Oscillations in non-autonomous systems

Linear systems with periodic forcing.

$f(t) = f_0 \cos \Omega t$

Harmonic oscillator. Energy is not conserved (time dependence)

$m\ddot{x} + kx = f_0 \cos \Omega t \implies \ddot{x} + \omega^2 x = a \cos \Omega t$

$\omega^2 = \frac{k}{m}, \quad a = \frac{f_0}{m}.$

Prop A general solution is obtained as

$$x(t) = x_p(t) + x_a(t)$$

where $x_p(t)$ is (any) particular solution and $x_a(t)$ is a general solution of the autonomous system with $a=0$.

Proof If $x_1(t)$ and $x_2(t)$ are solutions, then

$\ddot{x}_i + \omega^2 x_i = a \cos \Omega t, \quad i = 1,2$

$\Rightarrow \quad (x_1 - x_2)^{\cdot\cdot} + \omega^2 (x_1 - x_2) = 0.$

for Any $x(t) = x_p(t) + x_a(t)$, we have

$$\ddot{x} + \omega^2 x = \underbrace{\ddot{x}_p + \omega^2 x_p}_{a \cos \Omega t} + \underbrace{\ddot{x}_a + \omega^2 x_a}_{0} = a \cos \Omega t.$$

$\square$

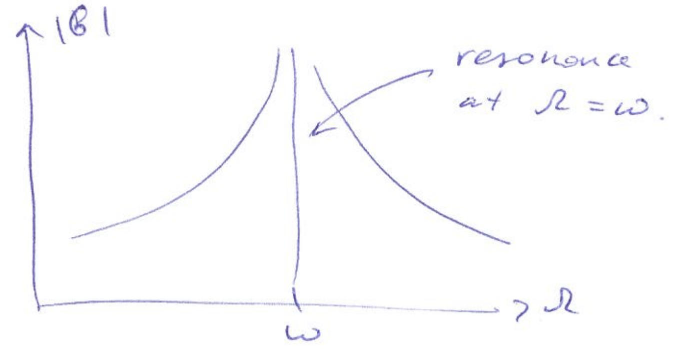
General solution: $x_a(t) = c \cos(\omega t + \varphi)$

for any $c \in \mathbb{R}$, $\varphi \in S^1$

Particular solution: $x_p = b\cos\Omega t$

$$\Rightarrow \quad -b\Omega^2\cos\Omega t + b\omega^2\cos\Omega t = a\cos\Omega t$$

$$\Rightarrow \quad b = \frac{a}{\omega^2-\Omega^2}$$



Oscillations near the resonance from the rest:

$$x(0)=0, \quad \dot{x}(0)=0.$$

$$x(t) = c\cos(\omega t+\varphi) + b\cos\Omega t,$$

$$\Omega = \omega+\varepsilon, \qquad \underline{\varepsilon \ll \omega.}$$

~~resonance~~

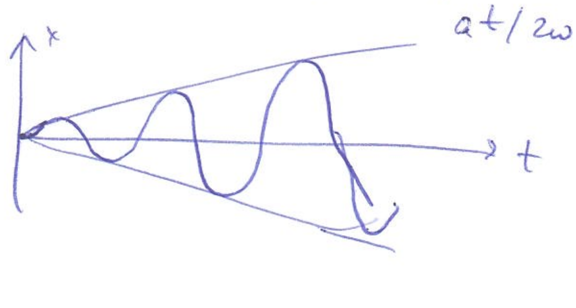
$$b\cos\Omega t = \frac{a}{\omega^2-\Omega^2}\cos\Omega t = \frac{a}{\omega^2-(\omega+\varepsilon)^2}\cos((\omega+\varepsilon)t)$$

$$= -\frac{a}{\underbrace{2\omega\varepsilon+\varepsilon^2}_{O(\varepsilon)}}\left(\cos\omega t\underbrace{\cos\varepsilon t}_{1+o(\varepsilon)} - \sin\omega t\underbrace{\sin\varepsilon t}_{\varepsilon t+o(\varepsilon)}\right)$$

$$\approx -\frac{a}{2\omega\varepsilon}\left(\underbrace{\cos\omega t}_{} - \varepsilon t\sin\omega t\right)$$

