Explosive ripple instability due to incipient wave breaking

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Introduction and Motivation

modeling a breaker in the surf zone

Basic mechanisms underlying wave breaking

Linear approximation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Weak nonlinearity (inviscid Burgers equation):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + bu\frac{\partial u}{\partial x} = 0$$

Weak nonlinearity and dispersion (KdV equation):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + bu\frac{\partial u}{\partial x} + c\frac{\partial^3 u}{\partial x^3} = 0$$

(cnoidal waves)

Full model

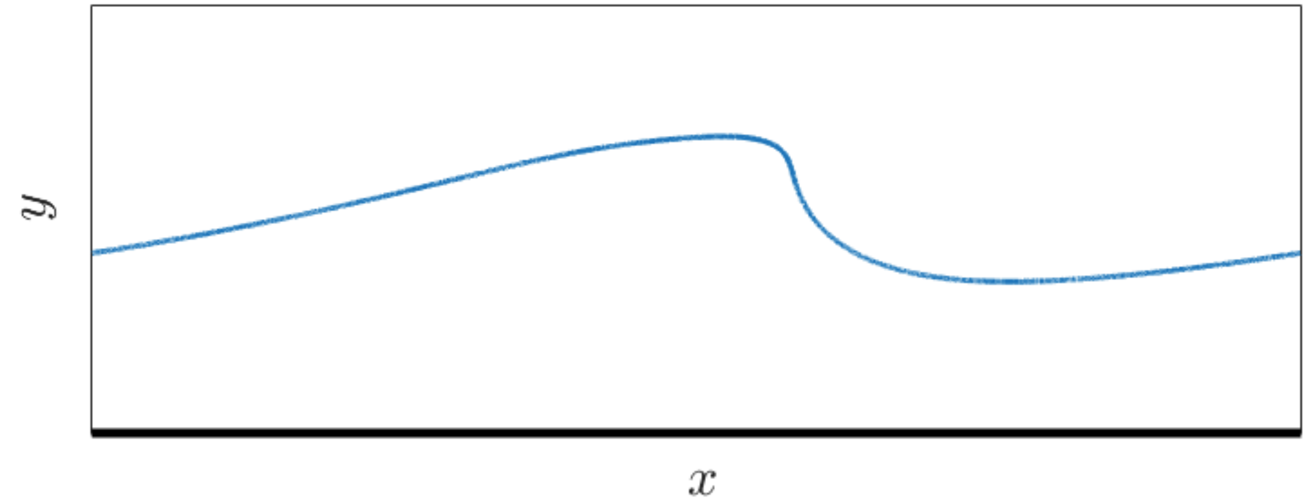
Incompressible Euler equations: 2D (x, y) potential ideal flow for water speed $\mathbf{v} = (u, v)$

Typical spatial scales [m] >> viscous and capillary scales [mm]

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p/\rho + \mathbf{g}, \qquad \mathrm{div}\, \mathbf{v} = 0$$

Boundary conditions at rigid bottom

$$y = -H : \quad v = 0$$



Boundary conditions at free surface

$$y = F(x, t) : \quad v = F_t + F_x u, \quad p = P_{atm}$$

Complex potential

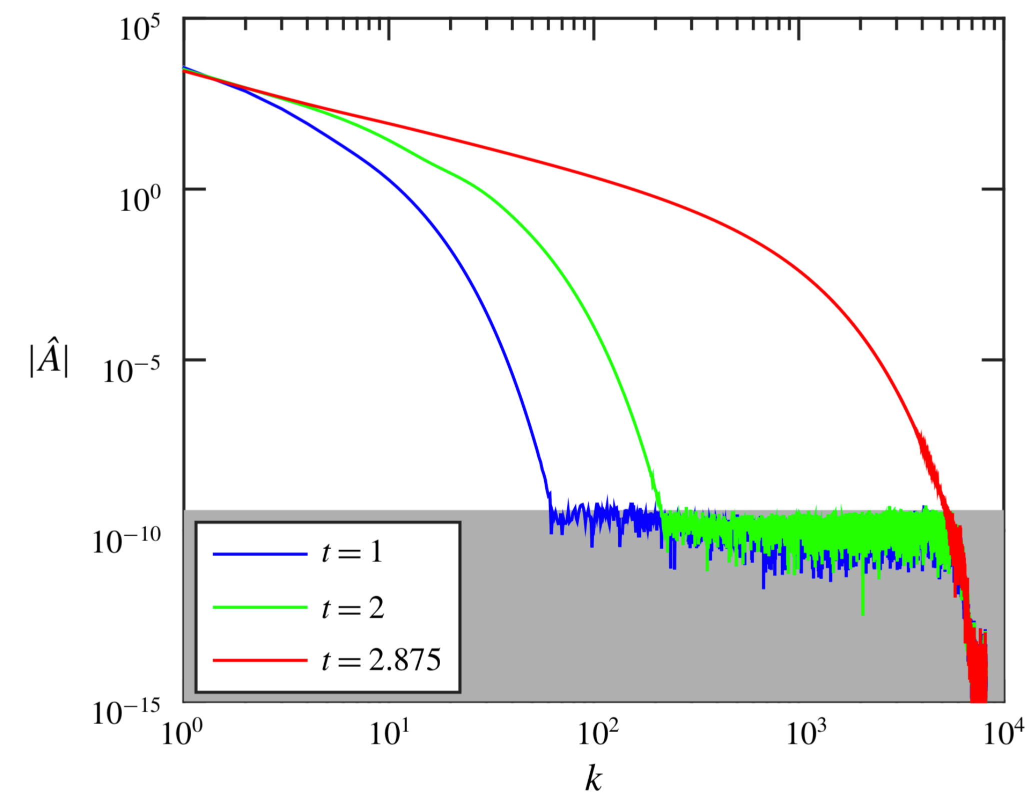
$$\Phi(z) = \varphi + i\psi, \quad z = x + iy$$ (holomorphic function)

$$u = \varphi_x = \psi_y, \qquad v = \varphi_y = -\psi_x$$

Numerical model

Dimensionless units (unit density, gravity and depth).

2π-periodic boundary condition in horizontal direction.



Conformal mapping: $z(\zeta, t)$ with $\zeta = \xi + i\eta$

from a horizontal strip $-K \leqslant \eta \leqslant 0$
to the fluid domain.

