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The Ehrenfest Lottery.

"Is it possible to reconcile time reversibility and recurrence with 'observable' physical behavior"

① The stochastic gas experiment

② Thermodynamic description.

③ Irreversible approach to equilibrium.

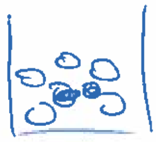
④ Recurrence Time.

⑤ Analogy with the deterministic setting.

cf | Kac, Probability and related topic in Physical science, 1959

| Cadeia de Markov ....

(2) Ehrenfest lottery: a stochastic gas experiment



Description

- N balls in an urn, labelled $1, 2 \dots N$.
- $\begin{cases} \# \bullet = A \\ \# \circ = B \end{cases}$ $\quad N = A+B$.
- $t \to t+1$. Pick a ball at random. Replace it by a ball of opposite colors.

No doubt that in a sense to be defined $\bar{A} \to N/2$ as $t \to +\infty$.

How can we characterize the dynamical evolution?

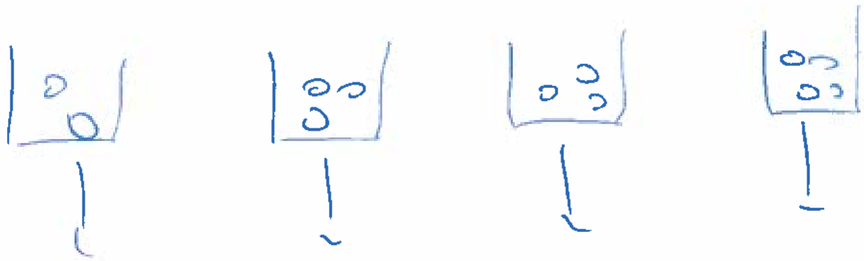
The equilibrium fluctuations?

Fine-grained:

- Experiment: ω.
- Configuration space: $\mathcal{E} = \{0,1\}^N$
- Configuration: $(t, i, \omega) \longrightarrow X(t,i,\omega) = 1$ if ith ball is ... $= 0$ otherwise
- Define: $\| f - g \| = \sum_{i=0}^{N} |f_i - g_i|$ for $f, g \in \mathcal{E}^2$.
- One experimental realization consists in drawing at random a sequence $[X(t,\omega)]_{t \in \mathbb{N}}$ such that $\forall t \; \| X(t+1) - X(t) \| = 1$ (*)
- Time-Reversibility: if $[X(t,\omega)]_{t \in [0,M]}$ satisfies (*), then $[\tilde{X}(M-t,\omega)]$ also satisfies (*) [!].
- The sequence $[X(t,\omega)]_{t \in \mathbb{N}}$ is a Markov-Chain, satisfying the defining property:

$$\forall x, y \in \mathcal{E}^N : \; P[X(t) = x \mid \forall i < t,\, X(t_i) = x_i] = P[X[t] = x \mid X(t-1) = \dots$$

P is to be thought as an average over infinitely many realizations of the same dynamics run in parallel.



number of experiment where $X_t = x$ is obtained.

$$P[X_t = x] = \lim_{N\to\infty} \frac{\#_x^N}{N}$$

The analogue of the Liouville equation is the

Chapman-Kolmogorov equation:

$$\boxed{\begin{aligned} P[X_t = x] &= \sum_{x'\in\mathcal{E}} P[X_t = x \mid X_{t-1} = x']\, P[X_{t-1} = x'] \\ &= \sum_{x'\in\mathcal{E}} \chi_{x'x}\, P[X_{t-1} = x'] \end{aligned}}$$

here

$(\chi_{x'x}) = \left(\frac{1}{N} \mathbb{1}_{\|x-x'\|=1}\right)$ is the matrix of transition rates.

