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MECHANICS

On Singularities of Boundaries for Parametric Resonance

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The theory of parametric resonance has broad applications in science and technology [1]. The basic issue of this theory is constructing stability domains in the space of parameters. The boundary of the stability domain may possess singularities, whose occurrence causes computational problems and affects the physical properties of a system.

In this study, a classification of generic-position singularities of stability-domain boundaries is performed for systems of general-form linear differential equations with coefficients periodic in time and dependent on two or three parameters. A constructive approach is proposed allowing the stability domain in the vicinity of a point of its boundary to be found in the first approximation. This approach is based on information available at that point, namely, the values of multipliers, the eigenvectors and adjoint vectors of the monodromy matrix, and the first derivatives of the system operator with respect to parameters. It is worth noting that, unlike some previous studies [1], the closeness of a periodic system to a stationary one was not assumed in the present paper. Singularities of stability-domain boundaries for stationary systems have been studied in [2–4].

1. We consider a system of linear homogeneous differential equations with periodic coefficients

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}, \qquad (1)$$

where $\mathbf{x}$ is the real vector of the dimension m, and $\mathbf{G} = \mathbf{G}(t)$ is a real-valued matrix function of the dimension $m \times m$, which is continuous in time t and periodic with minimum period T, i.e., $\mathbf{G}(t + T) = \mathbf{G}(t)$.

A matrix-valued function $\mathbf{X}(t)$ obeying the equation $\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}$, $\mathbf{X}(0) = \mathbf{I}$, where $\mathbf{I}$ is the unit matrix, is referred to as a matriciant, with its value $\mathbf{F} = \mathbf{X}(T)$ being the monodromy matrix [1]. The eigenvalues of the monodromy matrix $\mathbf{F}$ are said to be multipliers.

The stability of system (1) is determined by the following conditions imposed on the multipliers $\rho_1, \ldots, \rho_m$ [1]: If all multipliers lie inside the unit circle $|\rho_j| < 1$, $j = 1, \ldots, m$, system (1) is asymptotically stable. If at least one multiplier lies outside the unit circle $|\rho_j| > 1$, system (1) will be unstable (parametric resonance).

The matrix $\mathbf{G}$ in (1) and period T will be considered to depend smoothly on the vector $\mathbf{p} = (p_1, \ldots, p_n)$ of real parameters. Then, the monodromy matrix is a smooth function of parameters $\mathbf{F}(\mathbf{p})$, and its first derivatives are of the form [5]

$$\frac{\partial \mathbf{F}}{\partial p_k} = \mathbf{F}\int_0^T \mathbf{Y}^T \frac{\partial \mathbf{G}}{\partial p_k}\mathbf{X}\,dt + \mathbf{G}(T)\mathbf{F}\frac{\partial T}{\partial p_k}, \qquad (2)$$

where $\mathbf{Y}(t)$ is the matriciant of the conjugate system $\dot{\mathbf{Y}} = -\mathbf{G}^T\mathbf{Y}$, $\mathbf{Y}(0) = \mathbf{I}$. The matriciants $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are bound by the relation $\mathbf{X}(t)^T\mathbf{Y}(t) = \mathbf{I}$.

2. The asymptotic-stability condition divides the space of parameters R^n into a stability region and instability one (the region of parametric resonance). The transition from the stability region to the instability one is accompanied by the emergence of certain multipliers from the unit circle. In particular, when real multipliers go through the points 1 and –1, it is said to be the primary resonance; the emergence of a complex-conjugate pair through the unit circle at the points $\exp(\pm i\omega)$ ($\omega \neq \pi k$, $k \in Z$) is spoken of as a combinative resonance. Thus, the stability-domain boundary is determined by the presence of multipliers in the unit circle, providing the rest of the multipliers to lie inside the unit circle. Generally, the stability-domain boundary is a smooth hypersurface in a parameter space with certain singularities (points where the smoothness is lost). Singularities of the generic position (typical singularities) are of particular interest. The emergence of such singularities is always expected in studies of particular systems. As for the singularities of the nongeneric position, they are a consequence of a certain degeneracy or a symmetry of a system and disappear if the system is subjected to a perturbation as small as is wished [2].