Free surface parametrization: $x + iy = z(\xi, t)$

Using complex analysis equation are reduced to 1D form
(Dyachenko et al. 1996a; Zakharov et al. 2002) :

$$K_t = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathsf{R}\hat{\varphi}_\xi}{|\hat{z}_\xi|^2}\, d\xi,$$

$$\hat{A}_t = \left[(\mathsf{R}\hat{A}_\xi) - \left(1+\hat{A}_\xi\right)\mathsf{T}\right]\frac{\mathsf{R}\hat{\varphi}_\xi}{|\hat{z}_\xi|^2},$$

$$\hat{\varphi}_t = -\hat{\varphi}_\xi \mathsf{T}\frac{\mathsf{R}\hat{\varphi}_\xi}{|\hat{z}_\xi|^2} - \frac{|\hat{\varphi}_\xi|^2 - |\mathsf{R}\hat{\varphi}_\xi|^2}{2|\hat{z}_\xi|^2} - g\hat{y},$$

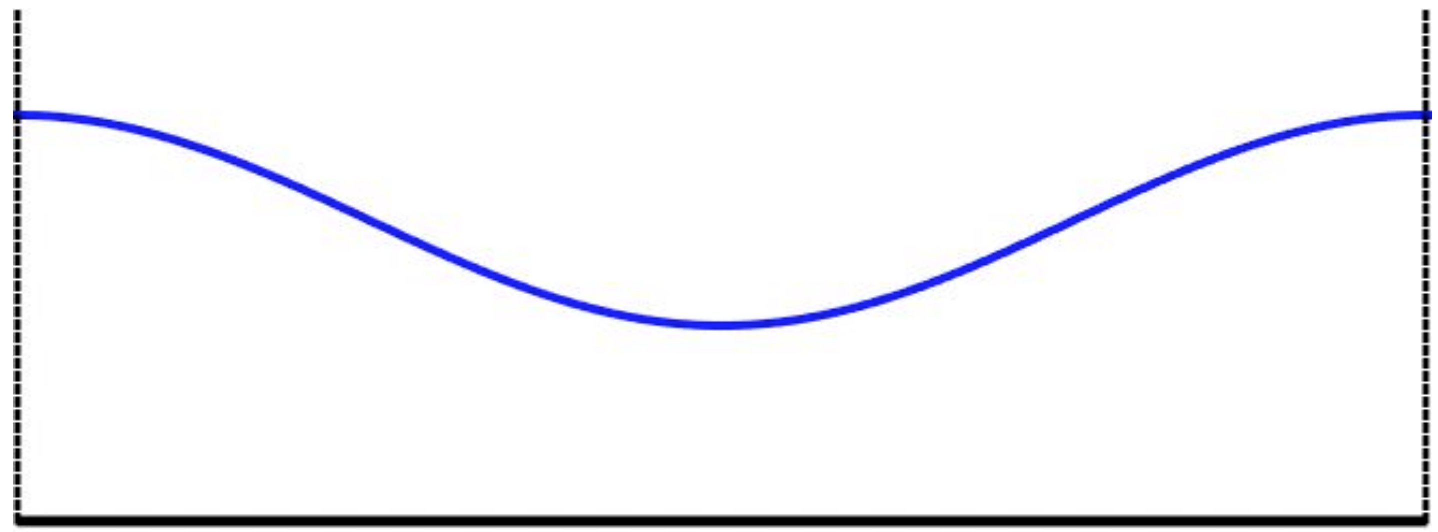
$$\mathsf{R}\hat{f}(\xi) = \sum_{m\in\mathbb{Z}} i\tanh(Km) f_m e^{im\xi},$$

$$\mathsf{T}\hat{f}(\xi) = -\sum_{m\neq 0} i\coth(Km) f_m e^{im\xi},$$

$$\hat{x}(\xi, t) = \xi + \hat{A}(\xi, t),$$

$$\hat{y}(\xi, t) = K(t) - 1 + \mathsf{R}\hat{A}(\xi, t).$$

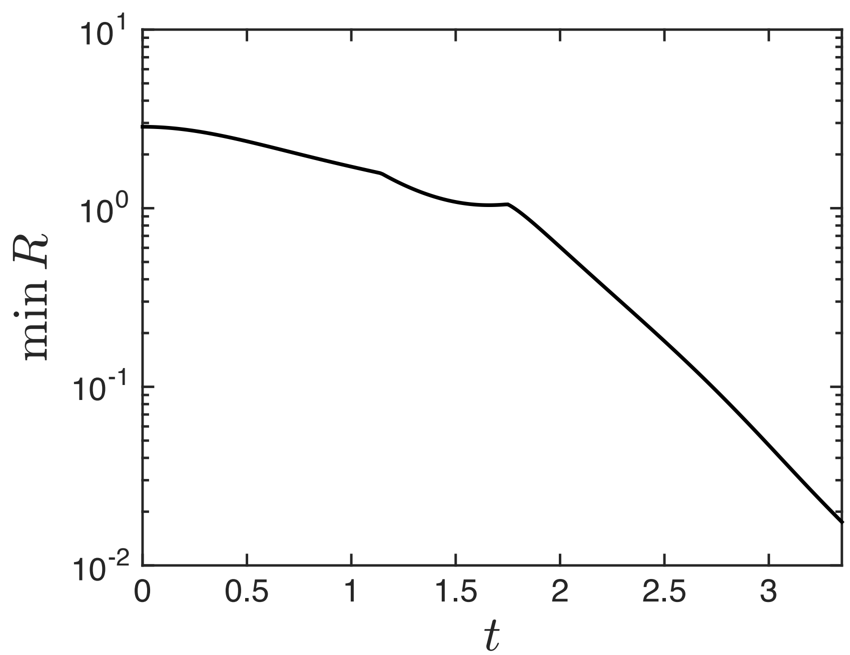
Numerical simulation



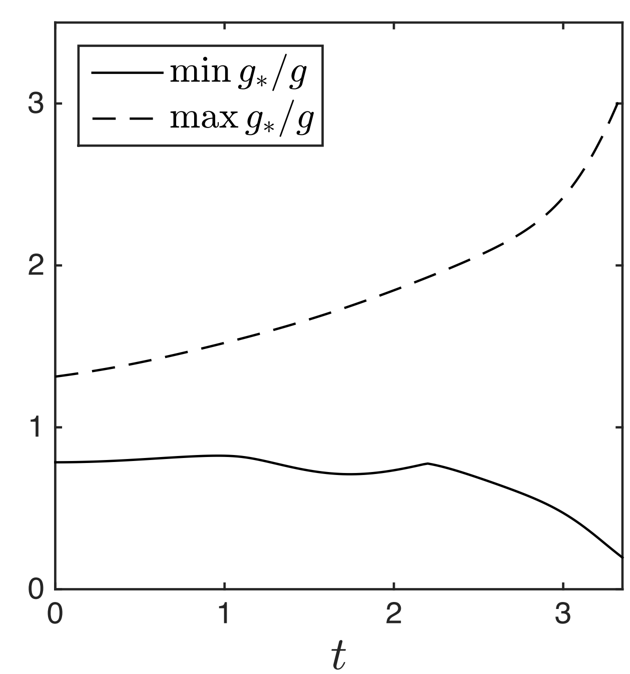
Initial conditions (linear wave): $\varphi = (0.35/\sqrt{\tanh 1})\sin x, \quad y = 0.35\cos x$

RK4 in time, pseudo-spectral in space, adaptive spatial step (final 2M grid), round-off-level accuracy.