χ has dimension $\#\mathcal{E} \times \#\mathcal{E}$ and is stochastic: $\sum_x \chi_{x'x} = 1$

Here, it is independent of t = this define a homogeneous MC

Exo: Show $\forall t \ \sum_x P[X_t = x] = 1$

Show $\forall t, s > 0 \quad P[X_{t+s} = x] = \sum_{x'\in\mathcal{E}} \chi_{x'x}(s)\, P[X_t = x']$

with $\chi_{x'x}(s) := \chi^s_{x'x}$.

The evolution of P is then entirely prescribed by
- the transition matrix χ
- the initial condition P_0

Microscopic Reversibility translate into $\chi_{x,y} = \chi_{y,x} \quad \forall (x,y)\in\mathcal{E}^2$

③ Macroscopic description

- In terms of the observable:

$$A(t,\omega) \doteq \sum_{i=1}^{N} \chi(\sigma_i;\omega) = \#\bullet \text{ at time } t \text{ in exp } \omega.$$

$[A(t)]_{t\in\mathbb{N}}$ is still a homogeneous Markov chain,

with configuration space $\tilde{\mathcal{E}} = [\![0;N]\!]$

and transition rates

$$\chi_{i\,i+1} = 1-\frac{i}{n} \quad [\text{choose a } \circ \text{ ball at step } i].$$

$$\chi_{i\,i-1} = \frac{i}{n} \quad [\text{pick a } \bullet \text{ ball}].$$

That is $\quad \chi_{ij} = \delta[j-i-1]\left[1-\frac{i}{n}\right] + \delta[j-i+1]\,\frac{i}{n}$

$$\chi = \begin{bmatrix} 0 & 1 & & \\ 1-\frac{1}{n} & \ddots & \frac{1}{n} & \\ & \ddots & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix}$$

- Chapman-Kolmogorov

$$P[j,t+s] = \sum_{i=0}^{N} \chi_{ij}^{(s)}\, P[j,t].$$

with $\quad \chi_{ij}[s] = (\chi^s)_{ij}$

$P[j,t] = \mathbb{P}[A_t = j]$.

$t,s \geqslant 0 \quad \in \mathbb{N}$.

- graphical:

$$0 \underset{1}{\overset{1-1/n}{\rightleftarrows}} 1 \underset{1/n}{\overset{1-2/n}{\rightleftarrows}} 2 \rightleftarrows 3 \qquad \underset{1-\frac{1}{n}}{\overset{1}{\rightleftarrows}} n$$

Clearly $X_{ij} \neq X_{ji}$ Coarse graining "killed" Reversibility

- **Reversibility**

Would require $P[j,t|i] = P[j|i,t] \quad t \in \mathbb{N}.$

$$\underbrace{P[j,t|i]}_{X_{ij}(t)} = P[j|i,t]$$

Yet $\quad P[j|i,t] = \dfrac{P[j \,\&\, i,t]}{P[i,t]} = \dfrac{p^{o}[j]}{P[i,t]} X_{ji}(t)$

Depends on p^{o}

- **Detailed balance:**

Choose $\quad p^{o}(j) = W(j) \doteq 2^{-N}\binom{N}{j}$

Then

① $P(i,1) = W(i) \qquad \left(= W(i-1,i) X_{i-1,i} + X_{i+1,i} W(i) \right)$

② $P(i,t) = W(i)$

③ $P[j|i,t] = X_{ji}(t) \dfrac{W(j)}{W(i)} = X_{ij}(t) = P[i,t|$

$$W(i) X_{ij}(t) = W(j) X_{ji}(t).$$

The Condition:

$$\boxed{W(i) X_{ij}(t) = W(j) X_{ji}(t) \text{ is the detailed balance condition}}$$

Exo: Show that W is the unique stationary distribution, ie satisfying $\sum_i X_{ij} W(i) = W(j)$

④ Irreversible approach to equilibrium

In general, the probability $P(t)$ is <u>not</u> an equilibrium, nor is it time-reversible. We show now that it converges to equilibrium as $t \to \infty$, i.e. $\forall i \in [0, N]$ $p_i \to$ ...

The proof employs convex functions.

<u>Reminder:</u> $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is convex & twice differentiable.

