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Evaluation of Multiple Eigenvalues and Jordan Chains of Vectors for Matrices Depending on Parameters

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The reduction of a matrix to the Jordan form is an unstable operation if the matrix has multiple eigenvalues. We can make all eigenvalues simple by an arbitrarily small variation of the matrix. However, multiple eigenvalues are unremovable when we consider families of matrices smoothly depending on parameters, and they play an important role in problems of stability and optimization of eigenvalues [1–3]. Arnol'd [4, 5] obtained local normal forms of families of complex matrices (versal deformations) and applied them to a qualitative study of bifurcation diagrams (sets of parameter values corresponding to matrices with multiple eigenvalues). In this paper, we apply the theory of versal deformations to obtain explicit formulas that make it possible to find multiple eigenvalues and the corresponding Jordan chains of vectors for complex and real families of matrices under the conditions of precise calculations. In numerical calculations with the use of standard procedures, it is necessary to take into account that round-off errors can sometimes lead to large errors in evaluation of simple eigenvalues used in formulas. In this paper, we consider the case of multiple eigenvalues that correspond to a single Jordan chain of vectors (one Jordan block). The eigenvalues of this form are most typical [4, 5].

Many authors considered the problem of evaluating multiple eigenvalues of matrices (see, e.g., [6–9]). However, the methods described in the literature do not apply to matrices depending on parameters; in addition, the formulas of this paper give more complete information about multiple eigenvalues.

1. Consider a complex $m \times m$ matrix $\mathbf{A}$ analytically depending on a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ of complex parameters. By λ^d, we denote the set of values of the parameter vector $\mathbf{p}$ such that the matrix $\mathbf{A}(\mathbf{p})$ has an eigenvalue of algebraic multiplicity d corresponding to a single Jordan chain of vectors (single Jordan block) and the other eigenvalues are simple. In the case of general position, the set λ^d (stratum) is a smooth manifold of codimension $d - 1$ [4]. Let $\mathbf{p}' \in \lambda^d$ be a point in this manifold. By λ', we denote a multiple eigenvalue of the matrix $\mathbf{A}' = \mathbf{A}(\mathbf{p}')$, and by $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_d$, the corresponding Jordan chain of vectors satisfying the equations

$$\mathbf{A}'\mathbf{u}'_1 = \lambda'\mathbf{u}'_1, \quad \mathbf{A}'\mathbf{u}'_2 = \lambda'\mathbf{u}'_2 + \mathbf{u}'_1, \quad \ldots, \quad \mathbf{A}'\mathbf{u}'_d = \lambda'\mathbf{u}'_d + \mathbf{u}'_{d-1}. \tag{1}$$

For convenience, we represent the Jordan chain as the $m \times d$ matrix $\mathbf{U}' = [\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_d]$.

Let $\mathbf{p}_0$ be a point in the parameter space lying in the vicinity of the set λ^d. In the case of general position, the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has only simple eigenvalues. By λ_1, $\lambda_2, \ldots, \lambda_d$, we denote the eigenvalues of $\mathbf{A}_0$ transforming into the multiple eigenvalue λ' as $\mathbf{p}_0 \to \mathbf{p}' \in \lambda^d$. Let $\mathbf{u}_i$ and $\mathbf{v}_i$ be the right and left eigenvectors corresponding to λ_i, i.e., satisfying the equations

$$\mathbf{A}_0\mathbf{u}_i = \lambda_i\mathbf{u}_i, \quad \mathbf{v}_i^T\mathbf{A}_0 = \lambda_i\mathbf{v}_i^T, \quad \mathbf{v}_i^T\mathbf{u}_i = 1; \tag{2}$$

the last relation is a normalization condition. Consider the $m \times d$ matrices $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_d]$ and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d]$ and the row-vectors $\mathbf{n}_i$ and $\mathbf{n}$ of dimension n defined as