Minimun curvature radius



Min/max local intrinsic gravity



$$\mathbf{g}_* = \mathbf{g} - \mathbf{a}$$

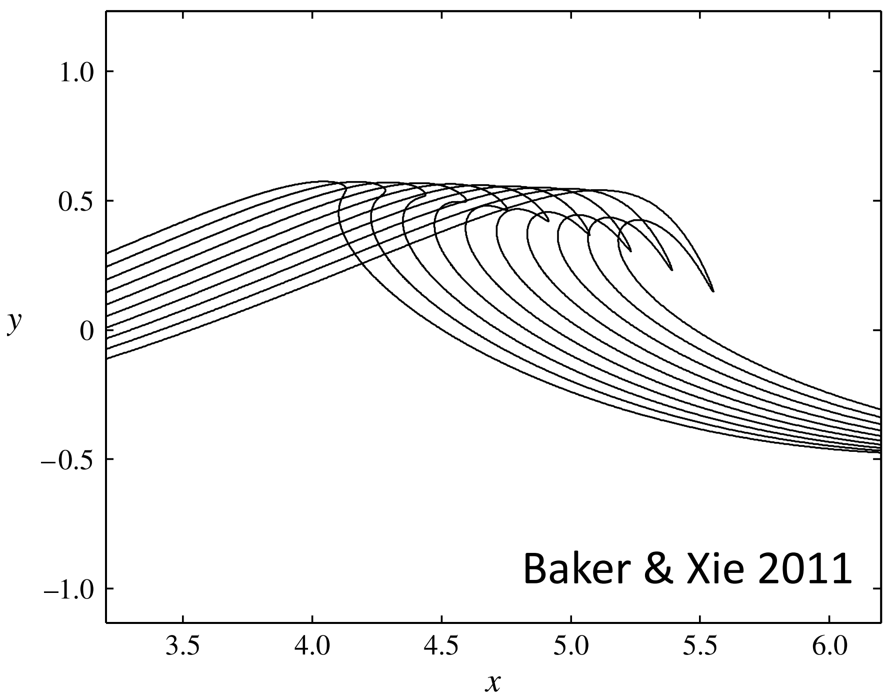
$$\mathbf{a} = -(\nabla p)/\rho + \mathbf{g}$$

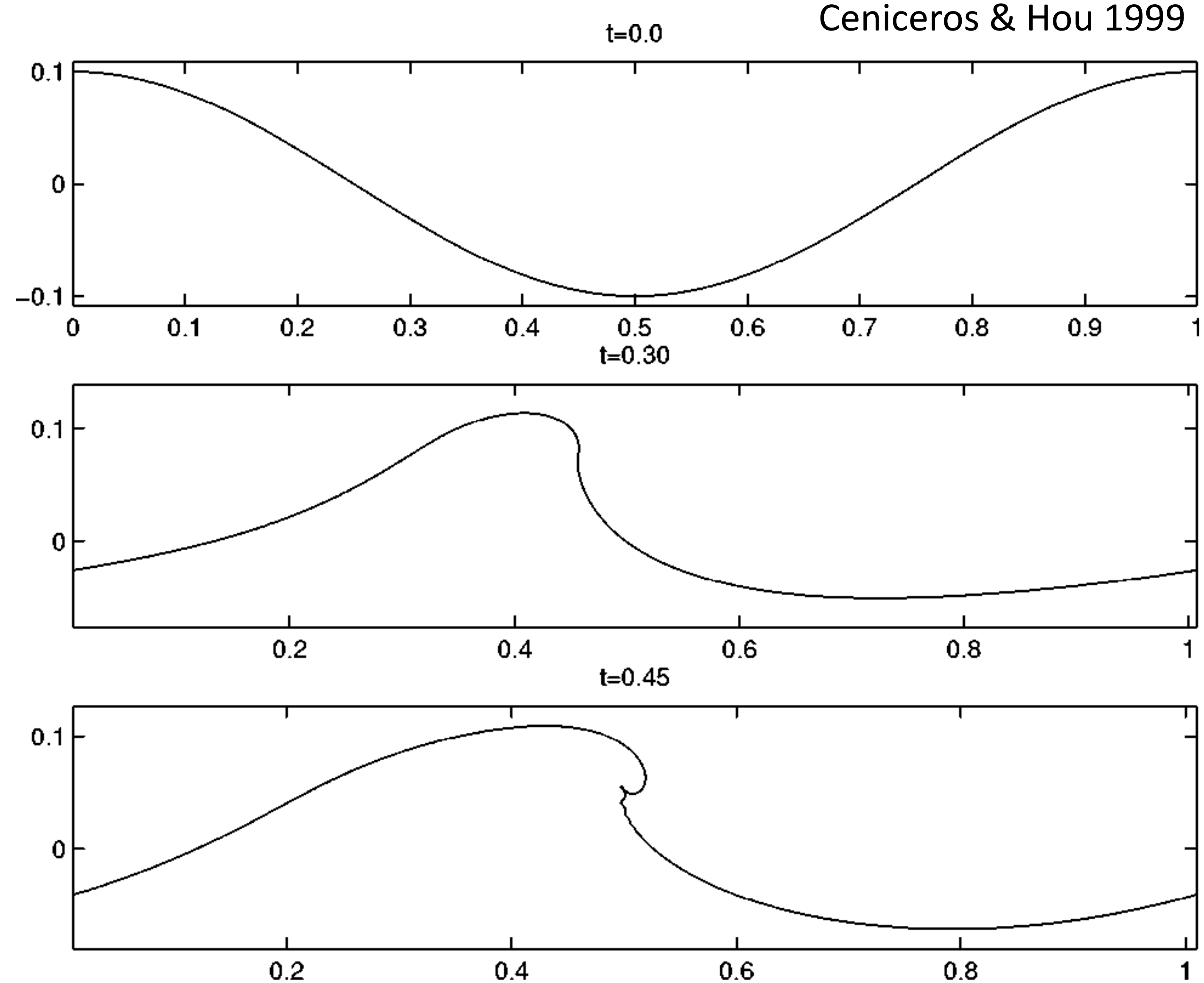
$$\mathbf{g}_* = \frac{\nabla p}{\rho} = -g_* \mathbf{n}$$

No Rayleigh-Taylor instability (Wu, 1997)

Too regular… (white caps!)

Many previous numerical studies show similar regular results:
(Peregrine 1983; Grilli & Svendsen 1990; Baker & Xie 2011, etc.)





Capillary effects and parasitic instability:
(Longuet-Higgins 1995, Ceniceros & Hou 1999, Dyachenko & Newell 2016, etc.)

