Vortex gas and negative Temperature.

① 2D Euler and Vortex gas.

② Negative Temperature.

③ Mean Field approach.

④ observations.

<u>Sources</u>:

Chavanis, 2002 Stat Mech of 2D vortices & stellar Σ.

Eyink & Spohn, 1993

Montgomery & Joyce 1974.

Chorin, Vorticity & Turbulence

U

① 2D Euler and Vortex gas

In the limit $Kn \to \infty$, we justified the formal derivation of the Euler equations. Neglecting fluctuations of the density, T & u decouple to yield

Incompressible Euler

$$\begin{cases} \partial_t u + u \cdot \partial_x u + \partial_x p = 0 \\ \partial_x \cdot u = 0 \end{cases}$$

Let us here consider (EE) in two-D $u = u(x,y)$ $p = p(x,y)$ ...

Upon taking the curl, (EE) becomes: $\vec{u} = (u_x, u_y)$

$$\boxed{\begin{aligned} &\partial_t \omega + u \cdot \nabla\omega = 0. \\ &\omega \doteq \partial_x u_y - \partial_y u_x = -\nabla^2\psi \\ &u = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} \psi = \nabla^\perp \psi. \end{aligned}}$$

Go! Bad notation abt (here ∂_x is scalar

obs: • In this form, (EE) resembles a transport equation, but the vorticity is active: it retroacts on the velocity field via the stream function.

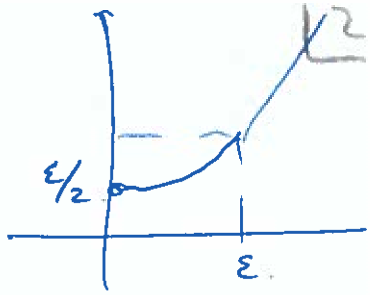
• (EE) admits "point-vortex" solutions, whereby $\omega = 0$ except on a discrete set of zero-measure:

$$\omega_\varepsilon(x) \doteq \sum_{\alpha=1}^{N} \gamma_\alpha \, \delta[x - X_\alpha^\varepsilon(t)].$$

eg

with $\dot{\vec{X}}_\alpha^\varepsilon(t) = \vec{U}_\alpha^\varepsilon = \nabla_\alpha^\perp \psi_\varepsilon$; $\psi_\varepsilon = -\sum_{\alpha=1}^{N} \frac{\gamma_\beta}{2\pi} \lg|x_\alpha - x_\beta|_\varepsilon$

$$|r|_\varepsilon \overset{\text{e.g.}}{\doteq} \begin{cases} \dfrac{r^2}{2\varepsilon} + \dfrac{\varepsilon}{2} & |r| \leq \varepsilon. \\ |r| & |r| \geq \varepsilon. \end{cases}$$



Then

Formally:
$$\frac{\partial}{\partial t}\omega_\varepsilon(x,t) = -\sum_{\alpha=1}^{N} \gamma_\alpha \nabla_x \delta[x - X_\alpha^\varepsilon(t)]\, \dot{X}_\alpha^\varepsilon(t).$$
$$= -\partial_x \cdot \sum_{\alpha=1}^{N} \delta(x - X_\alpha^\varepsilon(t))\, V_\varepsilon[X_\alpha(t)]$$
$$= -V_\varepsilon(x,t) \cdot \partial_x \underbrace{\sum_{\alpha=1}^{N} \delta(x - X_\alpha^\varepsilon(t))}_{\omega_\varepsilon(x,t).}$$

$\varepsilon \to 0$. $\quad \partial_t \omega(x,t) + V(x,t)\cdot\partial_x \omega = 0.$

↳ weakly (ie one has to integrate wrt obs).

⇒ the fluid equations (Euler) generate a new type of point particles, the vortex (point).