We denote the types of the boundary points using the product of multipliers located in the unit circle, with exponents equal to the dimensions of the corresponding Jordan boxes. For example, $1^2\exp(\pm i\omega_1)\exp(\pm i\omega_2)$ implies that the monodromy matrix $\mathbf{F}$ has twofold $\rho = 1$ with the Jordan box of the second order and two pairs

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of simple multipliers $\rho = \exp(\pm i\omega_1)$, $\exp(\pm i\omega_2)$, such that $\omega_1, \omega_2 \in (0, \pi)$, $\omega_1 \neq \omega_2$.

For convenience, we introduce a concise notation for certain types of boundary points

$$B_1(1), \quad B_2(-1), \quad B_3(\exp(\pm i\omega)), \tag{3}$$

$$C_1(1^2), \quad C_2((-1)^2), \tag{4}$$

$$D_1(1^3), \quad D_2((-1)^3), \quad D_3((\exp(\pm i\omega))^2), \tag{5}$$

and for their combinations as well

$$B_{12}(1(-1)), \quad B_{13}(1\exp(\pm i\omega)), \quad B_{23}((-1)\exp(\pm i\omega)),$$
$$B_{33}(\exp(\pm i\omega_1)\exp(\pm i\omega_2)), \tag{6}$$

$$\begin{gathered} B_{123}(1(-1)\exp(\pm i\omega)), \\ B_{133}(1\exp(\pm i\omega_1)\exp(\pm i\omega_2)), \\ B_{233}((-1)\exp(\pm i\omega_1)\exp(\pm i\omega_2)), \\ B_{333}(\exp(\pm i\omega_1)\exp(\pm i\omega_2)\exp(\pm i\omega_3)), \\ C_1B_2(1^2(-1)), \\ C_2B_1((-1)^2 1), \quad C_1B_3(1^2\exp(\pm i\omega)), \\ C_2B_3((-1)^2\exp(\pm i\omega)). \end{gathered} \tag{7}$$

Qualitative analysis of the stability domain in the vicinity of its boundary point is carried out on the basis of the theory of versal deformations [2, 6]. The result obtained is stated in the form of a theorem.

Theorem 1. *In the generic-position case, the stability-domain boundary for system* (1) *consists of* (a) *isolated points of* B_1-, B_2-, *and* B_3-*types* [*see* (3)] *corresponding to the primary resonance and combinative one in the case of one parameter*; (b) *smooth curves of type* (3) *intersecting each other transversally* (*at nonzero angle*) *at break points of types* (4), (6) *in the case of two parameters*; (c) *smooth surfaces of type* (3), *whose singularities are curves of types* (4), (6) (*i.e., the dihedral angle*) *and separate points of type* (5) D_1, D_2 (*i.e., the edge break*), D_3 (*i.e., dead end in the edge*), *and* (7), (*i.e., trihedral angle*) *in the case of three parameters.*

The stability domains in the vicinity of singular points of the above-listed types, up to the nondegenerate smooth change of parameters (diffeomorphism), are of the form presented in Figs. 1 and 2 (the stability domain is denoted by the letter *S*).

3. Consider a point of the stability-domain boundary $\mathbf{p} = \mathbf{p}_0$.

(a) Let ρ_0 be a simple multiplier of the matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$ and $\mathbf{u}_0$, $\mathbf{v}_0$ be the right and left eigenvectors, respectively, corresponding to the multiplier

$$\mathbf{F}_0\mathbf{u}_0 = \rho_0\mathbf{u}_0, \quad \mathbf{v}_0^T\mathbf{F}_0 = \rho_0\mathbf{v}_0^T. \tag{8}$$

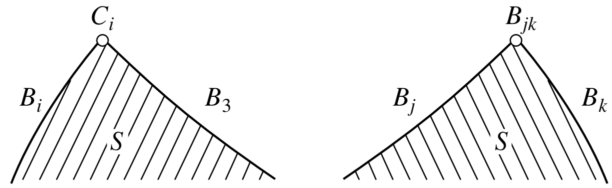

Fig. 1.