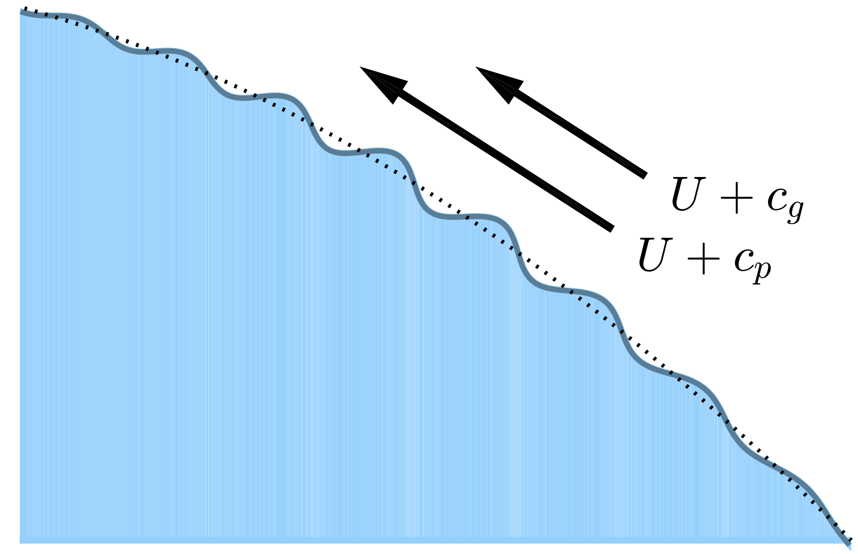
Overturn of a wave seems unnecessary for a white cap

Problem formulation

modeling of small ripples
riding on the surface
of a steepening breaking wave

Ripples dynamics

Surface length coordinate (local): *s*



Wavetrain ripples:

frequency	$\omega(s,t)$
wavenumber	$k(s,t)$
amplitude	$a(s,t)$

Regime of interest: appreciable changes are observed after many periods (2π/ω) and wavelengths (2π/k) (small amplitude, short wavelength)

Intrinsic frequency: $\Omega = \sqrt{g_* k}$ (deep-water approximation)

Intrinsic gravity: $\mathbf{g}_* = \mathbf{g} - \dfrac{D\mathbf{v}}{Dt} = \dfrac{\nabla p}{\rho} = -g_* \mathbf{n}$

Phase and group speed (in the Lagrangian reference frame):

$$c_p = \frac{\Omega}{k} = \sqrt{\frac{g_*}{k}}, \quad c_g = \frac{\partial \Omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g_*}{k}}$$

in Lagrangian frame

Doppler shift for the frequency: $\omega = Uk + \Omega$ (*U* is the medium's local speed)

Conservation of wave action on a free surface

First conservation law:

$$\omega = -\frac{\partial \theta}{\partial t}, \quad k = \frac{\partial \theta}{\partial s}$$

$\theta(s,t)$ is a ripple phase function

Consistency condition for second derivatives:

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial s} = 0$$

Use the Doppler-shifted frequency and expression for the phase speed:

$$\boxed{\frac{\partial k}{\partial t} + \frac{\partial}{\partial s}\left[(U + c_p)k\right] = 0}$$

Second conservation law:

E/Ω is the wave action density (energy density/intrinsic frequency)

The next equation is valid asymptotically in the adiabatic limit, i.e., for slow variations of the underlying flow

$$\boxed{\frac{\partial}{\partial t}\frac{E}{\Omega} + \frac{\partial}{\partial s}\left[(U + c_g)\frac{E}{\Omega}\right] = 0}$$

(Bretherton & Garrett 1968)

Explanation

Consider a **Hamiltonian system** with one degree of freedom (a linear oscillator).
Let parameters change slowly in time. Then the adiabatic invariant is conserved:

E/Ω = oscillator energy divided by its frequency

Example:

for a pendulum with slowly changing length,
the energy changes proportionally to frequency.

Euler equations is
a Hamiltonian system.
Ripple is a linear oscillator.

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial s}\left[(U + c_p)k\right] = 0$$

number of oscillators

$$\frac{\partial}{\partial t}\frac{E}{\Omega} + \frac{\partial}{\partial s}\left[(U + c_g)\frac{E}{\Omega}\right] = 0$$

adiabatic invariant

Adiabatic Lagrangian invariants

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial s}\left[(U + c_p)k\right] = 0 \qquad \frac{\partial}{\partial t}\frac{E}{\Omega} + \frac{\partial}{\partial s}\left[(U + c_g)\frac{E}{\Omega}\right] = 0$$

For short wavelength ripples: $c_p \ll U, \quad c_g \ll U$ $\qquad c_p = \frac{\Omega}{k} = \sqrt{\frac{g_*}{k}}, \quad c_g = \frac{\partial \Omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g_*}{k}}$

In the first approximation: $\frac{\partial k}{\partial t} + \frac{\partial}{\partial s}(Uk) = 0, \quad \frac{\partial}{\partial t}\left(\frac{E}{\Omega}\right) + \frac{\partial}{\partial s}\left(U\frac{E}{\Omega}\right) = 0.$

Marker density function: $\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial s}(U\sigma) = 0 \qquad t = 0: \quad \sigma(x) \equiv 1$

$$\frac{D}{Dt}\left(\frac{k}{\sigma}\right) = 0, \quad \frac{D}{Dt}\left(\frac{E}{\sigma\Omega}\right) = 0 \qquad \Rightarrow \qquad \boxed{\frac{k}{\sigma} = const, \qquad \frac{E}{\sigma\Omega} = const}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial s}$$

adiabatic Lagrangian invariants

(conserved along material trajectories on a surface)

Power-law for the change of ripple steepness

Lagrangian invariants:

$$\frac{k}{\sigma} = const, \qquad \frac{E}{\sigma \Omega} = const$$

$$E = \frac{1}{2}\rho g_* a^2$$ (mean oscillation energy)

$$\Omega = \sqrt{g_* k}$$ (intrinsic frequency)

Ripple amplitude:
$$\frac{a}{a_0} = \left(\frac{\sigma^3 g_{*0}}{g_*} \right)^{1/4}$$