$$\mathbf{n}_i = \left(\mathbf{v}_i^T\frac{\partial\mathbf{A}}{\partial p_1}\mathbf{u}_i, \mathbf{v}_i^T\frac{\partial\mathbf{A}}{\partial p_2}\mathbf{u}_i, \ldots, \mathbf{v}_i^T\frac{\partial\mathbf{A}}{\partial p_n}\mathbf{u}_i\right), \quad i = 1, 2, \ldots, d, \quad \mathbf{n} = \sum_{i=1}^{d}\frac{\mathbf{n}_i}{d}, \tag{3}$$

where the derivatives are evaluated at the point $\mathbf{p}_0$. Let us also introduce the $d \times d$ matrices $\mathbf{K} = \mathrm{diag}(k_1, k_2, \ldots, k_d)$, $\mathbf{R}$, $\mathbf{S} = \mathbf{R}^{-1}$, and $\mathbf{Y}$ with components k_i, r_{ij}, s_{ij}, and y_{ij}, where

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$$k_i = \frac{\sqrt{\mathbf{u}_1^T\bar{\mathbf{u}}_1}}{(\mathbf{u}_i^T\bar{\mathbf{u}}_1)}, \quad r_{ij} = \mu_j^{i-1}, \quad y_{ij} = -s_{id}\mu_j^d,$$
$$\mu_j = \lambda_j - \lambda_0, \quad \lambda_0 = \frac{\lambda_1 + \lambda_2 + \ldots + \lambda_d}{d} \tag{4}$$

(here, $\mu_j^0 = 1$ if μ_j is zero). Note that determining the vectors and matrices defined by (3) and (4) only requires an information on the family $\mathbf{A}(\mathbf{p})$ at the point $\mathbf{p}_0$. Having this information, we can evaluate in the first approximation [accurate to $O(\|\mathbf{p}_0 - \mathbf{p}'\|^2)$] the set λ^d near the point $\mathbf{p}_0$, the multiple eigenvalue, and the corresponding Jordan chain at the points of the stratum λ^d with the use of the formulas given by the following theorem.

Theorem 1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_d$ be simple eigenvalues of the matrix $\mathbf{A}_0$. Then, the first approximation of the stratum λ^d for which $\lambda_1, \lambda_2, \ldots, \lambda_d$ form a multiple eigenvalue λ' is determined by the following system of $d - 1$ linear equations in* $\mathbf{p}$:

$$q_j^0 + \nabla q_j(\mathbf{p} - \mathbf{p}_0)^T = 0, \quad j = 1, 2, \ldots, d-1,$$
$$q_j^0 = \sum_{i=1}^{d} s_{ij}\mu_i^d, \quad \nabla q_j = \sum_{i=1}^{d} \frac{s_{ij}(\mathbf{n}_i - \mathbf{n})}{s_{id}}. \tag{5}$$

The first approximations of the multiple eigenvalue λ' and of the corresponding Jordan chain $\mathbf{U}' = [\mathbf{u}_1', \mathbf{u}_2', \ldots, \mathbf{u}_d']$ at the points of stratum (5) *are determined by the relations*

$$\lambda' = \lambda_0 + \Delta\lambda, \quad \Delta\lambda = \mathbf{n}(\mathbf{p} - \mathbf{p}_0)^T, \tag{6}$$

$$\mathbf{U}' = (\mathbf{U}(\mathbf{K} + \mathbf{K}') + [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d])\mathbf{S},$$
$$\mathbf{w}_i = (\mathbf{A}_0 - \lambda_i\mathbf{I} - \bar{\mathbf{v}}_i\mathbf{v}_i^T)^{-1}(\mathbf{UKY} + \Delta\lambda\mathbf{UK} - \Delta\mathbf{AUK})^{\langle i\rangle},$$
$$\mathbf{K}' = \mathrm{diag}(k_1', k_2', \ldots, k_d'), \tag{7}$$
$$k_i' = -\mathbf{v}_i^T[\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d]\frac{\mathbf{S}^{\langle d\rangle}}{s_{id}},$$
$$\Delta\mathbf{A} = \sum_{i=1}^{n} \frac{\partial\mathbf{A}}{\partial p_i}(p_i - p_{0i}),$$

where all derivatives are evaluated at $\mathbf{p} = \mathbf{p}_0$, $\mathbf{S}^{\langle d\rangle}$ denotes the dth *column of the matrix* $\mathbf{S}$ *and* $\mathbf{I}$ *is the identity matrix.*

Theorem 1 makes it possible to examine the structure of the stratum λ^d and determine multiple eigenvalues and Jordan chains at the points of the stratum based on the information at the point $\mathbf{p}_0$, where the matrix has only simple eigenvalues.

