Hydrodynamics (From Boltzmann)

① Local Conservation laws.

② Hydrodynamic fields.

③ Dimensionless Boltzmann

④ Euler Equations.

⑤ Navier-Stokes. (via Chapman Enskog)

⑥ observations.

Sources
- Saint-Raymond, Hydro limits of the Boltzmann equations. (Lecture notes in Mathematics)
- Villani, review of mathematical topics in collisional kinetic theory.

(- Pottier, Non-Equilibrium statistical physics)

Useful conventions:

$k_B = 1$

$m = 1$

$(\vec{q}, \vec{p}) \longrightarrow (\vec{x}, \vec{v})$, (p-space

① Local Conservation Laws

Recall (once again) the

Boltzmann equation:

[BE].
$$\boxed{\begin{aligned} &\partial_t f + v\cdot\partial_x f = Q(f,f) \\ &Q(f,f) = \int dp^* \int_{\mathbb{R}^2} d\sigma\, \chi(\sigma, |p-p_*|)\left(f' f'_* - f f_*\right) \end{aligned}}$$

Obs: • To simplify, here the Boltzmann equation is considered in $\mathbb{R}^6$ without boundaries.

• $Q(f,f) = 0$ iff f is locally Maxwell,

$\xrightarrow{eg}$
$$f(x,v) \sim M_{loc}(T, u_0, \rho) \stackrel{\circ}{=} \frac{\rho(x)}{\sqrt{2T(x)\pi}^{\,3}} e^{-\frac{|v-u(x)|^2}{2T(x)}}$$

Notation: $M_{loc} \sim \tilde{M}(T, u_0, \rho)$.

• Assume Initial Conditions yield global equilibrium:

(glob)
$$f(x,v) \sim M(T_0, \rho) \stackrel{\circ}{=} \frac{\rho}{(2T_0\pi)^{3/2}} e^{-\frac{|v|^2}{2T_0}}.$$

The 5 global conservation laws

$i=1,2,3.$ $$\frac{d}{dt}\int_{p=(x,v)\in\mathbb{R}^3\times\mathbb{R}^3} \begin{pmatrix} 1 \\ v_i \\ v^2 \end{pmatrix} f = 0.$$

And the "H-thm"

$$\frac{d}{dt}\int_p f\log f = -D(f) \leq 0.$$

$$\boxed{D(f) = \int_p Q(f,f)\log f}$$

have the formal local counterparts; obtained by replacing $\int_p \to \int_v$

(Local Conservation)

$$\frac{d}{dt}\int_{v\in\mathbb{R}^3}\begin{bmatrix} 1 \\ v_i \\ |v|^2 \end{bmatrix} f + \partial_x \cdot \int_{v\in\mathbb{R}^3}\begin{bmatrix} 1 \\ v_i \\ |v|^2 \end{bmatrix}\vec{v} f = 0.$$

Divergence

Flux

$$\frac{d}{dt}\int_{v\in\mathbb{R}^3} f\log f + \partial_x \cdot \int_{v\in\mathbb{R}^3} f\log f\,\vec{v} = -D_x(f) \leq 0$$

$$D(x,f) \doteq \int_{v\in\mathbb{R}^3} Q(f,f)\log f.$$

2/ Direct use of the Boltzmann equation

+ the property that Q is local in space = i.e. invariants

stem from symmetry of X with respect to in/outgoing momenta.

([illegible])

To highlight connection with hydro, introduce:

② Hydrodynamic fields

$$\rho(x) \doteq \int_{v\in\mathbb{R}^3} f_1(x,v,t)\, dv$$

(→ In fact $\rho(x,t), u(x,t), T(x,t)$ ...)

$$\rho(x)\, u_i(x) \doteq \int_{v\in\mathbb{R}^3} f_1(x,v,t)\, v_i\, dv \qquad i=1,2,3.$$

[HYDRO].

$$\rho(x)\left[|u|^2(x) + 3_{\rho}T(x)\right] \doteq \int_{v\in\mathbb{R}^3} f_1\, |v|^2\, dv.$$

$$\rho(x)\, s(x) \doteq -\int_{v\in\mathbb{R}^3} f \log f\, dv$$

Obs: • $\rho(x)T(x) \doteq \frac{1}{3}\int_v f_1\, |v-u(x)|^2\, dv$, so that (III) is a definition of a non-equilibrium temperature.