↳ can be merged with $c\cos(\omega t+\varphi)$

$$\Rightarrow \quad x(t) \approx \frac{a}{2\omega}\, t\sin\omega t$$



Valid for $\varepsilon t \ll 1 \Rightarrow t \ll \frac{1}{\varepsilon}$

For large times:

$$x(t) = \underbrace{c\cos(\omega t+\varphi)}_{x_a} + \underbrace{b\cos((\omega+\varepsilon)t)}_{x_p}$$

$$= \mathrm{Re}\left[c\, e^{i(\omega t+\varphi)} + b\, e^{i(\omega+\varepsilon)t}\right] = \mathrm{Re}\Big[\underbrace{c e^{i\varphi} + b e^{i\varepsilon t}}_{A(t)\, e^{i\psi(t)}}\Big] e^{i\omega t}$$

$$= A(t)\, \mathrm{Re}\, e^{i(\omega t+\psi(t))} = \underline{A(t)\cos(\omega t + \psi(t))}$$

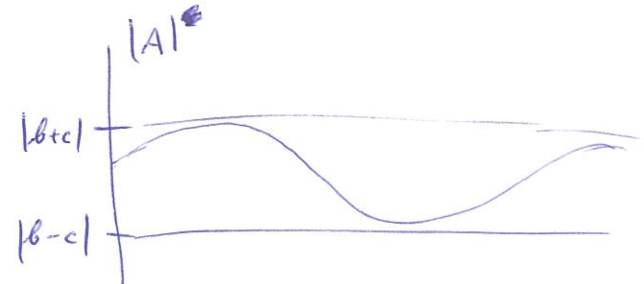
Functions $A(t)$ and $\psi(t)$ are slow (depend on εt).

$$|A|^2 = |c e^{i\varphi} + b e^{i\varepsilon t}|^2 = (c\cos\varphi + b\cos\varepsilon t)^2 + (c\sin\varphi + b\sin\varepsilon t)^2$$

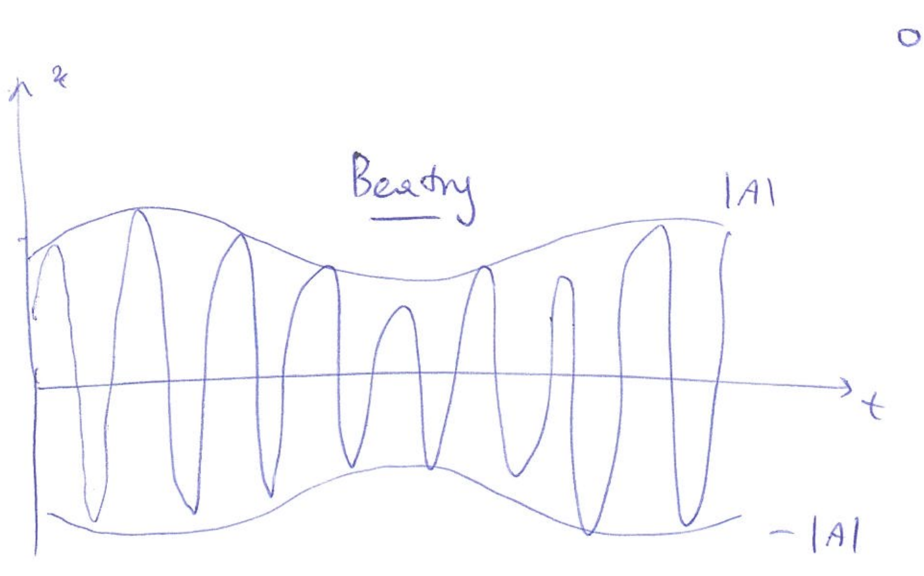
$$= c^2+b^2+2bc(\cos\varphi\cos\varepsilon t + \sin\varphi\sin\varepsilon t) =$$

$$= c^2+b^2+2bc\cos(\varepsilon t-\varphi).$$

$$\Rightarrow \quad |b-c| \le |A| \le |b+c|$$



Final ~~solution~~ form of solution:



Oscillator with dissipation:

$$m\ddot{x} + d\dot{x} + \omega^2 x = \cancel{f_0} \cos \Omega t \qquad /m$$

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = a\cos\Omega t\,, \qquad \gamma = \frac{d}{2m}.$$

General solution: $\underbrace{x_p(t)}_{\text{particular}} + \underbrace{x_a(t)}_{\text{general autonomous } (a=0)}$

$\underline{x_a(t)} = c\,e^{\lambda t} \quad (a=0)$

$$(\lambda^2 + 2\gamma\lambda + \omega^2)c = 0 \Rightarrow \lambda = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

$\mathrm{Re}\,\lambda < 0 \Rightarrow x_a(t) \to 0$ as $t \to \infty$. (assympt. stability)

$\underline{x_p(t)}$

We look for a complex solution of

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = a e^{i\Omega t} \quad \Big|\ \mathrm{Re} \to \text{our equation}$$

$$x = B e^{i\Omega t} \Rightarrow -\Omega^2 B e^{i\Omega t} + 2\gamma i\Omega B e^{i\Omega t} + \omega^2 e^{i\Omega t} = a e^{i\Omega t}$$

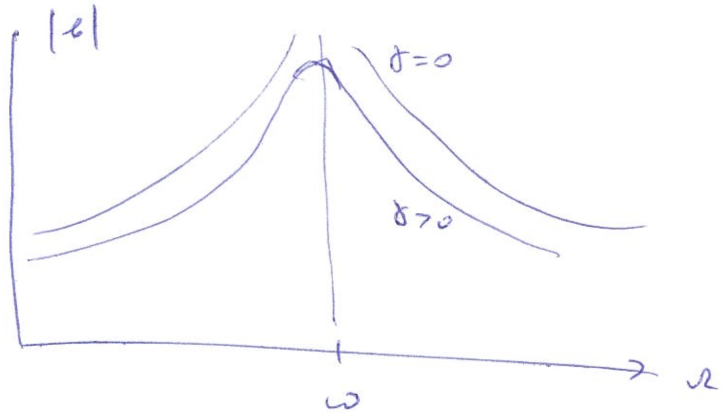
$$(-\Omega^2 + 2i\gamma\Omega + \omega^2) B = a$$

$$B = \frac{a}{\omega^2 - \Omega^2 + 2i\gamma\Omega} = b e^{i\xi}.$$

$$\Rightarrow x_p(t) = \mathrm{Re}\left(b e^{i\xi} e^{i\Omega t}\right) = b(\cos \Omega t + \xi)$$

At large times $x(t) \simeq x_p(t) = b\cos(\Omega t + \xi)$.

phase shift.



$$|b| = \frac{|a|}{\sqrt{(\omega^2-\Omega^2)^2 + 4\gamma^2\Omega^2}}$$

System with n degrees of freedom.

Small forced oscillations (linearized system)

$$M\ddot{x} + Cx = f_0 \cos\Omega t \qquad M = M^T > 0,\ C = C^T.$$

$$x = q - q_0 \in \mathbb{R}^n, \quad f_0 \in \mathbb{R}^n,$$

Consider a stable system: $C > 0$.