To derive relations (5)–(7), we have used the theory of versal deformations [4], according to which a family of matrices $\mathbf{A}(\mathbf{p})$ in a neighborhood of the point $\mathbf{p}'$ can be obtained from a family $\mathbf{A}'(\mathbf{q})$ of special form (versal deformation) by a smooth change of the basis and a smooth change $\mathbf{q} = \mathbf{q}(\mathbf{p})$ of the parameters. The stratum λ^d is determined by the equations $q_1 = q_2 = \ldots = q_{d-1} = 0$, where q_i are the parameters of the versal deformation selected in a special way. Using linear approximations of the functions $q_i(\mathbf{p})$ at the point $\mathbf{p}_0$, we obtain system of equations (5), where $q_i^0 = q_i(\mathbf{p}_0)$ and ∇q_i is the gradient (row-vector) evaluated at $\mathbf{p} = \mathbf{p}_0$. Equations (5) specify a plane σ in the parameter space; this plane is the first approximation of the stratum λ^d in a neighborhood of the point $\mathbf{p}_0$ (Fig. 1). The values of q_i^0 and ∇q_i, as well as the values of λ' and $\mathbf{U}'$, are determined by analyzing the relations between the family $\mathbf{A}(\mathbf{p})$ and the versal deformation at the point $\mathbf{p}_0$.

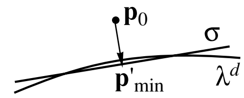

Fig. 1.

Let us write system (5) in the matrix form

$$\mathbf{Q}(\mathbf{p} - \mathbf{p}_0)^T = \mathbf{x}, \quad \mathbf{Q} = \begin{bmatrix} \nabla q_1 \\ \vdots \\ \nabla q_{d-1} \end{bmatrix},$$
$$\mathbf{x} = \begin{bmatrix} -q_1^0 \\ \vdots \\ -q_{d-1}^0 \end{bmatrix}, \tag{8}$$

where $\mathbf{Q}$ is a $(d - 1) \times n$ matrix and $\mathbf{x}$ is a vector of dimension $d - 1$.

Corollary 1. *Under the conditions of Theorem* 1, *the value of the vector $\mathbf{p}'_{\min} \in \lambda^d$ nearest to $\mathbf{p}_0$* (*with respect to the Euclidean norm*) *in the first approximation has the form*

$$\mathbf{p}'_{\min} = \mathbf{p}_0 + \mathbf{x}^T(\bar{\mathbf{Q}}\mathbf{Q}^T)^{-1}\bar{\mathbf{Q}}. \tag{9}$$

Consider the stratum $\lambda_1^{d_1}\lambda_2^{d_2}\ldots\lambda_s^{d_s}$ consisting of the values of $\mathbf{p}$ at which the matrix $\mathbf{A}(\mathbf{p})$ has s different multiple eigenvalues $\lambda_1', \lambda_2', \ldots\lambda_s'$, and each eigenvalue λ_i' corresponds to a single Jordan chain of length d_i. The stratum $\lambda_1^{d_1}\lambda_2^{d_2}\ldots\lambda_s^{d_s}$ is the transversal intersection of the strata λ^{d_i} with $i = 1, 2, \ldots, s$ [4].

Corollary 2. *Let $\mathbf{p}_0$ be a point in the parameter space at which the matrix $\mathbf{A}_0$ has only simple eigenvalues. Then the first approximation of the stratum*

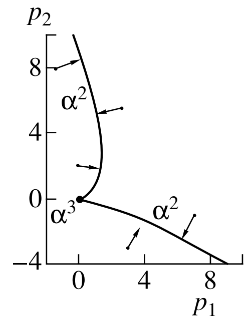

Fig. 2.