The fundamental observation at this point is that the dynamics of a system of P.V is hamiltonian.

$$\dot{Z} = \{Z, H\}.$$

With:
$$H_\varepsilon = -\sum_{\alpha,\beta} \frac{\gamma_\beta \gamma_\alpha}{4\pi} \log r^\varepsilon_{\alpha\beta} + \text{cst} \qquad r^\varepsilon_{\alpha\beta} = \|\vec{X}_\alpha - X_\beta\|_\varepsilon.$$

$$\{\ ,\ \} = \sum_\alpha \left[\frac{\partial}{\partial X_\alpha}\frac{\partial}{\partial Y_\alpha} - \frac{\partial}{\partial Y_\alpha}\frac{\partial}{\partial X_\alpha}\right]\frac{1}{\gamma_\alpha}.$$

Obs: Naturally, $\forall \alpha$ $\gamma_\alpha > 0$.

- H_ε defined up to a constant.
- Constants of motion include,

  Angular momentum $L = \sum_\alpha \Gamma_\alpha r_\alpha^2$.

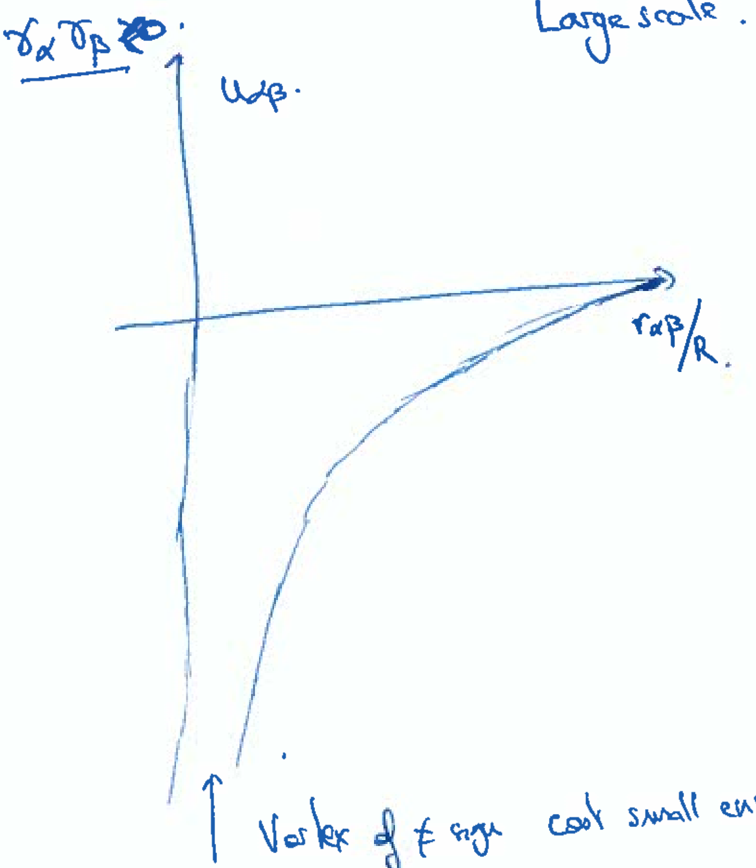
  Linear Impulses $\Pi_x = \sum_\alpha \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} \Gamma_\alpha$.

- Very peculiar Hamiltonian: no kinetic energy !!

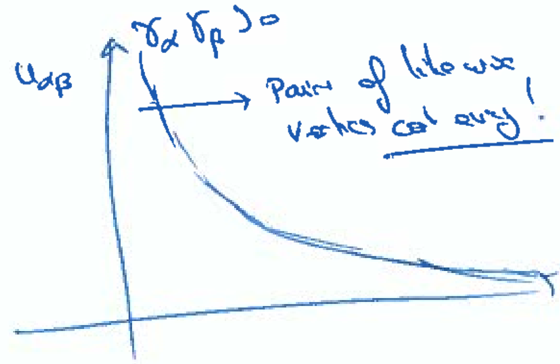
A vortex is very $\neq$ than a free particle: left alone, it stays still !!.

$$u_{\alpha\beta} = -\gamma_\alpha \gamma_\beta \log r_{\alpha\beta}^2 / R.$$

Large scale.