<u>Then</u> $\varphi''(x) > 0$.

$$\varphi(x') = \varphi(x) + (x' - x)\,\varphi'(x) + \frac{(x' - x)^2}{2}\,\varphi''(y) \qquad y \in]x, x'[$$

<u>This implies:</u> $\displaystyle\sum_i \alpha_i \varphi(x_i) \geq \varphi\Big[\sum_i \alpha_i x_i\Big]$

for $(\alpha_i)_{i \in [1,l]}$ $\sum \alpha_i = 1$

<u>"Markov - H - Thm"</u>

Define the "H function"

$$H_\varphi[t] = \sum_{i=0}^{N} W(i)\, \varphi\left[\frac{p_i(t)}{W(i)}\right]$$

where φ is convex & twice differentiable.

<u>Then</u> $H_\varphi(t) \xrightarrow[t \to \infty]{} c_\varphi < \infty$.

Proof relies on:

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① H bounded below } Direct calculation.

② H decreases }

③ Require $\chi_{ij}(s) > 0$ for s sufficiently large

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① Observe: $H[t] = \sum_i W(i)\varphi\left[\frac{p_i(t)}{W(i)}\right] \geq \varphi\left[\sum \frac{p_i}{W(i)} W(i)\right]$

$\geq \varphi(1)$ ✓.

② H decrease:

Compute: $H[t] - H[t+s] = \sum_i W(i)\varphi\left[\frac{p_i(t)}{W(i)}\right] - \sum_j W(j)\varphi\left[\frac{p_j[t+s]}{W(j)}\right]$

$= \sum_{i,j} W(i)\chi_{ij}^{(s)}\varphi\left[\frac{p_i(t)}{W_i}\right] - \sum_{i,j} W(j)\chi_{ji}^{(s)}\varphi\left[\frac{p_j(t+s)}{W(j)}\right]$

χ stochastic

$= \sum_{i,j} W(j)\chi_{ji}(s)\left[\varphi\left[\frac{p_i(t)}{W(i)}\right] - \varphi\left[\frac{p_j(t+s)}{W(j)}\right]\right]$. (*)

DB

$\geq \sum_{i,j} W(j)\chi_{ji}(s)\left[\frac{p_i(t)}{W(i)} - \frac{p_j(t+s)}{W(j)}\right]\varphi'\left[\frac{p_j(t+s)}{W(j)}\right]$

$= \sum_{i,j} \chi_{ji}\frac{W_j}{W_j}p_j \ldots$

$= \sum p_j\varphi' \ldots$

DB

$= \sum_{i,j} \chi_{ij}(s)\, p_i(t)\,\varphi'\left[\frac{p_j(t+s)}{W(j)}\right]$.

$= \sum_j p_j(t+s)\,\varphi'\left[\frac{p_j(t+s)}{W(j)}\right]$.

Chapman-Kolmogorov

≥ 0 ✓.

③ Refine (*) into: $\Delta H = \frac{1}{2}\sum_{ij} W(j)\chi_{ji}(s)\left[\frac{p_i(t)}{W(i)} - \frac{p_j(t+s)}{W(j)}\right]^2 \varphi''\left[\frac{p_j(t+s)}{W(j)}\right]$

choose s such that $\chi_{ji}(s) > 0$ [here $s = N+1$ works].

take $t \to \infty$ to obtain $0 = \frac{1}{2}\sum_{ij} W(j)\chi_{ji}(s)\left(\frac{p_i(t)}{W(i)} - \frac{p_j(t+s)}{W(j)}\right)^2\varphi''$

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This implies:

$$\frac{p_i(t)}{w(i)} - \frac{p_j(t+c)}{w(j)} \xrightarrow[t\to\infty]{} 0, \qquad \forall i \quad p_i(t) \to p_i^*$$

And

$$p_i^* (= p_j^* \frac{w(i)}{w(j)}$$

ie

$$1 = \frac{p_j^*}{w(j)}$$

D'où $\boxed{p_i(t) \xrightarrow[t\to\infty]{} w(i)}$

The proof used

① Regularity $\exists s \ X_{ij}(s) > 0 \quad \forall (i,j) \in \mathcal{E}$.

