The Kac Ring model

cf Wannier - Statistical Physics § 18.6. LO

Simple Deterministic setting to illustrate the relevance (or not) of the Chaotic Hypothesis

Setting

- n points $P_0 \dots P_{n-1}$ on a circle, say parametrised by their polar angle $\theta(i) = 2i\pi/n$ $\quad i \in [\![0 ; n-1]\!]$
- m markers at angle $\theta(i_1) \dots \theta(i_m)$ $\quad 0 \leq i_1 < \dots < i_m \leq n-1$
- At angles $\varphi(i) = 2\left(i + \frac{1}{2}\right)\frac{\pi}{n}$ $\quad i \in [\![0 ; n-1]\!]$ $\quad n$ balls

  Among which $W(0)$ are ~~black~~ white. $\quad$ | $\quad$ $B^{(0)} + W^{(0)} = n$.

  $B^{(0)}$ are black.

- Dynamics: $t \to t+1$ balls move collectively to the left

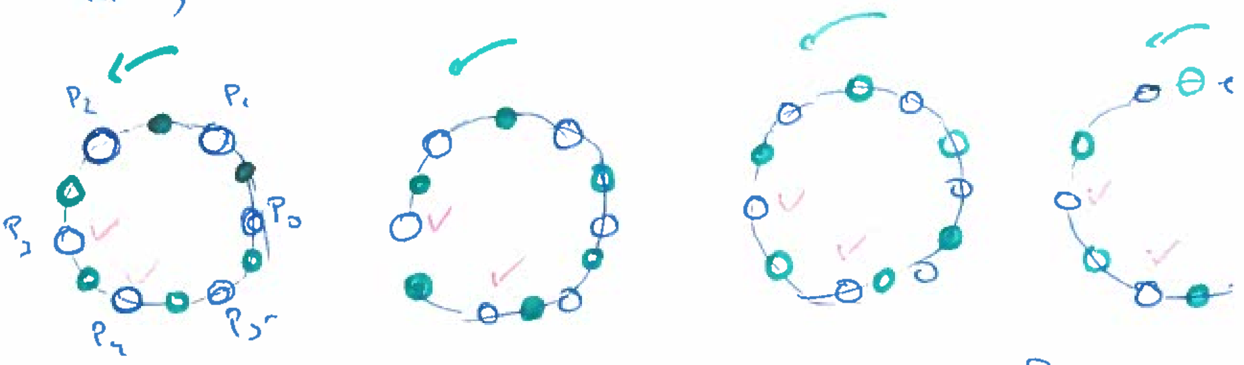
  $\varphi(i) \to \varphi(i+1)$

  if $i+1 \in \{i_1 \dots i_m\}$, i.e. if there is a marker in front of them, the balls change color.

Example:

$n = 6$

$m = 2$ (✓)



Question: number of white and black balls at time t?

(1) Approach through the chaotic hypothesis.

Let $w(t)$ the number of white balls with markers in front of them at t

$b(t)$ —— black ——

- Write evolution equation for $W(t)$ and $B(t)$.

- Using the hypothesis that $\dfrac{w(t)}{W(t)} = \dfrac{b(t)}{B(t)} = \mu = \dfrac{m}{n}$ independently of t, solve the equation evolution.

- Discuss the case $t \to \infty$.

- Discuss the reverse process where balls move collectively to the right, and relevance of the chaotic hypothesis in that case.

(2) Exact approach.

- Introduce $\varepsilon_i = 1$ if $i \notin \{i_1 - i_m\}$. (marker/no marker @ p_i?)
  $= -1$ otherwise.

  and $y_i(t) = +1$ if ball @ $p(i)$ is black at time t.
  $= -1$ otherwise.

- Write equation for $\{y_i(t)\}_{i \in [0, n-1]}$ as a function of $\{y_i(0)\}_{i \in [0, n-1]}$.

Stochastic setting:

- Instead of a single experiment, one then consider an ensemble of dynamics with same initial condition but stochastic distribution of markers. Specifically, for each realization of the dynamics, points P_i have probability p to be marked.

- Show that each realization of the dynamics is periodic with period $2n$.

- Show that $\langle B(t) - W(t) \rangle = (1-2p)^{n-|n-t|} \langle B(0) - W(0) \rangle$.

  (for $t \leq 2n$).

Discuss the relevance of the chaotic hypothesis in the light of this exact result, as $n \to \infty$.