

# Classical mechanics (brief intro.)

Objects: material points (size is negligible)

Description: time  $t$ , coordinates  $r = (x, y, z)$

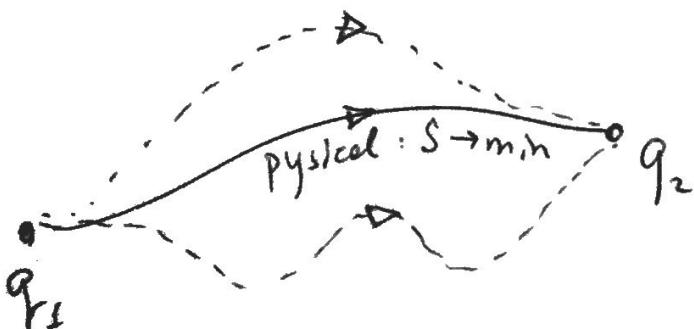
$N$  particles (material points):  $r_1, \dots, r_N$

Speed:  $v = \frac{dr}{dt} = \dot{r}$ , Acceleration:  $a = \frac{d^2 r}{dt^2} = \ddot{r}$

Least action principle (Hamilton's principle)

The action  $S := \int_{t_1}^{t_2} L(\underline{q}, \dot{\underline{q}}, t) dt \rightarrow \min$

for a physical trajectory  $\underline{q}(t) = (r_1(t), \dots, r_N(t))$   
among all functions with fixed end  
points  $\underline{q}(t_1) = q_1$  and  $\underline{q}(t_2) = q_2$



Obs:  $L$  is called  
the Lagrangian  
function.

## Lagrange's equations

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$S \rightarrow \min \Rightarrow \delta S = 0$  ~~from~~ for any infinitesimal perturbation  $\delta q(t)$  with  $\delta q(t_1) = 0$  and  $\delta q(t_2) = 0$ .

$$\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt = \text{(case } q \in \mathbb{R})$$

$$= \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} \delta q + \underbrace{\frac{\partial L}{\partial \dot{q}} \delta \dot{q}}_{\text{by parts}} \right] dt =$$

$$= \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right] \delta q dt = 0$$

$$\forall \delta q \Rightarrow \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0}$$

Obs :  $\frac{d}{dt}$  is the material derivative  $= \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} + \ddot{q} \frac{\partial}{\partial \dot{q}}$

Obs: In general, there are  $N$  eqs, for each  $q_i$ :

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0, \quad a = 1, \dots, N.$$

(exercise)

Obs Gauge invariance  $\rightarrow$  non-uniqueness of  $L$ .

For any  $L' = L + \frac{df(q,t)}{dt} := L + \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial t}$ ,

equations of motion do not change:

$$\begin{aligned} S' &= \int_{t_1}^{t_2} L' dt = \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \frac{\partial f}{\partial t} dt = \cancel{L(t_1, q_1, \dot{q}_1)} - \\ &= S + \underbrace{f(q_2, t_2) - f(q_1, t_1)}_{\text{fixed in the Ham. princ.}} \rightarrow \min \text{ for the same } q(t). \end{aligned}$$

### Space-time symmetries:

- |                |   |
|----------------|---|
| Galilean group | homogeneity of space and time $\Leftrightarrow$ all points are equivalent<br>$(r \mapsto r + r_0, t \mapsto t + t_0)$ |
|                | isotropy of space (rotations) $\Leftrightarrow$ all directions are equivalent   |
|                | Galilean transformation to a moving frame<br>$(r \mapsto t + Vt)$   |

Galilean principle: physical laws are invariant w.r.t. the Galilean group.

The corresponding coordinate systems  $(x, y, z, t)$  are called inertial frames of reference.

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Lagrangian  $L(r, \dot{r}, t)$  for free mat. point:

Homogeneity  $\Rightarrow L = L(\dot{r})$  no dep. on  $r$  and  $t$

Isotropy  $\Rightarrow L = L(v)$ ,  $v = \|\dot{r}\|$ , no dep. on directions.