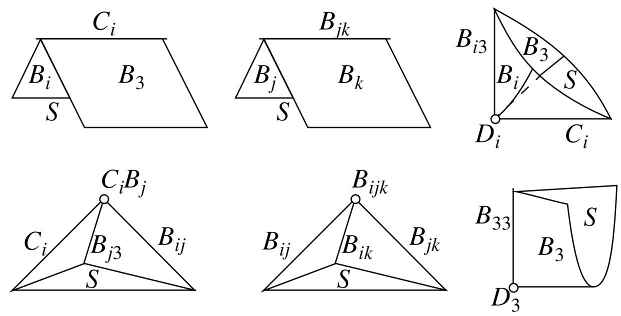

Fig. 2.

We introduce the real *n*-dimensional vectors $\mathbf{r}$ and $\mathbf{k}$ with the components

$$r^s + ik^s = \frac{\mathbf{v}_0^T \dfrac{\partial \mathbf{F}}{\partial p_s}\mathbf{u}_0}{\mathbf{v}_0^T\mathbf{u}_0}, \quad s = 1, \ldots, n, \tag{9}$$

where i is the imaginary unit. If ρ_0 is a real number, then $\mathbf{k} = 0$.

(b) We now consider the case of the twofold multiplier ρ_0 with the second-order Jordan box. Jordan chains for right and left eigenvectors and adjoint vectors are of the form

$$\begin{aligned} \mathbf{F}_0\mathbf{u}_0 &= \rho_0\mathbf{u}_0, \quad \mathbf{F}_0\mathbf{u}_1 = \rho_0\mathbf{u}_1 + \mathbf{u}_0; \\ \mathbf{v}_0^T\mathbf{F}_0 &= \rho_0\mathbf{v}_0^T, \quad \mathbf{v}_1^T\mathbf{F}_0 = \rho_0\mathbf{v}_1^T + \mathbf{v}_0^T. \end{aligned} \tag{10}$$

We consider the vectors $\mathbf{u}_0$ and $\mathbf{u}_1$ as fixed and introduce the normalization $\mathbf{v}_0^T\mathbf{u}_1 = 1$, $\mathbf{v}_1^T\mathbf{u}_1 = 0$ unambiguously defining the vectors $\mathbf{v}_0$, $\mathbf{v}_1$. We define the real vectors $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{q}_1$, and $\mathbf{q}_2$ with the components

$$\begin{gathered} f_1^s + i f_2^s = \mathbf{v}_0^T\frac{\partial \mathbf{F}}{\partial p_s}\mathbf{u}_0, \\ q_1^s + iq_2^s = \mathbf{v}_0^T\frac{\partial \mathbf{F}}{\partial p_s}\mathbf{u}_1 + \mathbf{v}_1^T\frac{\partial \mathbf{F}}{\partial p_s}\mathbf{u}_0, \quad s = 1, \ldots, n. \end{gathered} \tag{11}$$

If ρ_0 is a real number, then $\mathbf{f}_2 = \mathbf{q}_2 = 0$.