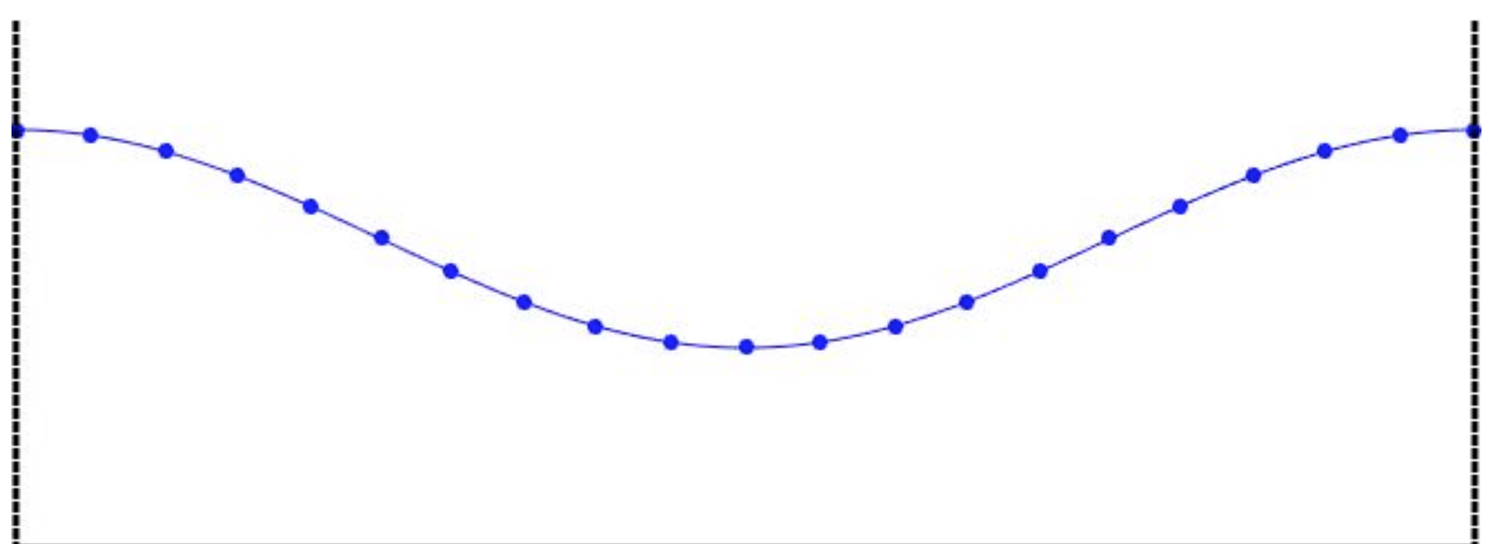
Ripple steepness:

$$S = \frac{2a}{\ell} = \frac{ak}{\pi} \quad \Rightarrow \quad \boxed{\frac{S}{S_0} = \left(\frac{\sigma^7 g_{*0}}{g_*} \right)^{1/4}}$$

Ripple steepness is fully determined by
- **the marker density (surface compression)**
- **and effective gravity.**

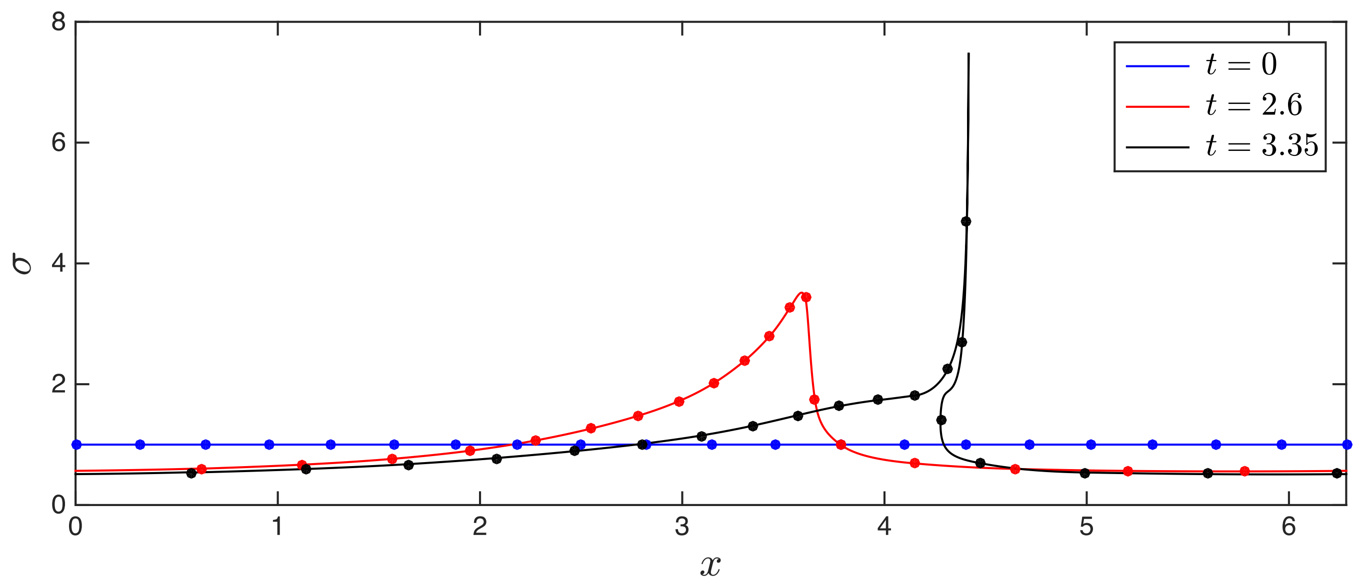
The law does not depend on ripple wavelength.

Evolution of markers (material points):

