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BRIEF COMMUNICATIONS

# ON TANGENT CONES TO THE STABILITY DOMAIN OF A FAMILY OF REAL MATRICES

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A smooth $n$-parameter family of real $(m \times m)$-matrices $A(p)$ $(p \in \boldsymbol{R}^n)$ is considered. A stability domain is a set of values of the parameter vector $p$ such that all the eigenvalues $\lambda$ of the matrix $A(p)$ have negative real parts. A tangent cone to the stability domain at a point of its boundary is called a set of direction vectors $e = dp/d\varepsilon\big|_{\varepsilon=0}$ of the curves $p(\varepsilon)$ $(\varepsilon > 0)$ which start at this point and lie in the stability domain [1]. A description of tangent cones (up to a nondegenerate linear transformation) is given in [1] depending on the Jordan structure of the matrix $A$. In our paper, the tangent cones are explicitly constructed by the first derivatives of the matrix $A$ with respect to the parameters $p_j$ $(j = 1, \ldots, n)$ and by its eigenvectors and associated vectors calculated at the boundary point being considered.

Let us consider a point $p = p_0$ of the boundary of the stability domain such that the matrix $A_0 = A(p_0)$ has the zero eigenvalue $\lambda = 0$ (only one Jordan block of order $k$ corresponds to it) and the other eigenvalues have negative real parts. Let $u_0, u_1, \ldots, u_{k-1}$ and $v_0, v_1, \ldots, v_{k-1}$ be the right and left eigenvectors and associated vectors corresponding to $\lambda = 0$ ($A_0u_0 = 0$, $A_0u_1 = u_0$, $\ldots$, $A_0u_{k-1} = u_{k-2}$, $v_0^TA_0 = 0$, $v_1^TA_0 = v_0^T$, $\ldots$, $v_{k-1}^TA_0 = v_{k-2}^T$) and satisfying the normalization conditions $v_0^Tu_{k-1} = 1$, $v_j^Tu_{k-1} = 0$ $(j = 1, \ldots, k-1)$. If $u_0, \ldots, u_{k-1}$ are given, then the vectors $v_0, \ldots, v_{k-1}$ satisfying this normalization are uniquely determined. According to [2, 3], an arbitrary smooth family of the matrices $A(p)$ is represented in the neighborhood of $p = p_0$ as follows:

$$A(p) = C(p)\,A'\big(\varphi(p)\big)\,C^{-1}(p) \tag{1}$$

Here $C(p)$ and $\varphi(p) = (\varphi_1, \ldots, \varphi_d)^T$ are an $m \times m$-matrix and a $d$-vector (both smoothly dependent on $p$) such that $\det C(p) \neq 0$ and $\varphi(p_0) = 0$. A family of matrices $A'(p')$, $p' = (p_1', \ldots, p_d')^T \in \boldsymbol{R}^d$ (the versal deformation of the matrix $A_0$) may be chosen in the form $A'(p') = J_0 + B(p')$, where $J_0$ is the Jordan upper triangular form of the matrix $A_0$ and $B(p')$ is a family of block diagonal matrices depending on the structure of $J_0$. The characteristic equations for $A(p)$ and $A'\big(\varphi(p)\big)$ are the same. In the case being considered, the first block of the family $A'(p')$ corresponding to $\lambda = 0$ is as follows [2–4]:

$$\begin{pmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ & \cdots & \cdots \\ 0 & 0 & 0 \cdots 1 \\ 0 & 0 & 0 \cdots 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ p_1' & p_2' & p_3' & \cdots & p_k' \end{pmatrix} \tag{2}$$

In a neighborhood of the point $p' = \varphi(p_0) = 0$, stability of the matrix $A'(p')$ is determined by block (2) (the other blocks correspond to the eigenvalues with negative real parts; therefore, for eigenvalues of these blocks the inequality $\operatorname{Re}\lambda < 0$ holds in a sufficiently small neighborhood of zero). The characteristic equation for (2) has the form $\lambda^k - p_k'\lambda^{k-1} - \cdots - p_1' = 0$. Using the results given in [1], we determine the tangent cone $K_0'$ to the stability domain of $A'(p')$ at the point $p' = 0$ ($e'$ is a direction in $\boldsymbol{R}^d$):

$$K_0' = \left\{e' = (e_1', \ldots, e_d')^T : \; e_1' = \cdots = e_{k-2}' = 0, \; e_{k-1}' \leq 0, \; e_k' \leq 0\right\} \tag{3}$$