Vortex of $\neq$ sign cost small energy
pair.



$\Longrightarrow$ Presence of long-range interaction will create structure for the equilibrium state!

(unlike GP!)

(2) Negative Temperature.

Assume that the gas (vortex) is confined in some ways.

→ Either by confining opposite sign vortices in a bounded container (sphere, Disk, etc)

→ or by considering gas of same-sign vortices in $\mathbb{R}^2$.

Then Hamiltonian structure allows to construct standard thermodynamic ensemble, and describe (putative) long-time state of the Z following standard procedure.

Yet, here Γ-space is very particular, unlike G.P

Γ is bounded

$$\Gamma = \{(z_1 \dots z_N) \in \mathcal{D}^N\}$$

This has a practical implication for temperature.

Consider microcanonical statistics:

$$P_{\Gamma,E} = \frac{\delta[H(\{z\}) - E]}{g(E)}$$

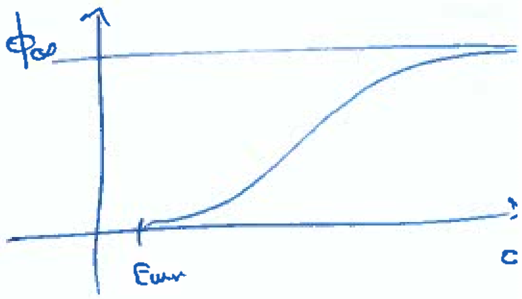
$$g(E) = \int_{\mathcal{D}^N = \Gamma} \delta[H\{z\} - E]\, dz_1 \dots dz_N.$$

Introduce $\phi(E) \doteq \int_{E_{min}}^{E} dE' \, g(E')$ [# state $\leqslant E$]

$\phi(E)$ ranges C^0 from 0 to finite value A^N as $E \to \infty$

ie

$$\lim_{E \to \infty} \phi(E) = \mathcal{A}^N < \infty .$$



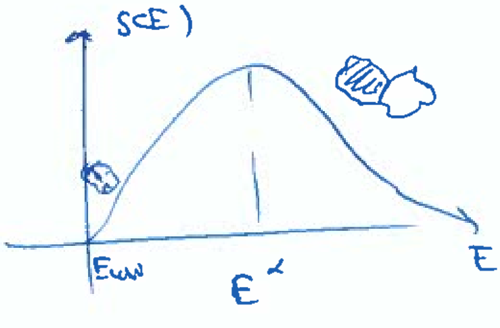
$$\mathcal{A}^N = \int_{E_{min}}^{\infty} dE \, g(E)$$

$\phi \nearrow$ $\exists$ inflexion point: $\phi''(E) = 0$.

ie $\exists E^*$ $g'(E^*) = 0$, eg $\underline{S(E^*) = 0}$.

For $E < E^*$ expect positive temperature . Low energy pairing of opposite ...

$E > E^*$ expect negative temperature : High Energy pairing of same-sign ...



Question in the relevant thermo limit, is $\underline{E^* < \infty}$? .

The answer is ... $\underline{\text{yes}}$! .

As the $\underline{\text{relevant}}$. thermo limit implies $N \to \infty$.

$E = -\gamma^2 \langle \log k_B \rangle N^2 \to 1$

ie $\gamma \sim 1/N$. ($\Rightarrow T = \gamma_N \to$ ...

<u>Obs</u> : φ of negative T.

A negative temperature is hotter than any positive temperature.

③ <u>Thermodynamic limit</u>

To describe the emerging macroscopic structure, one needs to determine the macrostate $f_1(r) = N\,p_1(r)$ = # vortices / unit $\Delta r = dx\,dy$.