• Rather than entropy one should call s local negentropy.

• $\int \rho(x)\, dx = N = M \qquad (m=1).$

Introduce, the

Pressure Tensor:

$$i,j\in 1,2,3 \qquad P_{ij} \doteq \int_{v\in\mathbb{R}^3} [v_i - u_i(x,t)]\,[v_j - u_j(x,t)]\, f(x,v,t).$$

eg (more compact)

$$P(x,t) = \int_v (v-u(x,t)) \otimes (v-u(x,t))\, f(x,v,t)$$

obs $\boxed{\rho T(x,t) \doteq \frac{1}{3} \mathrm{Tr}\, P.}$

In order to "derive" hydrodynamics, one needs to make appropriate assumptions. This is most conveniently done in terms of the

③ Dimensionless Boltzmann equation

Introduce t_0, L_0 macroscopic time & length scale such that

$$t_0 = \frac{L_0}{u_0}. \quad \text{with } u_0 = \text{bulk velocity.}$$

Introduce c_0 speed of sound here $c_0 \sim \left(\frac{5}{3}\theta_0\right)^{1/2}$.

then (BE) $\quad \partial_t f + v\,\partial_x f = Q.$ ← collision kernel

$$\partial_t f \sim \frac{N}{t_0}, \quad v\,\partial_x f \sim \frac{c_0 N}{L_0}, \quad Q \sim \frac{c_0^3 N^2 \, d^2 c_0}{(L_0 c_0)^6}$$

can be recast in terms of $\tilde{f}\left(\frac{x}{L_0}, \frac{v}{c_0}, \frac{t}{t_0}\right) = \frac{(L_0 c_0)^3}{N} f(x,v,t)$

as:

(RBE)
$$\boxed{\begin{gathered} Ma\, \partial_{\tilde t} \tilde f + \tilde v \cdot \partial_{\tilde x} \tilde f = \frac{1}{Kn} \tilde Q. \\ \text{with} \quad Ma = \frac{u_0}{c_0}. \quad \text{Mach number.} \\ Kn = \frac{\lambda}{L_0}. \quad \text{Knudsen number.} \end{gathered}}$$

hydro regime correspond to scaling with $Kn \ll 1$.

Obs: Sometimes, the Strouhal ($St = \frac{L_0}{t_0 c_0}$) number is used rather than the Mach number, if the bulk velocity is not introduced to relate the macro scale.

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the system of Local conservation laws + entropy is then translated into:

$$\partial_t \rho(x,t) + \partial_x \cdot (\rho u) = 0 .$$

$i = 1,2,3$ $$\partial_t \rho u_i(x,t) + \partial_{x_j} \cdot \left[P_{ij} + \rho u_i \vec{u}_j \right] = 0 .$$

[LOCAL ρ] $$\partial_t \left[\rho |u(x)|^2 + 3\rho T(x) \right] + \partial_x \cdot \left[2 u_0 P + \vec{u} \left[|u(x)|^2 \rho + T_r P \right] \right] = - \partial_x \cdot \int_{v \in \mathbb{R}^3} |v - u_0(x)|^2 (v - u_0(x)) f_1 \, dv$$

$$\partial_t \rho s(x) + \partial_x \cdot \int f \log f \vec{v} \leq 0 .$$

Obs: • The momentum equation is more compactly written as:

$$\partial_t \rho \vec{u} + \partial_x \cdot \left(P + \rho \vec{u} \otimes \vec{u} \right) = 0 .$$

• Apparition of Pressure tensor is through explicit decomposition, and patience! eg

$$\int v_i \vec{v} f = \left(v_i - u_i(x) + u_i \right) \left(\vec{v} - \vec{u}_0 + \vec{u}_0(x) \right) f .$$
$$= \int (v_i - u_i)(\vec{v} - \vec{u}_0) f + \int (v_i - u_i(x)) u_0 f .$$
$$+ \int u_i (v - u_0) f + \int u_i u_0 f$$
$$= P_{i,\cdot} + u_i \vec{u}(x) \rho(x)$$

~~[illegible]~~

• [illegible] [illegible] resemble Euler Equations but require to solve BE !