Galilean transform (with gauge symmetry)

$$\Rightarrow \boxed{L = \frac{m}{2} v^2}, \text{ where } m > 0 \text{ is called mass.}$$

Ob  $m=0$  or  $m < 0$  do not lead to  $S \rightarrow \min.$

System of  $N$  particles

$$q = (r_1, \dots, r_N) \in \mathbb{R}^{3N}, \quad v_a = \|\dot{r}_a\|.$$

$$L = \underbrace{\sum_{a=1}^N \frac{m_a}{2} v_a^2}_{\text{kinetic energy}} - \underbrace{U(q)}_{\text{potential energy}}$$

The potential energy  $U(r_1, \dots, r_N)$  describes interaction among particles and it must be homogeneous ( $r_a \rightarrow r_a + r_0$ ) and isotropic (rotations)

For example :  $U = \sum_{1 \leq a < b \leq N} U_2(\|r_a - r_b\|)$  - pair interactions

# Newton's laws

I free particle .  $L = \frac{mv^2}{2} = \frac{m}{2}(\dot{r}, \dot{r})$

$$\dot{r} = (\dot{x}, \dot{y}, \dot{z})$$

E.-L. equations :  $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$  , etc. for  $y$  and  $z$ .

$$\Rightarrow m\ddot{x} = 0 , m\ddot{y} = 0 , m\ddot{z} = 0 \Rightarrow \boxed{\dot{r} = \text{const}}$$

motion with constant speed and direction.

II  $L = \sum_{a=1}^N \frac{m_a v_a^2}{2} - U$

~~missed~~

Euler-Lagrange eqs:  $m_a \ddot{x}_a + \frac{\partial U}{\partial x_a} = 0$ , etc.

or  $\boxed{m_a \ddot{r}_a = F_a}$  with the force

$$F_a := - \frac{\partial U}{\partial r_a} := \left( -\frac{\partial U}{\partial x_a}, -\frac{\partial U}{\partial y_a}, -\frac{\partial U}{\partial z_a} \right)$$

III In a two-particle system  $L = \sum_{a=1}^2 \frac{m_a v_a^2}{2} - U(|r_1 - r_2|)$

$$\Rightarrow \boxed{F_1 = -\frac{\partial U}{\partial r_1} = \frac{\partial U}{\partial r_2} = -F_2}$$

For every action there is an equal and opposite reaction.

## Conservation laws

A diffeomorphism  $q' = h^s(q)$  depending on a parameter  $s \in \mathbb{R}$  (or  $S^1$  - circle) is a one-parameter symmetry group, if:

$$h^{s_1+s_2} = h^{s_1} \circ h^{s_2} \quad (\text{group})$$

and

$$L(q) \equiv L(h^s(q)) \quad (\text{symmetry})$$

### Theorem (Noether)

For any one-param. sym. group, there exists an invariant

$$I := \left( \frac{\partial L}{\partial \dot{q}}, \frac{\partial h^s}{\partial s} \Big|_{s=0} \right), \quad (\text{scalar product})$$

which is conserved along all physical trajectories  
 $q = q(t)$ .

Proof - exercise!

Classical conservation laws:

① Linear momentum  $P = (P_x, P_y, P_z)$ .

homogeneity of space  $\Rightarrow x' = x + s, \quad h^s(q) = (r'_1, \dots, r'_N)$   
 with  $r'_a = (x_a + s, y_a, z_a)$

$\frac{\partial h^s}{\partial s} \Big|_{s=0} = (1, 0, 0, 1, 0, 0, \dots) \Rightarrow P_x = I := \sum_{a=1}^N m_a \dot{r}_a$   
 and similarly for  $y$  and  $z$ .

$$P = \sum_{a=1}^N m_a \dot{r}_a$$

total momentum is conserved!

II Angular momentum  $M = (M_x, M_y, M_z)$

$M := \sum_{a=1}^N m_a \mathbf{r}_a \times \dot{\mathbf{r}}_a \Leftarrow$  isotropy and Noether's theorem applied to rotations around axes  $x, y$  and  $z$ .  
(optional exercise)

III Noether's theorem can be extended to the symmetry groups that change both  $\mathbf{q}$  and  $t$ .