General solution $x(t) = \underbrace{x_p(t)}_{\text{particular}} + \underbrace{x_a(t)}_{\text{general autonomous } (f_0 = 0)}$

$$x_a(t) = \sum_{i=1}^{n} c_i u_i \cos(\omega_i t + \varphi_i), \qquad u_i \in \mathbb{R}^n$$

$$(-\omega_i^2 M + C)u_i = 0$$

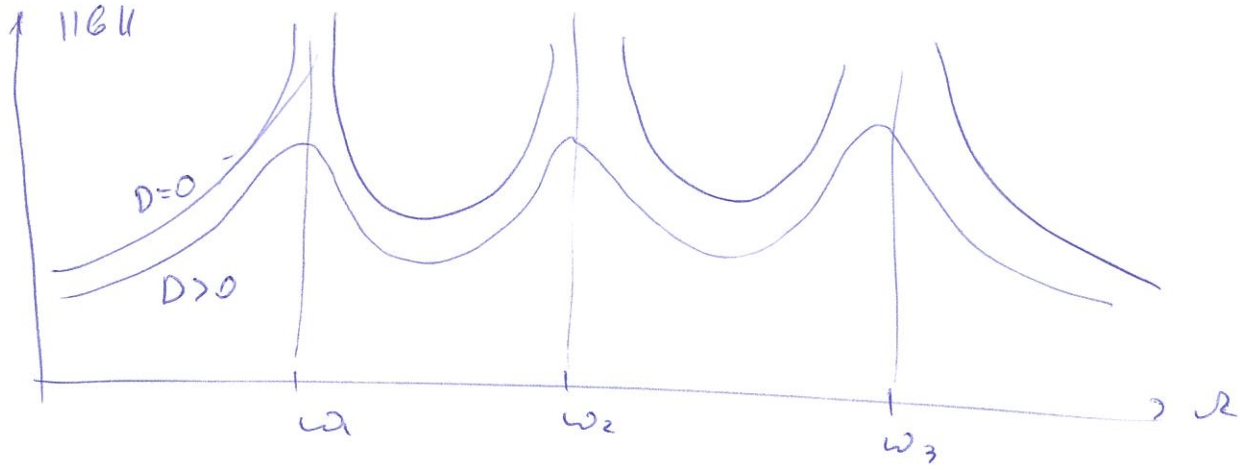
$$(\lambda^2 = -\omega^2)$$

Assume $x_p = b\cos\Omega t$, $b \in \mathbb{R}^n$

$$[-\Omega^2 M + C]\, b \cancel{\cos\Omega t} = f_0 \cancel{\cos\Omega t}$$

$$b = [-\Omega^2 M + C]^{-1} f_0 \quad \text{if} \quad \underbrace{\det(-\Omega^2 M + C) \neq 0}_{\Omega^2 \neq \omega_i^2 \text{ for all } i.}$$

Resonances: $\Omega^2 \approx \omega_i^2$.



System with dissipation: $M\ddot{x} + D\dot{x} + Cx = f_0 \cos \Omega t$

$$D = D^T > 0.$$

Complex form: $M\ddot{x} + D\dot{x} + Cx = f_0 e^{i\Omega t}$, $x = b e^{i\Omega t}$, $b \in \mathbb{C}^n$ } Real solution is $\mathrm{Re}(b e^{i\Omega t})$.

$$\Rightarrow \quad [-\Omega^2 M + i\Omega D + C]\, b\, \cancel{e^{i\Omega t}} = f_0 \cancel{e^{i\Omega t}}$$

Note that $\det[-\Omega^2 M + i\Omega D + C] \neq 0$ for any $\Omega \in \mathbb{R}$ (we proved that $\mathrm{Re}\,\lambda < 0$ for any $\lambda = i\Omega$ ~~[illegible]~~ satisfying the charact. eq. $\det[\lambda^2 M + \lambda D + C] = 0$)

$$\Rightarrow \quad b = [-\Omega^2 M + i\Omega D + C]^{-1} f_0$$

~~As~~ General solution is $x(t) = x_p(t) + x_a(t)$ and $x_a(t) \to 0$ as $t \to \infty$ (assymptotic stability).

Hence $x(t) \to x_p(t) = \mathrm{Re}(b e^{i\Omega t}) =$

$$= \mathrm{Re}\, b \cos \Omega t + \mathrm{Im}\, b \sin \Omega t.$$