Now we introduce the following vectors $f_j \in \boldsymbol{R}^n$ $(j = 0, \ldots, k-1)$:

$$f_j = \left(\sum_{r=0}^{j} v_r^T \frac{\partial A}{\partial p_1} u_{j-r}, \; \sum_{r=0}^{j} v_r^T \frac{\partial A}{\partial p_2} u_{j-r}, \; \ldots, \; \sum_{r=0}^{j} v_r^T \frac{\partial A}{\partial p_n} u_{j-r}\right)^T \tag{4}$$

Here the derivatives are taken at the point $p_0$. Similarly, we introduce the vectors $f'_j \in \boldsymbol{R}^d$ $(j = 0, \dots, k-1)$ for the family $A'(p')$ at the point $p' = 0$.

**Lemma.** *The vectors $f_j$ and $f'_j$ $(j = 0, \dots, k-1)$ are related as follows:*

$$f_j^T = f_j'^T D_\varphi \quad (j = 0, \dots, k-1) \tag{5}$$

*Here $D_\varphi$ is a Jacobi $(d \times n)$-matrix with the elements $d\varphi_r/dp_s$ $(r = 1, \dots, d;\ s = 1, \dots, n)$.*

In order to prove the lemma, we should substitute (1) into the expression for the vector $f_j$ and use properties of eigenvectors and associated vectors. It can be shown by direct calculations that $f'_j = (0, \dots, 0, 1, 0, \dots, 0)^T$, where 1 is in the $(j+1)$th position.

The directions $e \in \boldsymbol{R}^n$ and $e' \in \boldsymbol{R}^d$ are related by

$$e' = D_\varphi e \tag{6}$$

Let $p(\varepsilon)$ be an arbitrary curve drawn along the direction $e$ toward the stability domain. Then the curve $p'(\varepsilon) = \varphi(p(\varepsilon))$ with the direction $e'$ determined by (6) belongs to the stability domain, too. If the vectors $f_j$ $(j = 0, \dots, k-1)$ are linearly independent, then it can be shown (with the help of the lemma and the implicit function theorem applied to the relation $p' - \varphi(p) = 0$) that there exists a curve lying in the stability domain with an arbitrary direction $e$ such that $e' = D_\varphi e$, $e' \in K'_0$. Multiplying (5) scalarly by $e$ and taking into account (6), we obtain $(f_j, e) = (f'_j, e') = e'_{j+1}$. Then, using (3), we determine the following tangent cone to the stability domain of $A(p)$ at the point $p = p_0$:

$$K_0 = \{e: \ (f_0, e) = \dots = (f_{k-3}, e) = 0, \ (f_{k-2}, e) \le 0, \ (f_{k-1}, e) \le 0\} \tag{7}$$

In a similar manner we can consider the boundary point $p = p_0$ of the stability domain where the matrix $A_0 = A(p_0)$ has one complex-conjugate pair of the purely imaginary eigenvalues $\lambda = \pm i\omega$ such that only one Jordan block of order $k$ corresponds to each of them (for the other eigenvalues the inequality $\operatorname{Re}\lambda < 0$ holds). Suppose $g_j$ and $h_j$ $(j = 0, \dots, k-1)$ are vectors such that

$$g_j + ih_j = \left( \sum_{r=0}^{j} v_r^T \frac{\partial A}{\partial p_1} u_{j-r}, \ \sum_{r=0}^{j} v_r^T \frac{\partial A}{\partial p_2} u_{j-r}, \ \dots, \ \sum_{r=0}^{j} v_r^T \frac{\partial A}{\partial p_n} u_{j-r} \right)^T \tag{8}$$

where $u_j$ and $v_j$ $(j = 0, \dots, k-1)$ are the right and left eigenvectors and associated vectors corresponding to $\lambda = i\omega$ $(A_0 u_0 = i\omega u_0, \ \dots, \ A_0 u_{k-1} = i\omega u_{k-1} + u_{k-2}, \ v_0^T A_0 = i\omega v_0^T, \ \dots, \ v_{k-1}^T A_0 = i\omega v_{k-1}^T + v_{k-2}^T)$ and satisfying the normalization conditions $v_0^T u_{k-1} = 1$, $v_j^T u_{k-1} = 0$ $(j = 1, \dots, k-1)$; the derivatives are calculated at $p = p_0$. Instead of (2), we consider the block [2–4]