④ Compressible Euler

Taking the formal limit $Kn \to 0$ in the local conservation laws yield the compressible Euler equations:

$$\boxed{\begin{aligned}
&Ma\, \partial_t \rho + \partial_x \rho u_0 = 0. \\
&Ma\, \partial_t \rho \vec{u}_0 + \partial_x \cdot [P + \rho u \otimes u] = 0. \\
&Ma\, \partial_t (\rho |u|^2 + 3 T\rho) + \partial_x \cdot [2 u_0 P + u(|u|^2 \rho + Tr P)] \\
&\qquad = - \partial_x \int |v-u|^2 (v-u) f. \\
&Ma\, \partial_t\, \rho s(x) + \partial_x \int f \log f\, v\, \mathrm{d}v.
\end{aligned}}$$

with $f \sim \vec{M}(T_0, u_0, \rho) = \frac{\rho}{\sqrt{2\pi T}^3} e^{-\frac{|v-u_0|^2}{2T}}$

this hydro-limit is prescribed by the collision term being large, hence fast relaxation towards equilibrium.

Replacing v, P, s in the equations above with local Maxwellian for f yields:

$$P_{ij} = \int f (v_i - u_i)(v_j - u_j) = \rho T \delta_{ij}. \implies P = \rho T\, Id.$$

$$\int v f \log f = \int f \left[\log \frac{\rho}{(2\pi T)^{3/2}} - \frac{|v-u_0|^2}{2T}\right] v = u_0 \rho \log \frac{\rho}{(2\pi T)^{3/2}} - \int f \frac{|v-u_0|^2}{2T} \underbrace{(v-u_0)}_{0} \cdot u_0 \rho.$$

$$= u_0 \rho \left[-1 + \log \frac{\rho}{(2\pi T)^{3/2}}\right].$$

$$\int |v-u|^2 (v-u) f = 0.$$

(Euler)

Hence:

$$\begin{aligned}
&Ma\, \partial_t \rho + \partial_x \rho u_0 = 0. \\
&Ma\, \partial_t \rho u + \partial_x \cdot [\rho u \otimes u] = -\partial_x (T\rho) \\
&Ma\, \partial_t (\rho |u|^2 + 3 T\rho) + \partial_x [2 u T\rho + u(|u|^2 \rho + 3\rho T)] = 0.
\end{aligned}$$

Compressible Euler:

$$
(E1)\quad
\begin{cases}
\text{Ma}\quad \partial_t \rho + \partial_x \rho u = 0. \\
\text{Ma}\quad \partial_t \rho u + \partial_x\left(\rho\, u\otimes u + \rho T\right) = 0. \\
\text{Ma}\quad \partial_t\left(\rho|u|^2 + 3\rho T\right) + \partial_x\left(5T + |u|^2\right)\rho u = 0.
\end{cases}
$$

$\oplus$ Entropy:

$$
\text{Ma}\quad \partial_t\left[+\rho\log\frac{\rho}{(2\pi T)^{3/2}} - \rho\right] + \partial_x\left[u_0\rho\left(-1+\log\frac{\rho}{(2\pi T)^{3/2}}\right)\right] \leq 0.
$$

ie

$$
\boxed{\text{Ma}\quad \partial_t\, \rho\log\frac{\rho}{T^{3/2}} + \partial_x\left(\log\frac{\rho}{T^{3/2}}\right)\rho u_0 \leq 0.}
$$