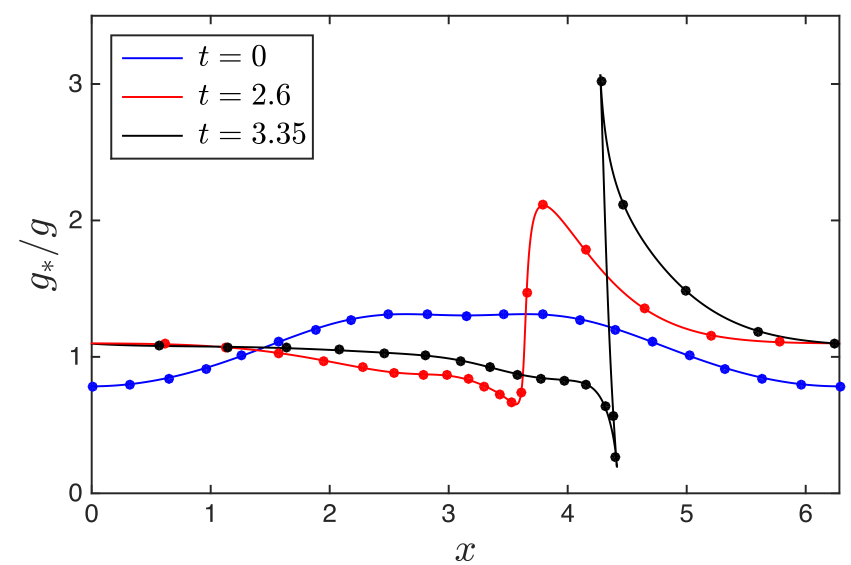


Ripple steepness amplification

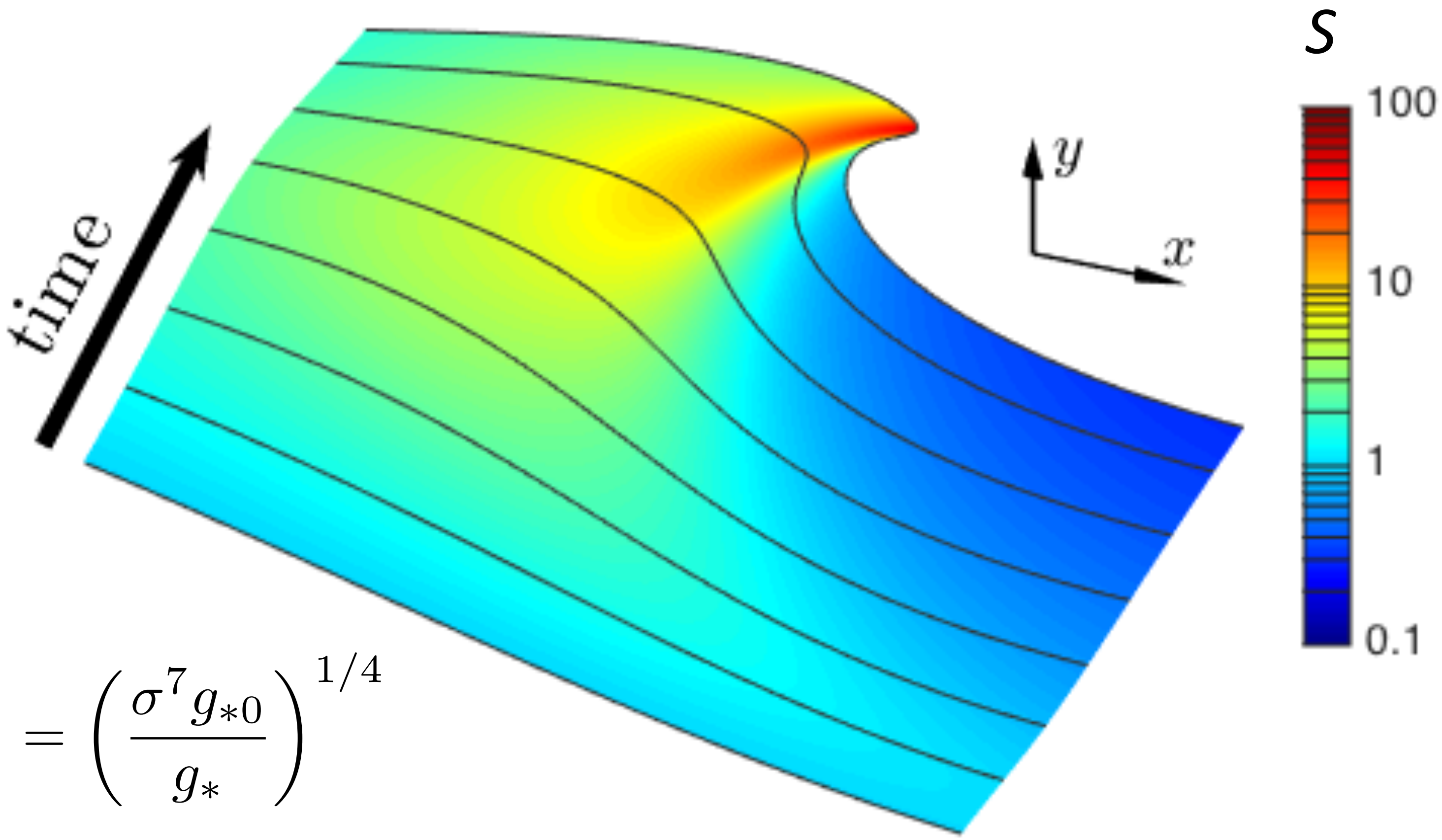
Marker density



Intrinsic gravity

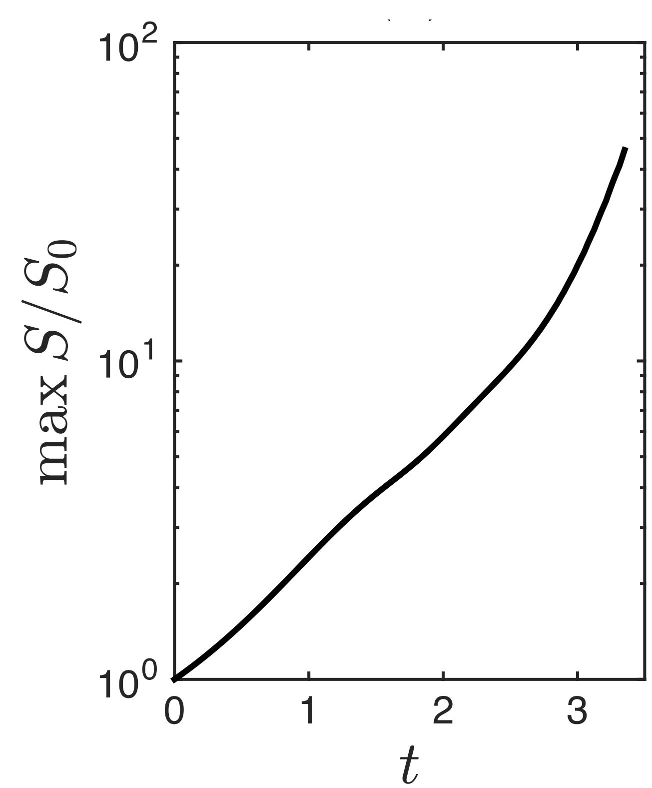


Ripple steepness
(log-color scale):

