

Macroscopic vs. microscopic (momentum, ang. mom.)

Isolated system (no box, no walls) has
7 additive invariants:

- energy $E = H(p, q) \in \mathbb{R}$
- momentum $\sum p_a \in \mathbb{R}^3$, $p_a = m_a \dot{r}_a$
- angular momentum $\sum r_a \times p_a \in \mathbb{R}^3$

These are also approximate invariants for any macroscopic subsystem.

Momentum: Consider a small macroscopic subsystem.

Center-of-mass velocity: $v_s := \frac{p_s}{m_s} \cancel{\text{def.}}$,

where $p_s = \sum_{a \in \text{subsys.}} p_a$, $m_s = \sum_{a \in \text{subsys.}} m_a$.

For identical particles: $m_a = m \Rightarrow v_s = \frac{\sum_{a \in \text{subsys.}} \dot{r}_a}{N_s}$,

the arithm. mean velocity.

Obs: $\underline{v_s} \approx \text{const}$ is conserved approximately in the equilibrium, with fluctuations, which are very small relative to the mean.

Galilean invariance: eqs. of motion are identical in inertial frames: $r_a = r'_a + v_s t$, $\dot{r}_a = \dot{r}'_a + v_s$

$$\Rightarrow H(p, q) = \frac{p_s^2}{2m_s} + H(p', q') \quad (\text{see exercise about kinetic energy in mech})$$

$\underbrace{\qquad\qquad\qquad}_{\text{same Hamiltonian}}$

Hence, statistical properties in the center-of-mass frame should not depend on the total mom. P_s .

Then, internal energy is defined as

$U_s := \overline{H(p^i, q^i)}$, the mean energy in the center-of-mass reference frame.

Angular momentum:

For a subsystem of size $L \times L \times L$, we have

$$N_s \sim L^3, E_s \sim \sum m \dot{r}_a^2 \sim N_s \sim L^3 \quad (\text{ftftft})$$

$$P_s \sim \sum r_a \dot{\phi} \sim N_s \sim L^3 \quad (\text{obs: } \dot{r}_a \sim V, r_a \sim L)$$

$$\text{ang. momentum} = \sum r_a \times p_a \sim L^4 \quad (r_a \sim L)$$

Hence, for a small subsystem $\frac{\text{ang. mom}}{\text{energy}} \rightarrow 0$, and we can neglect ang. momentum dependence for thermod. quantities such as ~~the~~ the internal energy, etc.

Obs Recall that $\underbrace{\log \rho}_{\substack{\text{probab.} \\ \text{density in} \\ \text{phase space.}} \sim \underbrace{\alpha + \beta E + \gamma M_x + \delta_1 M_y + \delta_2 M_z}_{\substack{\text{additive} \\ \text{function of energy and} \\ \text{angular momentum.}}}$

For a system not in ~~a~~ full equilibrium (e.g. slow motion), we can divide the system in small macroscopic subsystems, such that each subsystem is in local equilibrium. ~~The total subsystem is described by~~

State of a subsystem in local equilibrium is described by

- (1) position $r \in \mathbb{R}^3$ and center-of-mass velocity $v \in \mathbb{R}^3$
- (2) full set of thermodynamic variables, which determine ~~is~~ a local equilibrium. For example, temperature T and density ρ (for mixtures, also concentrations, etc.).

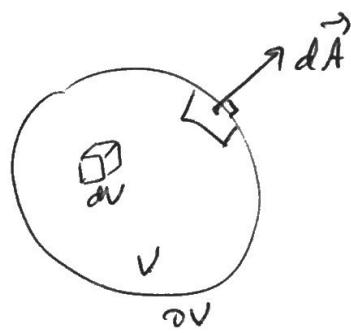
Fluids dynamics models macroscopic motion of fluid (or gas) with such variables:

$$v(r, t), \rho(r, t) \text{ and } T(r, t).$$

Continuity equation (mass conservation)

$$\underbrace{\frac{\partial}{\partial t} \int_V \rho dV}_{\text{change of mass}} = - \underbrace{\oint_{\partial V} \rho v \cdot d\vec{A}}_{\text{flow through the surface } \partial V}$$

(mass flux density)



fixed volume V
in physical space \mathbb{R}^3

Using the formulae $\oint_{\partial V} \rho v \cdot d\vec{A} = \int_V dN(\rho v) dV$,

$\int_V \left(\frac{\partial \rho}{\partial t} + dN(\rho v) \right) dV = 0$ for any fixed volume V . Hence,

$$\boxed{\frac{\partial \rho}{\partial t} + dN(\rho v) = 0}$$

Continuity equation.

The Euler equation (ideal fluid flow)

54

Ideal flow is a non-dissipative (reversible) flow. This is an approximation valid for sufficiently slow ~~fast~~ motion (adiabatic process). In this case, we can assume that (1) the system is in local equilibrium at any position $r \in \mathbb{R}^3$ and (2) there is no entropy production.