After some massaging and integrate, one gets:

$$
(E2)\quad
\begin{cases}
\text{Ma}\quad \partial_t \rho + \partial_x \rho u = 0. \\
\text{Ma}\quad \partial_t u + u\cdot\partial_x u + \dfrac{1}{\rho}\partial_x \rho T = 0. \\
\text{Ma}\quad \partial_t T + u\cdot\partial_x T + \dfrac{2}{3} T\,\partial_x\cdot u = 0. \\
\text{Ma}\quad \partial_t \log\dfrac{\rho}{T^{3/2}} + \cancel{\rho}\, u\cdot\partial_x \log\dfrac{\rho}{T^{3/2}} \leq 0.
\end{cases}
$$

Exercise: check that E1 and E2 are formally equivalent!

5) Navier-Stokes

A common route to make viscosity appear is to work at a non-perturbative expansion around a local Maxwell Perturbation.

Example of such strategy is the Chapman-Enskog expansion, whose solution is searched as:

$$f = f^{(0)} + Kn\, f^{(1)}\,[P[t,x]].[v]$$

$\sim M(\rho, T, U)$ conserved quantity.

At order 0, the conservation laws satisfy the compressible Euler equations

At order 1, the (scaled) BE becomes:

$$Ma\ \partial_t f^{(0)} + v\cdot\partial_x f^{(0)} = \frac{2\cancel{Kn}}{\cancel{Kn}} Q(f^{(1)}, f^{(0)})$$

To simplify the discussion, we assume that $Q[f^{(1)}, f^{(0)}]$ is simply of the form $-f^{(1)}/\tau$ [Relaxation time approx for the linearized collision operator].

The general case can be found in (Golse, §5)*, and the present simplified approach has the merit to be quite explicit.

The correction to the Maxwell-B. distribution is explicitly

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* Golse, the Boltzmann equation and its hydro limits.

We can now estimate the required terms in the Conservation laws,

that is:

- The Pressure Tensor

$$P = \int v_i v_j \left[f^0 + \varepsilon f^{(1)}\right].$$

K_n

$$= \rho T \delta_{ij} + 8 K_n \int v_i v_j f^{(0)} v^T \left(\frac{Tr\Lambda}{3T} - \frac{\Lambda}{T}\right) v$$

$$= \rho T \delta_{ij} + p^{(1)}$$

obs: $Tr\, p^{(1)} = 0$. $p^{(1)} \propto \Lambda$.

$$\Rightarrow p^{(1)} = - 2\eta\left[\Lambda - \frac{Tr\Lambda}{3} I\right].$$

i.e.

$$\boxed{P = \rho T I - 2\eta\left[\Lambda - \frac{Tr\Lambda}{3} I\right].}$$

$O(K_n)$

The momentum equation:

becomes:

$$Ma.\ \partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x \rho T = \frac{1}{\rho}\partial_x\left(\Lambda - \frac{Tr\Lambda}{3} I\right) \times 2 \tau_y K_n$$

$$\boxed{Ma\ \partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x \rho T = \eta\frac{2}{\rho}\partial_x\left(\Lambda - \frac{Tr\Lambda}{3} I\right)}$$

$$\left(\partial_{x_i} u_j + \partial_{x_j} u_i\right)\frac{1}{2}.$$

- The flux (heat)

$$- \partial_x \cdot \int v^2 v f^0 \left(1 + K_n \frac{f^{(1)}}{f^{(0)}}\right)$$

$$= - K_n \partial_x \cdot \int v^2 v\ v \cdot \frac{\partial_x T}{T}\left[\frac{5}{2} - \frac{|v|^2}{2T}\right].$$

$$= - \partial_x K(T) \cdot \partial_x T. \qquad [\text{Fourier Law}].$$

$$\Rightarrow \text{From } 2u \cdot P = 2u\rho T - 2\eta\, u \cdot \left[\Lambda - \frac{Tr\Lambda}{3} I\right].$$

$$f^{(1)} = -\tau\,(\Pi a \partial_t + v\partial_x) f^{(0)} = -\tau f^{(0)} [\Pi a\, \partial_t + v\partial_x] \log f^{(0)}$$