(c) Next, we consider the case of the threefold real multiplier ρ_0 with the third-order Jordan box. The Jor-

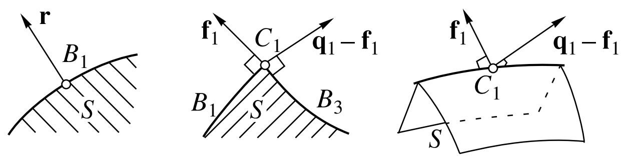

Fig. 3.

dan chains for 1 and left eigenvectors and adjoint vectors are of the form

$$\mathbf{F}_0\mathbf{u}_0 = \rho_0\mathbf{u}_0, \quad \mathbf{F}_0\mathbf{u}_1 = \rho_0\mathbf{u}_1 + \mathbf{u}_0,$$

$$\mathbf{F}_0\mathbf{u}_2 = \rho_0\mathbf{u}_2 + \mathbf{u}_1, \quad \mathbf{v}_0^T\mathbf{F}_0 = \rho_0\mathbf{v}_0^T, \qquad (12)$$

$$\mathbf{v}_1^T\mathbf{F}_0 = \rho_0\mathbf{v}_1^T + \mathbf{v}_0^T, \quad \mathbf{v}_2^T\mathbf{F}_0 = \rho_0\mathbf{v}_2^T + \mathbf{v}_1^T.$$

Considering the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$ as fixed, we impose the normalization conditions $\mathbf{v}_0^T\mathbf{u}_2 = 1$, $\mathbf{v}_1^T\mathbf{u}_2 = 0$, and $\mathbf{v}_2^T\mathbf{u}_2 = 0$ on $\mathbf{v}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$. Since ρ_0 is a real number, the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ can be chosen to be real. We introduce the n-dimensional vectors $\mathbf{g}$, $\mathbf{h}$, and $\mathbf{t}$ and a matrix $\mathbf{R}$ with the dimension $n \times n$ of the form

$$g^s = \mathbf{v}_0^T\frac{\partial}{\partial p_s}(\mathbf{F})\mathbf{u}_0, \quad h^s = \mathbf{v}_0^T\frac{\partial}{\partial p_s}(\mathbf{F})\mathbf{u}_1 + \mathbf{v}_1^T\frac{\partial}{\partial p_s}(\mathbf{F})\mathbf{u}_0,$$

$$t^s = \mathbf{v}_0^T\frac{\partial}{\partial p_s}(\mathbf{F})\mathbf{u}_2 + \mathbf{v}_1^T\frac{\partial}{\partial p_s}(\mathbf{F})\mathbf{u}_1 + \mathbf{v}_2^T\frac{\partial}{\partial p_s}(\mathbf{F})\mathbf{u}_0, \quad s = 1, \ldots, n,$$

$$\mathbf{R} = \left[\mathbf{v}_0^T\frac{\partial}{\partial p_i}(\mathbf{F})[\mathbf{F}_0 - \rho_0\mathbf{I} - \mathbf{v}_0\mathbf{v}_2^T]^{-1}\frac{\partial\mathbf{F}}{\partial p_j}\mathbf{u}_0 - \mathbf{v}_0^T\frac{\partial^2\mathbf{F}}{\partial p_i\partial p_j}\mathbf{u}_0\right], \qquad (13)$$

$$i, j = 1, \ldots, n.$$

Expressions for the second derivatives of the matrix $\mathbf{F}$ are presented in [5].

4. We draw a smooth curve $\mathbf{p} = \mathbf{p}(\varepsilon)$ from the point of the boundary of a parametric resonance $[\mathbf{p}_0 = \mathbf{p}(0), \varepsilon \geq 0]$. In this case, some curves will lie (for small $\varepsilon > 0$) in the stability domain and others, in the instability one. We select a direction $\mathbf{e} = \left.\frac{d\mathbf{p}}{d\varepsilon}\right|_{\varepsilon = 0}$ of the curves lying in the stability domain. The set of such directions forms a cone tangent to the stability domain at the point of its boundary $\mathbf{p} = \mathbf{p}_0$. The tangent cone is an approximation for the stability domain in the vicinity of point of its boundary.