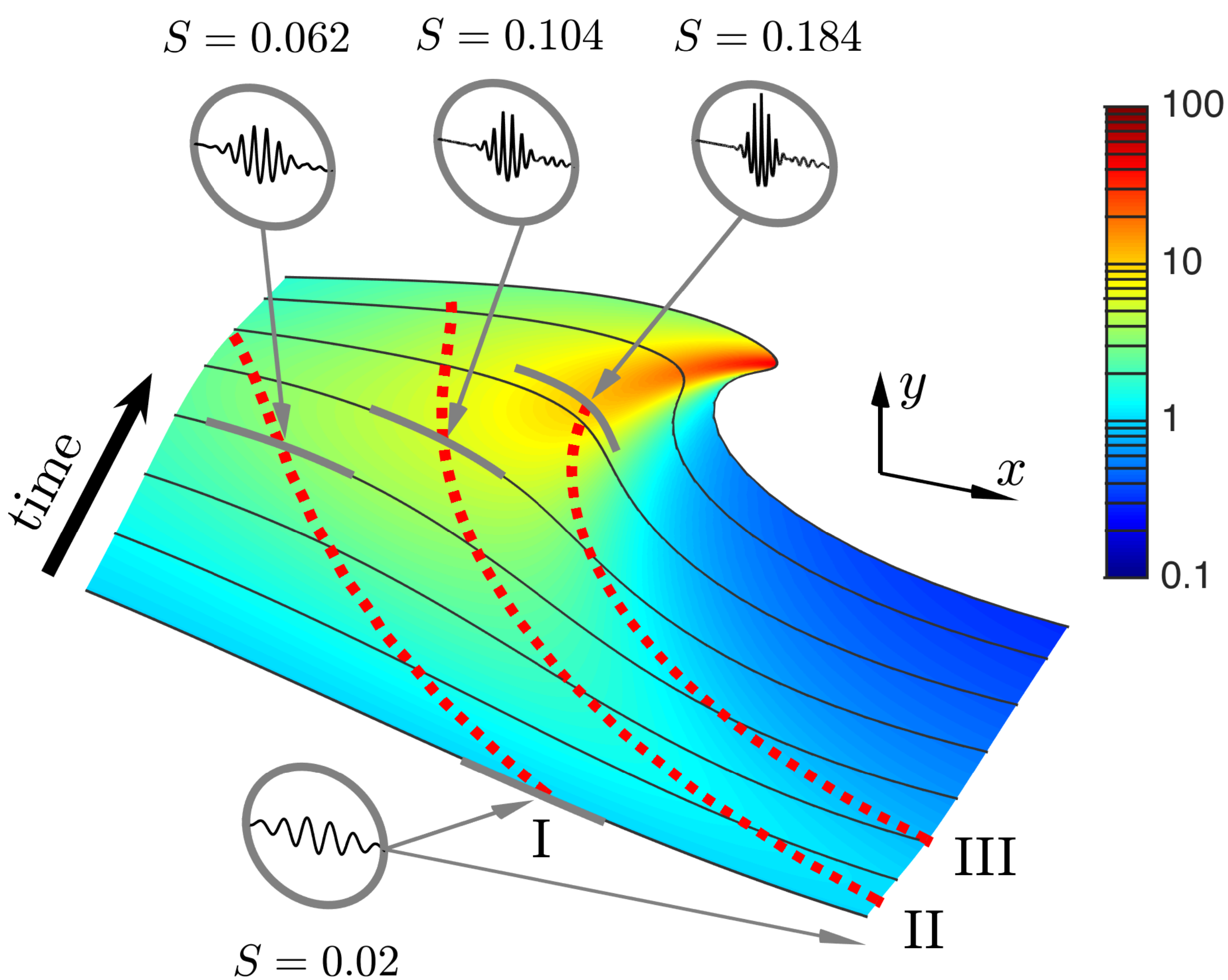


$$\frac{S}{S_0} = \left(\frac{\sigma^7 g_{*0}}{g_*}\right)^{1/4}$$

Super-exponential growth of ripple steepness

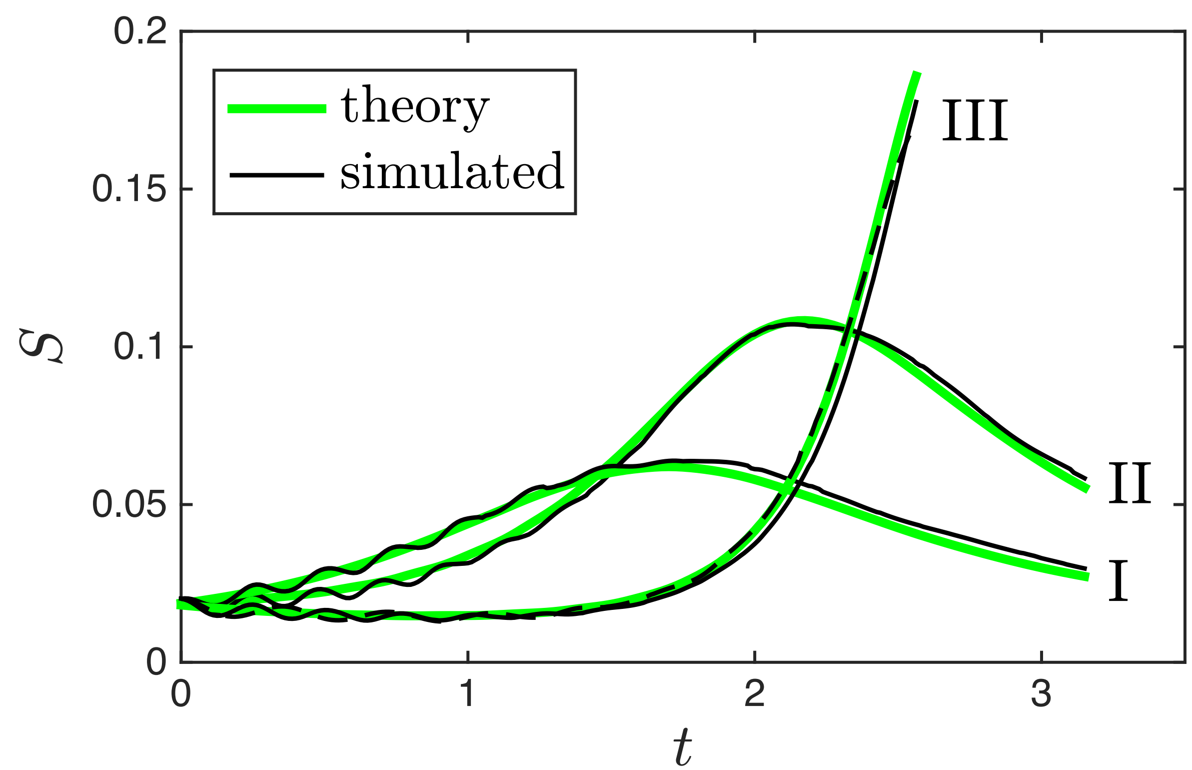


Gaussian ripples (theory vs. simulation)



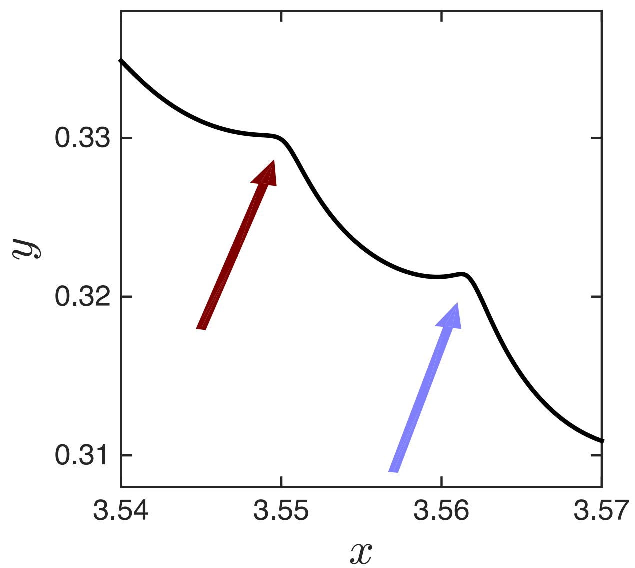
$$\frac{S}{S_0} = \left(\frac{\sigma^7 g_{*0}}{g_*}\right)^{1/4}$$