$$\begin{pmatrix} i\omega & 1 & \cdots & 0 \\ 0 & i\omega & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & i\omega \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ p'_1 + ip'_2 & p'_3 + ip'_4 & \cdots & p'_{2k-1} + ip'_{2k} \end{pmatrix} \tag{9}$$

corresponding to $\lambda = i\omega$ (the block corresponding to $\lambda = -i\omega$ consists of numbers which are complex conjugate to the elements in (9)). Then we can prove properties (5) and (6) for the vectors $g_j$, $g'_j$, $h_j$, $h'_j$ and the directions $e$, $e'$. The tangent cone to the stability domain of the family $A'(p')$ at the point $p' = 0$ is determined by the characteristic equation of block (9)

$$\mu^k - (p'_{2k-1} + ip'_{2k})\,\mu^{k-1} - \dots - (p'_1 + ip'_2) = 0, \quad \mu = \lambda - i\omega$$

and has the form [1]

$$K'_{i\omega} = \{e': \ e'_1 = \dots = e'_{2k-4} = 0, \ e'_{2k-3} \le 0, \ e'_{2k-2} = 0, \ e'_{2k-1} \le 0\}$$

If the vectors $g_j$ and $h_r$ $(j = 0, \dots, k-1;\ r = 0, \dots, k-2)$ are linearly independent, then for the family $A(p)$ at the point $p = p_0$ we obtain

$$K_{i\omega} = \{e: \ (g_0, e) = \dots = (g_{k-3}, e) = 0, \ (g_{k-2}, e) \le 0, \ (g_{k-1}, e) \le 0, \ (h_0, e) = \dots = (h_{k-2}, e) = 0\} \tag{10}$$

When the matrix $A(p_0)$ has several eigenvalues of the type $\lambda = \pm i\omega$ and (or) the eigenvalue $\lambda = 0$ such that only one Jordan block corresponds to each of them, then the versal deformation $A'(p')$ consists of blocks of types (9) and (or) (2). Combining the methods used for proving (7) and (10), we can prove the following theorem.

**Theorem.** *Let us assume that at a boundary point $p = p_0$ of the stability domain the matrix $A(p_0)$ has the eigenvalues $\lambda = \pm i\omega_s$ $(s = 1, \ldots, l)$ such that only one Jordan block of order $k_s$ corresponds to each of them and (or) the eigenvalue $\lambda = 0$ such that a Jordan block of order $k$ corresponds to it (for the other eigenvalues the inequality* $\operatorname{Re}\lambda < 0$ *holds). Then if the vectors $g_j^s$, $h_r^s$ $(j = 0, \ldots, k_s - 1;\ r = 0, \ldots, k_s - 2)$ calculated by formulas* (8) *for $\lambda = i\omega_s$ with each $s = 1, \ldots, l$ and (or) the vectors $f_j$ $(j = 0, \ldots, k - 1)$ calculated by formulas* (4) *for $\lambda = 0$ are linearly independent, then the tangent cone to the stability domain of the family $A(p)$ at the point $p = p_0$ has the form*

$$K = \{\, e : (g_0^s, e) = \cdots = (g_{k_s-3}^s, e) = 0,\ (g_{k_s-2}^s, e) \le 0,\ (g_{k_s-1}^s, e) \le 0,\ (h_0^s, e) = \cdots = (h_{k_s-2}^s, e) = 0$$
$$(s = 1, \ldots, l) \quad \text{and (or)} \quad (f_0, e) = \cdots = (f_{k-3}, e) = 0,\ (f_{k-2}, e) \le 0,\ (f_{k-1}, e) \le 0 \,\}$$

In the generic case, the condition of linear independence of the vectors is fulfilled (by arbitrary small perturbations of the family $A(p)$, we can be rid of the points where this condition is not valid [2, 3]).

This paper generalizes the results given in [5], where the tangent cones for generic two- and three-parameter families were found.

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