For a small ~~page~~ fluid parcel:

$$\int \underbrace{\rho dV}_{\text{mass}} \underbrace{\frac{dv}{dt}}_{\text{acceleration}} = \text{Force} := - \oint p \vec{dA} \quad (2^{\text{nd}} \text{ Newton's law})$$

pressure



~~the Stokes'~~ ~~Raman's~~ formula: $\oint p \, dA = \int \nabla p \, dV$



$$\int \left(\rho \frac{dv}{dt} + v \cdot \nabla p \right) dv = 0 \quad \Rightarrow \quad \int v \frac{dv}{dt} = -\nabla p.$$

$$\text{Material der Nat. Ne: } \frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} + \frac{\partial V}{\partial z} \dot{z} = \frac{\partial^2 V}{\partial t^2} + (\mathbf{v} \cdot \nabla) V$$

Euler's equation: $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}$

Here \mathbf{g} is any extra (body) force per unit mass,
e.g. gravity.

Isentropic condition

(55)

Adiabatic process implies that entropy is approximately conserved in any subsystem. For the entropy density s (per unit of mass), this conservation condition is written as

$$\frac{ds}{dt} = 0 \quad \text{for the material derivative } \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

Hence,

$$\boxed{\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0}$$

entropy
conservation eq.

Usually, one assumes that $s = \text{const}$ in the whole volume of fluid (isentropic motion).

In equilibrium, $S = S(P, V, N)$ - thermodynamics, where S , P and V are proportional to N . Hence,

$s := \frac{S}{mN} = s(p, g)$ is a function of pressure and density. $s = \frac{mN}{V}$.

Isentropic condition $s(p, g) = s_0 (\text{const})$ yields the constitutive law $P = P(g)$.

Similarly, one can write $s(T, g) = s_0 \Rightarrow T = T(g)$, i.e., all thermodynamic variables are determined by the density g (for a given fixed s_0).

For example : in ideal gas, $P \propto g^\delta$ with $\delta = C_p/C_v$.
Also, ~~T~~ $T \propto g^{\delta-1}$. (Exercise : derive.)

(56)

Enthalpy: $H_{(SP)} = U + PV$, $dH = Tds + Vdp$.

Enthalpy per unit mass $h = \frac{H}{mN}$

Internal energy per unit mass, $e = \frac{U}{mN}$

Density : $\rho = \frac{mN}{V}$

Hence, $h = e + \frac{P}{\rho}$ and $dh = Tds + \frac{1}{\rho} dp$

For isentropic flow $ds = 0 \Rightarrow dh = \frac{dp}{\rho} \Rightarrow \nabla h = \frac{\nabla p}{\rho}$

Another form of Euler's equation: $\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla h + \vec{g}$

Boundary conditions:

on a rigid wall: $v_n = (\vec{v}, \vec{n}) = 0$ (no penetration)

normal vector to a boundary
normal speed component

With $v_n = 0$, there is no need for ^{extra} conditions for p or s .

Ques Other possibilities: moving boundary or moving interface between different fluids.

Moving boundary with normal speed $\vec{v}_6 \parallel \vec{n} = v_6 \vec{n}$:

B.C. \Rightarrow

$$(\vec{v} - \vec{v}_6, \vec{n}) = 0$$

no penetration

Exercise:

Derive B.C. for interface

(extra condition for equality of pressures)

(57)

Conservation of energy in ideal flow

$$\cancel{\frac{\partial}{\partial t} \left(\frac{\rho v^2}{2} + \rho e \right)} + \cancel{dV} \left[\rho v \left(\frac{v^2}{2} + h \right) \right] = 0$$

total energy per unit mass energy flux density

Here e is the internal energy per unit mass, $h = e + \frac{P}{\rho}$ is the enthalpy.

Proof : We will use

$$(1) \frac{\partial s}{\partial t} + v \cdot \nabla s = 0, \quad (2) \frac{\partial P}{\partial t} + dV(\rho v) = 0, \quad (3) \frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{\nabla P}{\rho}$$

(no body forces)

Also $dU = TdS - PdV$ per unit mass (4) $de = Tds + \frac{P}{\rho^2} dP$,
 $dH = TdS + VdP$ $\left(\vec{V} = \frac{m}{\rho} \vec{v} \right) \rightarrow$ (5) $dh = Tds + \frac{1}{\rho} dP$.

Hence,

$$\frac{\partial}{\partial t} (\rho e) \stackrel{(4)}{=} \rho T \frac{\partial s}{\partial t} + \frac{P}{\rho} \frac{\partial P}{\partial t} + e \frac{\partial P}{\partial t} = \rho T \frac{\partial s}{\partial t} + \cancel{e} \underbrace{\left(e + \frac{P}{\rho} \right)}_h \frac{\partial P}{\partial t} \stackrel{(1,2)}{=} \\ = -\rho T v \cdot \nabla s - h dV(\rho v). \quad (A)$$

$$\frac{\partial}{\partial t} \left(\frac{\rho v^2}{2} \right) \stackrel{(2,3)}{=} -\frac{v^2}{2} dV(\rho v) - \rho v \cdot (v \cdot \nabla v) - v \cdot \nabla P \quad (B)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\rho v^2}{2} + \rho e \right) &\stackrel{(A+B)}{=} - \cancel{\left(\frac{v^2}{2} + h \right)} dV(\rho v) - \rho v \cdot \underbrace{\left(T \nabla s + \frac{1}{\rho} \nabla P \right)}_{\nabla h} - \\ &- \rho v \cdot (v \cdot \nabla v) = -dV(\rho v h) - \underbrace{\left[\frac{v^2}{2} dV(\rho v) + \rho v \cdot (v \cdot \nabla v) \right]}_{\text{check that this} = dV(\rho v \frac{v^2}{2})} \end{aligned}$$

Obs We did not use the extra assumption that $s = \text{const.}$!