The key idea is to replace the time derivate by a space derivative, using the zeroth order estimate for the local conservation laws.

$$\boxed{f^{(1)} = -\tau f^{(0)} [\Pi a\, \partial_t + u\partial_x] \log f^{0} - \tau f^{(0)} (v-u)\partial_x \log f^{(0)}}$$

And the EE

$\partial_t \rho + \partial_x \rho u.$

$\partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x \rho T = \ldots$

$\partial_t T + u\partial_x T + \frac{2}{3} T \partial_x \cdot v = \ldots$

$\partial_t s(x) + u\,\partial_x s(x) = 0.$

Recalling $\log f^{0} = \underbrace{\log \frac{\rho(x)}{(2\pi T)^{3/2}}}_{s(x)} - \frac{|v-u|^2}{2T}$

one obtains:

$$-(\Pi a\, \partial_t + u\partial_x) \log f^{(0)} = +(\Pi a\, \partial_t + u\partial_x) \frac{|v-u|^2}{2T}$$

$$= -\frac{1}{T}\underbrace{(\Pi a\, \partial_t + u\partial_x\, v)}_{= -\frac{1}{\rho}\partial_x \rho T}\cdot(v-u) - \frac{|v-u|^2}{2T^2}\underbrace{(\Pi a\, \partial_t + u\partial_x T)}_{-\frac{2}{3}T\partial_x \cdot v}.$$

$$= \frac{(v-u)}{T}\cdot\left[\partial_x T + \frac{T}{\cancel{\rho}}\cancel{\partial_x \rho}\right] + \frac{|v-u|^2}{3T}(\partial_x \cdot v).$$

$$-(v-u)\,\partial_x \log f^{(0)} = -(v-u)\cdot\frac{\partial_x \cancel{\rho}}{\cancel{\rho}} + (v-u)\cdot\frac{\partial_x T}{T}\frac{3}{2} + \left(\partial_x \frac{|v-u|^2}{2T}\right)\cdot(v-u)$$

$$= -\cancel{2}(v-u)_i(\partial_{x_j} u_i)\frac{(v-u)_j}{2\cancel{T}} - \ldots \frac{\partial_x T}{2T^2}\ldots$$

eg

$$\frac{f^{(1)}}{\tau f^{(0)}} = (v-u)\cdot\frac{\partial_x T}{T}\left[\frac{5}{2}\right] + \frac{|v-u|^2}{3T}(\partial_x \cdot v) - (v-u)^T(\partial_x u)\underbrace{\frac{(v-u)}{T}}_{\left(\frac{1}{2}(\partial_{x_i}u_j) + (\partial_{x_j}u_i)\right)} - (v-u)|v-u|^2 \ldots$$

$$= V\cdot\frac{\partial_x T}{T}\frac{5}{2} + \frac{V^2}{3T}\mathrm{Tr}\,\Lambda - V^T\Lambda\frac{V}{T} - V|V|^2\cdot\frac{\partial_x T}{2T^2}$$

$$\boxed{\frac{f^{(1)}}{\tau f^{(0)}} = V\cdot\frac{\partial_x T}{T}\left[\frac{5}{2} - \frac{|V|^2}{2T}\right] + V^T\left[\frac{\mathrm{Tr}\,\Lambda}{3T} - \frac{\Lambda}{T}\right]V.}$$

where $V = v - u$, $\Lambda = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

The equation for the temperature becomes:

$$\boxed{\partial_t T(x) + u \cdot \partial_x T + \frac{2}{3} T \,\mathrm{Tr}\,\Lambda = -\partial_x\left(\kappa(T)\,\partial_x T\right)\frac{1}{3\rho}}$$

The equation for the energy is:

$$\partial_t\left[\rho |u|^2 + 3\rho T\right] + \partial_x \cdot \left[2uT\rho + u\left[|u|^2\rho + 3T\rho\right]\right]$$
$$= 2\eta\, \partial_x u \cdot \left(\Lambda - \frac{\mathrm{Tr}\,\Lambda}{3} I\right)$$
$$- \partial_x\left(\kappa(T)\,\partial_x T\right)\frac{4}{3}.$$

The exact expression for κ and η are dependent upon the integrals. Here, it makes little sense to give an expression for these, as we have relied on the simplistic relaxation approximation Ansatz.