For example, homogeneity of time implies

$$L = L(\mathbf{q}, \dot{\mathbf{q}}) \quad - \text{independence of } t.$$

$\Rightarrow$  there is an invariant

$$E := \left( \frac{\partial L}{\partial \dot{\mathbf{q}}}, \dot{\mathbf{q}} \right) - L \quad \text{called } \underline{\text{Energy}}$$

Proof: exercise.

In the case of  $N$  particles:

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = (m_1 \dot{\mathbf{r}}_1, m_2 \dot{\mathbf{r}}_2, \dots, m_N \dot{\mathbf{r}}_N), \dot{\mathbf{q}} = (\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N)$$

$$\Rightarrow E = \sum_{a=1}^N m_a v_a^2 - L = \sum_{a=1}^N \frac{m_a v_a^2}{2} + U$$

total energy (kinetic + potential) is conserved!

## Discrete symmetries

(T) Time reversibility : if  $q(t)$  is a solution, then  $q(-t)$  is also a solution.

This holds for S.L. equations with  $L = \sum_a \frac{m_a v_a^2}{2} - U$   
(check!)

(P) Parity : if  $q(t) = (r_1(t), \dots, r_N(t))$  is a solution, then ~~if~~  $q_p(t) = (-r_1(t), \dots, -r_N(t))$  is a solution.

This holds for many (but not all) physical theories.

Exercise : suggest a potential  $U$  for a system, which is not P-symmetric

## Center of mass

~~R :=~~ 
$$\frac{\sum_{a=1}^N m_a r_a}{\sum_{a=1}^N m_a}$$

$$\dot{R} = \frac{P}{M_{\text{total}}} \left( \frac{\text{momentum}}{\text{total mass}} \right) = \text{const}$$

(center of mass moves with const. speed)

Kinetic energy :  $\frac{1}{2} \sum_{a=1}^N \frac{m_a v_a^2}{2} = \frac{m_{\text{tot}} |\dot{R}|^2}{2} + \sum_{a=1}^N \frac{m_a \tilde{v}_a^2}{2}$

(exercise)

where  $\tilde{v}_a = \|\dot{r}_a - \dot{R}\|$  is a relative speed.

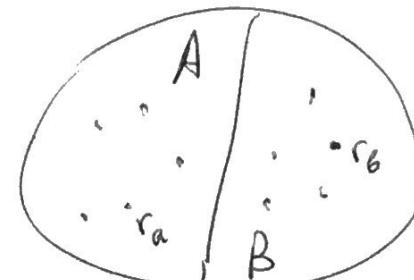
# Work done by external forces on a system

Consider a system splitted into 2 parts :  $q = (q_A, q_B)$

Lagrangian :  $L = L_A + L_B - U_{int}$

with  $L_A = \sum_{a \in A} \frac{m_a v_a^2}{2} - U_A(q_A)$

$$L_B = \sum_{b \in B} \frac{m_b v_b^2}{2} - U_B(q_B)$$



$$\begin{aligned} q_A &= \{r_a\} \\ q_B &= \{r_b\} \end{aligned}$$

and the interaction potential  $U_{int}(q_A, q_B)$ .

Note that  $U = U_A + U_B + U_{int}$ .

From the 2<sup>d</sup> Newton's law :

$$m_a \ddot{r}_a = F_a = - \frac{\partial U}{\partial r_a} = - \frac{\partial U_A}{\partial r_a} - \frac{\partial U_{int}}{\partial r_a} \text{ for } a \in A.$$

Hence, for the energy of system A, we have

$$E_A := \sum_{a \in A} \frac{m_a v_a^2}{2} + U_A(q_A)$$

and

$$\begin{aligned} \frac{dE_A}{dt} &\stackrel{\text{use}}{=} \sum_{a \in A} \left( m_a \ddot{r}_a + \frac{\partial U_A}{\partial r_a}, \dot{r}_a \right) = \\ &\stackrel{v_a^2 = (\dot{r}_a, \dot{r}_a)}{=} \sum_{a \in A} \left( - \frac{\partial U_{int}}{\partial r_a}, \dot{r}_a \right) \end{aligned}$$