Theorem 2. *Cones tangent to the stability domain at its boundary points of the type* (3)–(5) *are determined by the relations*

$$K_{B_1} = \{\mathbf{e}: (\mathbf{r}, \mathbf{e}) \leq 0\}, \quad K_{B_2} = \{\mathbf{e}: (\mathbf{r}, \mathbf{e}) \geq 0\},$$

$$K_{B_3} = \{\mathbf{e}: (\mathbf{r}\cos\omega + \mathbf{k}\sin\omega, \mathbf{e}) \leq 0\},$$

$$K_{C_1} = \{\mathbf{e}: (\mathbf{f}_1, \mathbf{e}) \leq 0, (\mathbf{q}_1 - \mathbf{f}_1, \mathbf{e}) \leq 0\},$$

$$K_{C_2} = \{\mathbf{e}: (\mathbf{f}_1, \mathbf{e}) \leq 0, (\mathbf{q}_1 + \mathbf{f}_1, \mathbf{e}) \geq 0\},$$

$$K_{D_1} = \{\mathbf{e}: (\mathbf{g}, \mathbf{e}) = 0, (\mathbf{h}, \mathbf{e}) \leq 0, (\mathbf{t} - \mathbf{h}, \mathbf{e}) \leq 0\}, \qquad (14)$$

$$K_{D_2} = \{\mathbf{e}: (\mathbf{g}, \mathbf{e}) = 0, (\mathbf{h}, \mathbf{e}) \leq 0, (\mathbf{t} + \mathbf{h}, \mathbf{e}) \geq 0\},$$

$$K_{D_3} = \{\mathbf{e}: (\mathbf{f}_2\cos 2\omega - \mathbf{f}_1\sin 2\omega, \mathbf{e}) = 0,$$

$$(\mathbf{f}_1\cos 2\omega + \mathbf{f}_2\sin 2\omega, \mathbf{e}) \leq 0,$$

$$(\mathbf{q}_1\cos\omega + \mathbf{q}_2\sin\omega - \mathbf{f}_1\cos 2\omega - \mathbf{f}_2\sin 2\omega, \mathbf{e}) \leq 0\},$$

where vectors are calculated for the multiplier determining the type of a boundary point. The tangent cones for the combined types (6), (7) *are produced by the intersection of tangent cones found for each of the subtypes. For example, in the case* C_1B_2, *we have* $K_{C_1B_2} = K_{C_1} \cap K_{B_2}$.

In the cases B_1, B_2, B_3, C_1, C_2, and D_3, all smooth curves drawn in $\mathbf{e}$-direction satisfying rigorous inequalities (14) lie in the stability domain for sufficiently small $\varepsilon > 0$. In the cases D_1 and D_2, curves satisfying additional conditions

$$(\mathbf{Re}, \mathbf{e}) - (\mathbf{t} - \mathbf{h}, \mathbf{e})(\mathbf{h}, \mathbf{e}) < (\mathbf{g}, \mathbf{d}) < (\mathbf{Re}, \mathbf{e})$$

and

$$(\mathbf{Re}, \mathbf{e}) < (\mathbf{g}, \mathbf{d}) < (\mathbf{Re}, \mathbf{e}) - (\mathbf{t} + \mathbf{h}, \mathbf{e})(\mathbf{h}, \mathbf{e})$$

respectively, $\left(\text{where } \mathbf{d} = \frac{1}{2}\left.\left(\frac{d^2\mathbf{p}}{d\varepsilon^2}\right)\right|_{\varepsilon = 0}\right)$ lie in the stability domain. It is worth noting that the tangent cones K_{D_1}, K_{D_2}, and K_{D_3} are degenerate (are two-dimensional angles in the three-dimensional parameter space).

