{\text{I}}(q, Q)) \end{aligned}$$

If you use different ~~coordinates~~ generalized coordinates $(q, Q) \rightarrow (q', Q') \Rightarrow \mathcal{U}_{\text{I}}(q, Q) = \mathcal{U}'_{\text{I}}(q', Q')$ are equal at corresponding configurations.

$$\begin{aligned} \Rightarrow \delta A &= - (\mathcal{U}'_{\text{I}}(q' + \delta q', Q') - \mathcal{U}'_{\text{I}}(q', Q')) \\ &\approx - \frac{\partial \mathcal{U}'_{\text{I}}}{\partial q'} \delta q' = \text{CAL. } \delta A' \end{aligned}$$

Virtual work is the same for different generalized coordinates:

$$F_{\text{ext} \cdot q} \delta q = F'_{\text{ext} \cdot q'} \delta q' \quad (\text{D'Alembert's principle})$$

Obs Useful for deriving expression for F_{ext} : 52

Use a model for F_{ext} in some convenient coordinates. When changing coordinates, find F_{ext}' by computing its virtual work.

Lecture 6 ~~Obs~~ Recall U, F_{diss} .

Dissipative Forces:

q are macroscopic variables (centers of masses)

Q are microscopic variables (relative ~~motion~~ positions of particles)

e.g. friction, macroscopic motion \rightarrow ~~heat~~ micro, (heat)

$F_{diss} = 0$ if $\dot{q} = 0 \Rightarrow$ for small \dot{q}

$F_{diss} \approx -D(q) \dot{q}$, where $D(q)$ is an $n \times n$ matrix of dissipation coefficients.

Statistical mechanics $\Rightarrow D = D^T$ is symmetric

and $D > 0$ is positive definite ($\dot{q}^T D \dot{q} > 0$ if $\dot{q} \neq 0$).

$$\frac{d}{dt} \frac{\partial L_s}{\partial \dot{q}_i} - \frac{\partial L_s}{\partial q_i} = - (D(q) \dot{q})_i$$

$$E = \sum_{i=1}^n \frac{\partial L_s}{\partial \dot{q}_i} \dot{q}_i - L_s \quad L_s = L_s(q, \dot{q})$$

$$\frac{dE}{dt} = \frac{d}{dt} \left(\sum_{i=1}^n \frac{\partial L_s}{\partial \dot{q}_i} \dot{q}_i - L_s \right) =$$

$$= \sum_{i=1}^n \frac{d}{dt} \left(\frac{\partial L_s}{\partial \dot{q}_i} \right) \dot{q}_i + \cancel{\frac{\partial L_s}{\partial \dot{q}_i} \ddot{q}_i} - \cancel{\frac{\partial L_s}{\partial \dot{q}_i} \dot{q}_i} - \cancel{\frac{\partial L_s}{\partial \dot{q}_i} \ddot{q}_i}$$

$$= \sum_{i=1}^n \left(\frac{d}{dt} \frac{\partial L_s}{\partial \dot{q}_i} - \frac{\partial L_s}{\partial q_i} \right) \dot{q}_i = - \sum_{i=1}^n (D(q) \dot{q})_i \dot{q}_i =$$

$$= - \dot{q}^T D(q) \dot{q} \leq 0 \quad (< 0 \text{ if } \dot{q} \neq 0)$$

$$\begin{matrix} \dot{q}^T & & \dot{q} \\ \hline (1 \times n) & \boxed{D(q)} & (n \times 1) \\ & (n \times n) & \end{matrix}$$

Dissipation!

Recall

Rayleigh dissipation function: $f(q, \dot{q}) = \frac{1}{2} \dot{q}^T D(q) \dot{q}$.

Ex: viscous dissipative force \Rightarrow

$$- (D(q) \dot{q})_i = - \frac{\partial f}{\partial \dot{q}_i}, \quad i = 1, \dots, n$$

$$\frac{dE}{dt} = - 2f$$