The message is that various asymptotics of the Boltzmann equation yields various hydrodynamic equations!

<u>Obs</u>: The expansion is not rigorous and from a mathematical perspective the hydro. behavior of the Boltzmann Equation is still an open question. Modern treatment bypass expansion to prove convergence of Boltzmann towards various smooth solution of hydro.dynamic type. The only rigorous case is the incompressible case obtained in the limit $Ma \ll 1$, $Kn \ll 1$, as a perturbation around global equilibrium (see Saint-Raymond for details)

6 Add on / Final observations.

1 Sound-speed: Linearize Euler around global Equilibrium

say $\rho_0 = 1$, $u_0 = 0$, $T_0 = 1$

The perturbations $(\tilde{\rho}, \tilde{u}, \tilde{T})$ satisfy the Linearized Euler equations.

$$\partial_t \tilde{\rho} + \partial_x \cdot \tilde{u} = 0$$
$$\partial_t \tilde{u} + \partial_x(\tilde{\rho} + \tilde{T}) = 0$$
$$\partial_t \tilde{T} + \frac{2}{3} \partial_x \cdot \tilde{u} = 0$$

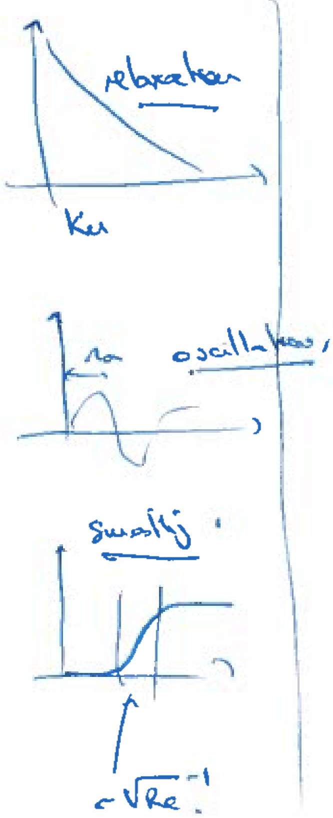
$$\Rightarrow \partial^2_{tt}(\tilde{\rho} + \tilde{T}) = \frac{5}{3} \partial^2_{xx}(\tilde{\rho} + \tilde{T})$$
$$\partial^2_{tt}(\nabla \cdot \tilde{u}) = \frac{5}{3} \partial^2_{xx}(\partial_x \cdot \tilde{u})$$

$\Rightarrow$ speed of sound is $c = \sqrt{\frac{5}{3}}$.

2 Incompressible limits:

They are obtained by considering various asymptotics from Boltzmann E

cf ST Raymond: the B. equation & its formal hydro limits.



Scaled Boltzmann.

$$Ma\, \partial_t f + v\, \partial_x f = Q.$$

$\downarrow$ $Kn \ll 1$

Compressible Euler + Kn corrections.

$Kn \ll Ma$.

Incompressible Euler.

$$\partial_t u + u \partial_x u + \partial_x p = 0.$$
$$\partial_t \theta + (u \partial_x)\theta = 0$$

$Kn \sim Ma$.

Incompressible N.S.

$$\partial_t u + u \partial_x u + \partial_x p = \nu \Delta_x u.$$
$$\partial_t \theta + u \cdot \partial_x \theta = \kappa \Delta_x \theta.$$

$$Re = \frac{Ma}{Kn}$$