Theorem 2 enables us to find, in the first approximation, the stability domain in the vicinity of a point of the stability-domain boundary from information at this point [using first derivatives of the operator $\mathbf{G}$ from (1) with respect to parameters and values of multipliers and corresponding eigenvectors and adjoint vectors calculated for $\mathbf{p} = \mathbf{p}_0$]. Relations (14) provide a clear idea of the stability domain. For example, in the case of non-

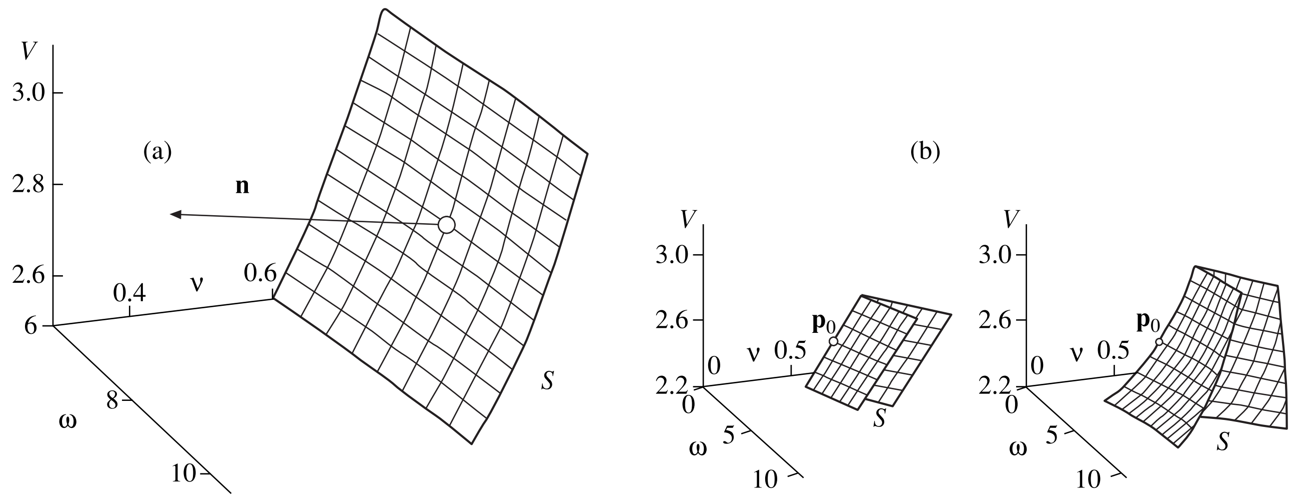

Fig. 4.

singular point B_1, a tangent cone is determined by the inequality $(\mathbf{r}, \mathbf{e}) \le 0$. Therefore, a plane tangent to the stability-domain boundary is specified by the equation $(\mathbf{r}, \mathbf{e}) = 0$, and the vector $\mathbf{r}$ is a normal to the boundary lying in the region of a parametric resonance (see Fig. 3). For the case of the singular point C_1, the tangent cone is the intersection of the half planes (half spaces) $(\mathbf{f}_1, \mathbf{e}) \le 0$ and $(\mathbf{q}_1 - \mathbf{f}_1, \mathbf{e}) \le 0$. These inequalities determine a two-dimensional (dihedral) angle in a space of two (three) parameters, which is the first approximation to the stability domain (Fig. 3).

5. As an example, we consider two-dimensional vibrations of a two-link pipe, along which a fluid with the linear mass m and pulsating velocity

$$u(t) = U(1 + \nu \sin \omega t)$$

flows. The pipe parts are connected by elastic hinges having the rigidity coefficients equal to c; they have the length l and linear mass $M = 2m$. The right end is free. We choose the angles φ and ψ of the deviation of the pipe parts from a horizontal axis as generalized coordinates. The linearized equations for the motion of the system in terms of dimensionless variables are of the form [7]