The long-range nature of the interaction potential is particularly useful, in that it prescribes the MF limit.

f_1^* is determined by solving

$$\boxed{-\int f_1 \log f_1 \longrightarrow \sup. \qquad \begin{aligned} \psi(r) &= \int G(r,r')\langle\omega(r')\rangle\, d \\ E_0 &= \int G(r,r')\langle\omega(r)\rangle \langle\omega \\ \Gamma_0 &= \int \langle\omega(r)\rangle\, dr \\ L_0 &= \int \langle\omega(r)\rangle\, r^2\, dr \end{aligned}}$$

<u>Obs</u> $\rightarrow$ $\omega(r) = \gamma f_1(r) = N\gamma\, p_1(r) = \underline{\Gamma_0\, p_1(r)}$

$N\gamma = \Gamma_0 = \text{cst}$

This prescribes:

$$f_1 = A e^{-\beta\psi'} \qquad \psi' = \psi + \gamma r^2$$

And A is solution to the Poisson-Boltzmann equation:

$$\boxed{\langle\omega\rangle = \tilde{A} e^{-\beta\psi} = -\nabla^2\psi} \quad (*) \qquad + \text{B.C.}$$

In gral solving (*) is not trivial.

(4) Obs: - Detailed Justification of M.F asymptotics rely on Kinetic theory ⊕ BBGKY Type argument.

- One can prove $\beta > \beta_c = -\frac{8\pi}{N\gamma^2} = -\frac{8\pi N}{\Gamma^2}$.

$$g(E,V) = \int^R \int^R \delta\left(E + \frac{\gamma^2}{4\pi}\sum_{i,j} \log(\,)\right)$$

$$\tilde{x}_i = \frac{x_i}{R} \quad = |\partial|^N \int_0^1\!\!\int_0^1 \delta\left(E + \frac{\gamma^2}{4\pi}\sum_{i,j}\log R + \gamma^2 \sum_i \log|x_i - x_j|\right)$$

$$= |\partial|^N g(E',1) \qquad E' = E + \gamma^2 \frac{N^2}{8\pi} \log|\partial|$$

$$S(E,V) = N\log|\partial| + S(E',1)$$

$$P = T\frac{\partial S}{\partial V} = \left(\frac{\partial S}{\partial E'}\frac{\gamma^2 N^2}{\pi\, 8|\partial|} + \frac{N}{|V|}\right) +$$

$$P = \frac{NT}{V}\left(1 + \frac{\Gamma}{N8\pi}\beta\right)$$

$$P \Rightarrow \boxed{\beta = -\frac{N 8\pi}{\Gamma}}$$

In fact negative-temperature states signals self-organization.

One can rely on stat mech tools without having to (arbitrarily) discretize ω into discrete vortices.

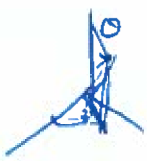
Pb remains: Kinetic theory in gal no H-Thm guarantees convergence towards global (or local even) eq state $\Longrightarrow$ Σ can be trapped in q.s state.

Pb: Kinetic theory to dominant order yields collisionless Boltzmann

$$\partial_t f_1 + \langle u \rangle \partial_x f_1 = 0 .$$

To have collisions appear one needs to take into account finite-N effects $\Longrightarrow$ more subtle!

6b) Unit sphere: $H_d = -\dfrac{1}{4\pi} \sum_{\alpha \neq \beta} \Gamma_\alpha \Gamma_\beta \log(1 - \cos\gamma_{\alpha\beta})$



$$\cos\gamma_{\alpha\beta} = \cos\theta_\alpha \cos\theta_\beta + \sin\theta_\alpha \sin\theta_\beta \cos(\phi_\alpha - \phi_\beta) .$$

colatitude longitude

Canonical variables: $\left| \begin{array}{l} q_\alpha = \Gamma_\alpha \cos\theta_\alpha . \\ p_\alpha = \phi_\alpha . \end{array} \right.$