② $N < \infty$.

③ Detailed balance.

④ Unicity of W.

⑤ Strict positivity of W.

For more in depth study of MC, see Lacoin, Cadeia de Markov!

⑤ Recurrence Time

(8).

How likely is it to see a return to an ordered state

A_{ω} or A_{NN} in a sequence $(X_0, X_1, \dots X_t \dots)_{t \in \mathbb{N}}$?

The answer is: very unlikely if N is large!

- Flavor: For s such that $X_{ij}(s) > 0$

$$\frac{X_{ij}(s)}{X_{ji}(s)} = \frac{W(j)}{W(i)}$$

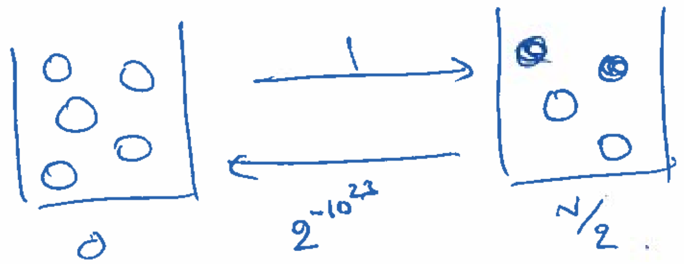
Ex: $i = \frac{N}{2}$; $j = 0$.

$$\frac{X_{\frac{N}{2}0}(s)}{X_{0\frac{N}{2}}} = \frac{W(0)}{W(\frac{N}{2})} = \frac{(N/2)!^2}{N!}$$

$$\Rightarrow \log \frac{X_{N/2\,0}}{X_{0\,N/2}} \sim N \log \frac{N}{2} - N \log N = -N \log 2$$

$$\Rightarrow X_{N/2\,0} \sim X_{0\frac{N}{2}} \, 2^{-N}.$$

ie $N = 10^{23}$?



- More convincingly, we have shown that ensemble of dynamic converge towards W. However, if one follows one realization only, and define

$$f_i^{\omega}(t) = \frac{1}{t} \sum_{s=1}^{t} \delta[A^{\omega}(s) - i].$$

Then?

$$f_i(t, \omega) \xrightarrow{P} \pi(i) \quad \text{and here} \quad f_i(t,\omega) \xrightarrow[t \to \infty]{\text{dist.}} \pi(i)$$

ie $\Pr[|f_i(t,\omega) - \pi(i)| > \varepsilon] \xrightarrow[t\to\infty]{} 0 \quad \forall \varepsilon.$

Proof

① show $\langle \varphi_i \rangle \to \pi(i)$

② that $\mathrm{Var}\, \varphi_i \to 0$.

③ B.C.: $\Pr[|\varphi_i - \pi_i| > \varepsilon] \leq \frac{\mathrm{Var}\, \varphi_i(t)}{\varepsilon^2} \to 0.$

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Among a sequence of length s, starting from eq, or for s "long enough", the mean recurrence time of state i can then be estimated as

length

$$\tau_{\text{recurrence}}(i) = \frac{s}{s \times w(i)} \sim \frac{1}{w(i)}$$

In particular: $\tau_{rec}\left(\frac{N}{2}\right) \sim \frac{1}{w\left(\frac{N}{2}\right)} \sim \sqrt{N}$

$\tau_{rec}(0) \sim 2^N$

For $N \sim 10^{23}$ this is again the estimate obtained from equilibrium computation of entropie.

6 Analogy & Summary

- Stochastic gas fleshes the idea of irreversible convergence towards equilibrium.
- Thermo: applies for $t \ll t_{universe}$. (!)
- Interactions are crucial to establish equilibrium [But Estimates with Perfect Gas are ok!].
- How to extend those ideas to deterministic settings?

$\Longrightarrow$ The "analog" of the c.g Chapman-Kolmogorov will turn out to be the Boltzmann Eq.