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}, \quad \mathbf{x} = (\varphi, \psi, \dot{\varphi}, \dot{\psi})^T,$$

$$\mathbf{G}(\tau) = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{B} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 4 & 1.5 \\ 1.5 & 1 \end{pmatrix}, \quad (15)$$

$$\mathbf{B} = v(\tau)\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 - f(\tau) & -1 + f(\tau) \\ -1 & 1 \end{pmatrix},$$

where $\mathbf{O}$ is the zero matrix of the dimension 2×2, and differentiation with respect to the dimensionless time τ is denoted by a dot. The following notation is used:

$$\tau = \alpha t, \quad f(\tau) = \dot{v}(\tau) + v^2(\tau),$$

$$v(\tau) = V(1 + \nu \sin w\tau), \quad w = \frac{\omega}{\alpha},$$

$$V = \frac{U}{\alpha l}, \text{ and } \alpha^2 = \frac{c}{ml^3}.$$

The matrix operator $\mathbf{G}(\tau)$ with the period $T = 2\pi/w$ smoothly depends on the dimensionless parameters $\mathbf{p} = (w, \nu, V)$.

We consider the point $\mathbf{p}_0 = (8, 0.737, 2.8)$ in the parameter space. We calculate the monodromy matrix $\mathbf{F}_0$ at this point and find the multipliers of the matrix:

$$\rho_{1,2} = \exp(\pm 0.882i), \quad \rho_3 = 0.535, \quad \rho_4 = 0.152.$$

Since ordinary complex-conjugate multipliers lie in the unit circumference $|\rho_{1,2}| = 1$, and the other multipliers are inside the unit circle, the point $\mathbf{p}_0$ is the regular point of the stability-domain boundary of the B_3-type. According to (14), the K_{B_3} cone tangent to the stability domain at the point $\mathbf{p}_0$ is described by the expression $(\mathbf{r}\cos\omega + \mathbf{k}\sin\omega, \mathbf{e}) \le 0$, where $\omega = \arg\rho_1$; the vectors $\mathbf{r}$ and $\mathbf{k}$ are calculated according to (2), (9). The vector $\mathbf{n} = \mathbf{r}\cos\omega + \mathbf{k}\sin\omega = (0.03, -2.05, 0.51)$ is the normal to the stability-domain boundary and lies in the parametric-resonance region. The numerically calculated boundary of the stability domain and the vector $\mathbf{n}$ are shown in Fig 4a.

Finally, we consider the point $\mathbf{p}_0 = (3.643, 0.5555, 2.6)$, at which the monodromy matrix $\mathbf{F}_0$ has the multipliers $\rho_1 = \rho_2 = -1$, $\rho_3 = 0.225$, $\rho_4 = 0.026$, second-order Jordan chains (10) being relevant to a twofold multiplier (we denote it by $\rho_0 = -1$). Since the twofold multiplier $\rho_0 = -1$ belongs to the boundary, and two others lie inside the unit circle, from Theorem 1, a singularity of the stability-domain boundary of the C_2 type (i.e., of the dihedral-angle type) is realized at the point $\mathbf{p}_0$. According to (2), (11), we find the vectors $\mathbf{f}_1$ = (–5.15, 45.2, –7.77), $\mathbf{q}_1$ = (4.49, –31.1, 3.16). From Theorem 2, these vectors define the cone tangent to the stability domain

$$K_{C_2} = \{\mathbf{e}: (\mathbf{f}_1, \mathbf{e}) \le 0, (\mathbf{q}_1 + \mathbf{f}_1, \mathbf{e}) \ge 0\}.$$

The vectors $\mathbf{f}_1$ and $-(\mathbf{q}_1 + \mathbf{f}_1)$ are normals to the sides of the dihedral angle, and they lie in the region of the parametric resonance. The vector $\mathbf{e}_\tau$ tangent to the edge of the dihedral angle is equal to $\mathbf{e}_\tau = (\mathbf{q}_1 + \mathbf{f}_1) \times \mathbf{f}_1$. In Fig. 4b (to the left), a dihedral angle is shown, which is an approximation for the stability-domain boundary in the vicinity of the point $\mathbf{p}_0$. For comparison, at the right of Fig. 4b, the numerically calculated stability-domain boundary is shown confirming the presence of the singularity and reasonable agreement of the results obtained.

Our results can be employed in solving various problems of stabilization of parameter-dependent periodic systems with the use of gradient methods.

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