$\lambda_1^{d_1}\lambda_2^{d_2}\ldots\lambda_s^{d_s}$ *in a neighborhood of* $\mathbf{p}_0$ *is the intersection of planes* (5) *for all multiple eigenvalues. The multiple eigenvalues and the corresponding Jordan chains at the points of stratum are defined by relations* (6), (7) *for every* λ_i'.

2. Consider the stratum λ^d in the space of all complex $m \times m$ matrices. Every matrix element can be treated as an independent parameter. Replacing the parameter vector $\mathbf{p}$ by a matrix $\mathbf{A}$ and taking into account that the derivative of a matrix with respect to a parameter contains only one nonzero element, which equals 1, we obtain the following assertion from Theorem 1 and Corollary 1.

Theorem 2. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_d$ *be simple eigenvalues of the matrix* $\mathbf{A}_0$. *Then, in the space of complex matrices, the first approximation of the stratum* λ^d *for which* $\lambda_1, \lambda_2, \ldots, \lambda_d$ *form a multiple eigenvalue* λ' *is determined by the following system of* $d - 1$ *linear equations in* $\mathbf{A}$:

$$q_j^0 + \operatorname{tr}(\mathbf{Q}_j(\mathbf{A}-\mathbf{A}_0)^T) = 0, \quad j = 1, 2, \ldots, d-1,$$
$$q_j^0 = \sum_{i=1}^{d} s_{ij}\mu_i^d, \quad \mathbf{Q}_j = \sum_{i=1}^{d} \frac{s_{ij}(\mathbf{N}_i - \mathbf{N})}{s_{id}}, \tag{10}$$

where $\operatorname{tr}(\mathbf{A})$ *is the trace of the matrix* $\mathbf{A}$, *and* $\mathbf{N}_i$ *and* $\mathbf{N}$ *are the following* $m \times m$ *matrices*:

$$\mathbf{N}_i = \mathbf{v}_i\mathbf{u}_i^T, \quad \mathbf{N} = \sum_{i=1}^{d} \frac{\mathbf{N}_i}{d}. \tag{11}$$

The first approximation of the multiple eigenvalue λ' *and the corresponding Jordan chain* $\mathbf{U}'$ *are determined at the points of stratum* (10) *by the relations*

$$\lambda' = \lambda_0 + \Delta\lambda, \quad \Delta\lambda = \operatorname{tr}(\mathbf{N}(\mathbf{A}-\mathbf{A}_0)^T), \tag{12}$$
$$\mathbf{U}' = (\mathbf{U}(\mathbf{K}+\mathbf{K}') + [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d])\mathbf{S},$$
$$\mathbf{w}_i = (\mathbf{A}_0 - \lambda_i\mathbf{I} - \bar{\mathbf{v}}_i\mathbf{v}_i^T)^{-1}$$
$$\times(\mathbf{UKY} + \Delta\lambda\mathbf{UK} - (\mathbf{A}-\mathbf{A}_0)\mathbf{UK})^{\langle i\rangle}, \tag{13}$$
$$\mathbf{K}' = \operatorname{diag}(k_1', k_2', \ldots, k_d'),$$
$$k_i' = -\mathbf{v}_i^T[\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d]\frac{\mathbf{S}^{\langle d\rangle}}{s_{id}}.$$

Corollary 3. *Under the conditions of Theorem* 2, *the matrix* $\mathbf{A}'_{\min} \in \lambda^d$ *nearest to* $\mathbf{A}_0$ (*with respect to the Euclidean norm* $\|\cdot\|_E$) *in the first approximation has the form*

$$\mathbf{A}'_{\min} = \mathbf{A}_0 + \sum_{j=1}^{d-1}\overline{\mathbf{Q}}_j y_j, \quad \mathbf{y} = \mathbf{P}^{-1}\mathbf{x}, \tag{14}$$

where $\mathbf{P}$ *is the* $(d-1)\times(d-1)$ *matrix with the elements* $p_{ij} = \operatorname{tr}(\mathbf{Q}_i^T\overline{\mathbf{Q}}_j)$, *and* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors-columns of dimension* $d - 1$ *with the components* $x_j = -q_j^0$ *and* y_j, *respectively.*

Note that, according to Theorem 2, determining the stratum λ^d in a neighborhood of the matrix $\mathbf{A}_0$, the multiple eigenvalue, and the corresponding Jordan chain only involves the simple eigenvalues and eigenvectors of the matrix $\mathbf{A}_0$.