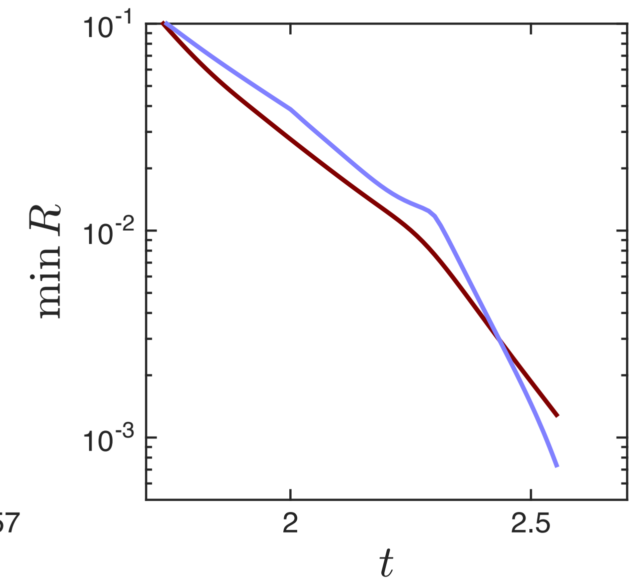
Evolution of the steepness evaluated at the center of the Gaussian ripple



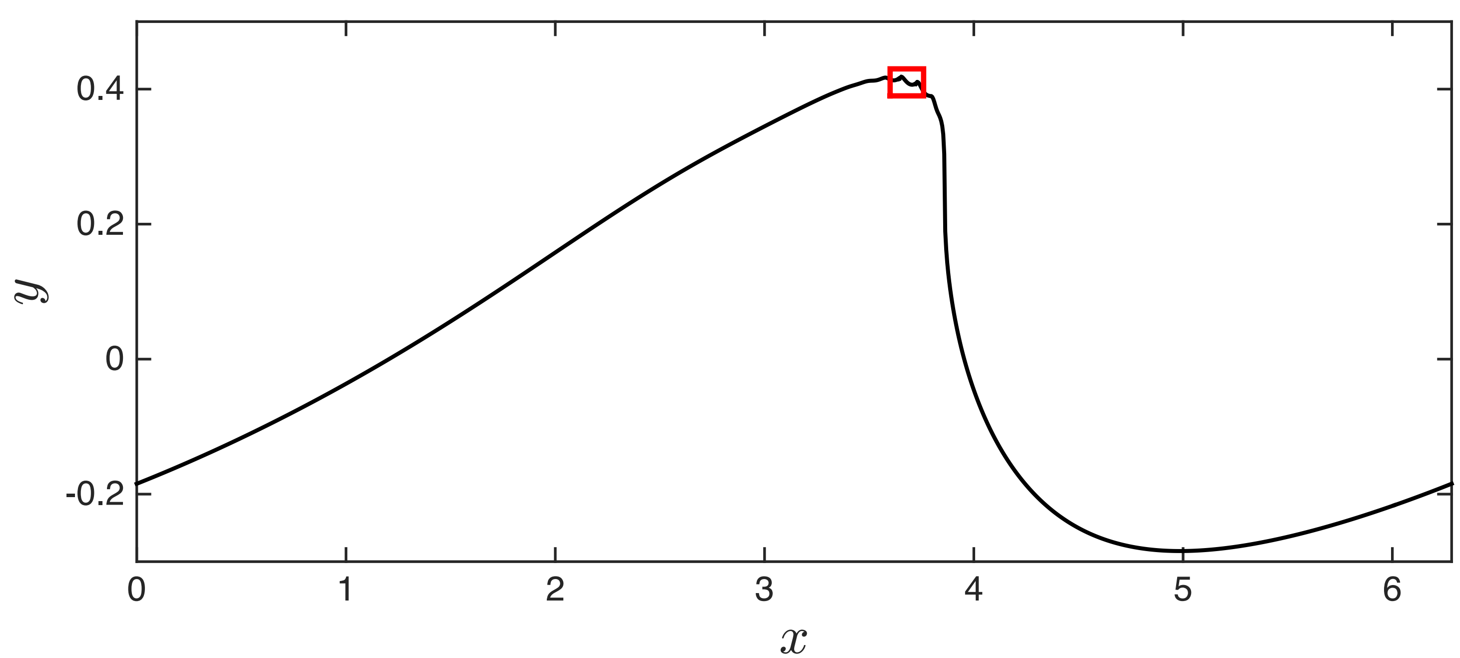
What is next?

Strongly nonlinear regime (formation of angles at ripple's crests)

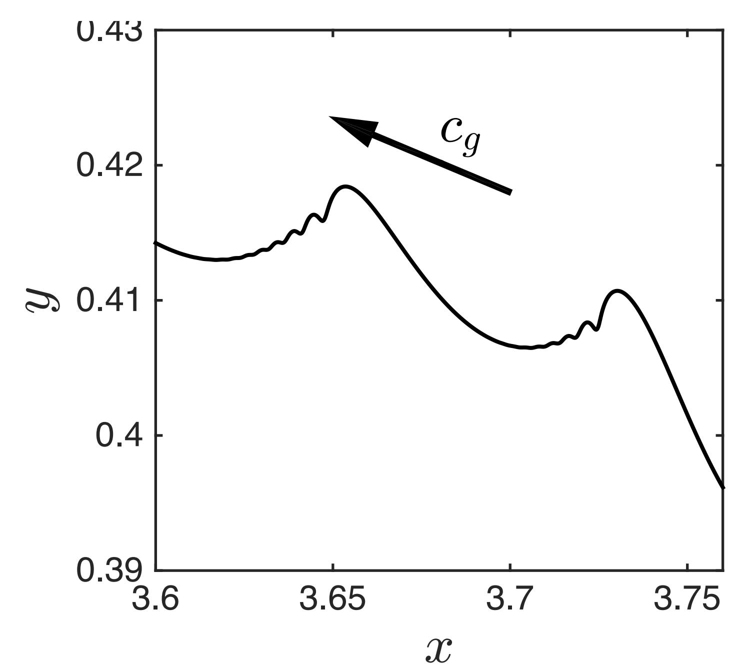




Capillary effects: stress balance at free surface $p = P_{atm} + \gamma/R$



onset of parasitic instability



Conclusions

Ripples steepness on the slope of a breaking wave is governed by a simple formula:

$$\frac{S}{S_0} = \left(\frac{\sigma^7 g_{*0}}{g_*}\right)^{1/4}$$ (depends only on the marker density and effective gravity)

The theory is in good agreement with numerical simulations.
It predicts the super-exponential increase of ripple steepness near the wave tip.

We observed numerically a start of the secondary "ripple breaking" generating the parasitic capillary instability.

Ripple instability may be an integral part of the multi-scale wave breaking phenomenon.

Спасибо!

Thank you!

Obrigado!

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