Hence,  $\frac{dE_A}{dt} = \sum_{a \in A} (F_a^{\text{ext}}, \dot{r}_a)$ ,  $F_a^{\text{ext}} = - \frac{\partial U_{int}}{\partial r_a}$  (external force)

$$E_A(t_2) - E_A(t_1) = W := \int_{t_1}^{t_2} \sum_{a \in A} (F_a^{\text{ext}}, \dot{r}_a) dt = \int_{t_1}^{t_2} \sum_{a \in A} (F_a^{\text{ext}}, dr_a) \quad \begin{array}{l} \text{WORK} \\ \text{External force} \times \text{displacement} \end{array}$$

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## Hamilton's equations

Consider a Lagrangian  $L = \sum_a \frac{m_a \dot{r}_a^2}{2} - U(r_1, \dots, r_N, t)$

(we added time  
for a more general  
setting)

Momentum of particle :  $p_a := \frac{\partial L}{\partial \dot{r}_a} = m_a \dot{r}_a$

Hamiltonian function :

$$H(r_1, \dots, r_N, p_1, \dots, p_N) := \underbrace{\sum_a (p_a, \dot{r}_a)}_{m_a \dot{r}_a^2} - L =$$

$$= \underbrace{\sum_a \frac{p_a^2}{2m_a}}_{\text{kinetic energy}} + \underbrace{U(r_1, \dots, r_N, t)}_{\text{potential energy}}$$

Hamilton's equations :

$$\dot{r}_a = \frac{p_a}{m} = \frac{\partial H}{\partial p_a} \quad , \quad \dot{p}_a = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}_a} \stackrel{\text{E.L.}}{=} \frac{\partial L}{\partial r_a} = - \frac{\partial U}{\partial r_a} = \frac{\partial H}{\partial r_a}$$

$\dot{r}_a = \frac{\partial H}{\partial p_a} \quad , \quad \dot{p}_a = - \frac{\partial H}{\partial r_a}$

H. eqs are symmetrized w.r.t. positions and speeds :  
 $(r_a, p_a, H) \leftrightarrow (p_a, r_a, -H)$ .

Obs In general,  $L = L(q, \dot{q}, t)$ ,  $p := \frac{\partial L}{\partial \dot{q}}$  (11)

$H(q, p, t) = (p, \dot{q}) - L$  with  $\dot{q}$  expressed in terms of  $p$  and  $q$  from the Eq.

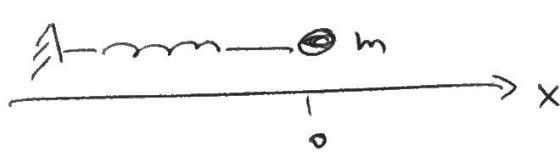
Hamilton's eqs:  $\dot{q} = \frac{\partial H}{\partial p}$ ,  $\dot{p} = -\frac{\partial H}{\partial q}$

In our case:  $q = (r_1, \dots, r_N)$ ,  $p = (p_1, \dots, p_N) \in \mathbb{R}^{3N}$

Obs Hamiltonian function is equal to the energy, which is conserved, IF the Hamiltonian does not depend explicitly on time,  $H = H(q, p)$ .

$$\frac{\partial H}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = \left(\frac{\partial H}{\partial q}, \dot{q}\right) + \left(\frac{\partial H}{\partial p}, \dot{p}\right) = (-\dot{p}, \dot{q}) + (\dot{q}, \dot{p}) = 0$$

Example (linear oscillator)



$$L = \frac{m\dot{x}^2}{2} - U(x)$$

$$U(x) = \frac{kx^2}{2}, \quad k > 0 \text{ is a stiffness coefficient}$$

Euler-Lagrange eq:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \omega^2 x = 0$$

$$\omega = \sqrt{k/m}$$

Solution:  $x(t) = A \cos \omega t + B \sin \omega t = C \cos(\omega t + \phi)$

Conserves the energy  $E = \frac{m\dot{x}^2}{2} + U$ . (Not momentum - why?)

Momentum  $p = m\dot{x}$ , Hamiltonian  $H = \frac{p^2}{2m} + \frac{kx^2}{2}$ .

Hamilton's eqs:  $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$ ,  $\dot{p} = -\frac{\partial H}{\partial x} = -kx$ .