3. Consider a real matrix $\mathbf{A}$ smoothly depending on a vector $\mathbf{p}$ of real parameters. In this case, we should distinguish between real and complex multiple eigenvalues λ'. The corresponding strata in the parameter space are denoted by α^d and $(\alpha \pm i\omega)^d$ and have codimensions $d - 1$ and $2(d - 1)$, respectively [4].

Theorems 1 and 2 can be extended to the case of the stratum $(\alpha \pm i\omega)^d$. In this case, relations (5) and (10) determine systems of $2(d - 1)$ linear equations (each equality determines two equations for the real and imaginary parts). The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of the matrix $\mathbf{A}_0$, which transform into a multiple complex eigenvalue λ' as $\mathbf{p}_0 \to \mathbf{p}' \in (\alpha \pm i\omega)^d$, should have imaginary parts of the same sign [this is so at the points $\mathbf{p}_0$ sufficiently close to $(\alpha \pm i\omega)^d$].

In the case of the stratum α^d, Theorems 1 and 2 are valid if the constants k_i that determine the diagonal elements of matrix $\mathbf{K}$ have the form

$$k_i = \frac{1}{(\mathbf{u}_i^T\mathbf{u}')}, \quad \mathbf{u}' = \operatorname{Re}\left(\frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^T\mathbf{u}_1}}\right). \tag{15}$$

Relations (5) and (10) with conditions (15) taken into account determine systems of $d - 1$ real linear equations, and relations (6), (7), (11), and (12) determine the real values λ' and $\mathbf{U}'$. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ of the matrix $\mathbf{A}_0$, which transform into a multiple real eigenvalue λ' as $\mathbf{p}_0 \to \mathbf{p}' \in \alpha^d$, must be real or complex conjugate.

4. As an example, consider the two-parameter family of real matrices

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 1 & 3 & 0 \\ p_1 & 1 & p_2 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \tag{16}$$

By Theorem 1, the equation of the stratum α^2 in the first approximation has the form

$$q_1^0 + \nabla q_1(\mathbf{p} - \mathbf{p}_0)^T = 0, \tag{17}$$

where $q_1^0 = \mu^2$, $\nabla q_1 = \mu(\mathbf{n}_2 - \mathbf{n}_1)$, and $\mu = \dfrac{\lambda_2 - \lambda_1}{2}$. The vector $\mathbf{p}'_{\min} \in \alpha^2$ nearest to $\mathbf{p}_0$ is determined by Corollary 1 in the form

$$\mathbf{p}'_{\min} = \mathbf{p}_0 - \frac{q_1^0}{\nabla q_1 \nabla q_1^T} \nabla q_1. \tag{18}$$

Figure 2 shows the bifurcation diagram (the strata α^2 and α^3) in the parameter space. The arrows denote the vectors $\mathbf{p}'_{\min} - \mathbf{p}_0$, where $\mathbf{p}_0$ are various values of the parameter vector and $\mathbf{p}'_{\min}$ are the nearest points of the stratum α^2 found approximately by (18). In calculations, λ_1 and λ_2 were set to be equal to complex conjugate eigenvalues of the matrix $\mathbf{A}_0$. If the matrix $\mathbf{A}_0$ had three real eigenvalues, then all the three possible pairs λ_1, λ_2 were considered, and the nearest vector $\mathbf{p}'_{\min}$ was selected among the three obtained vectors (every time, the value $\mathbf{p}'_{\min}$ at which the given pair of eigenvalues formed a double eigenvalue λ' was determined). It is seen from Fig. 2 that formula (18) gave a good approximation of the nearest vector $\mathbf{p}'_{\min} \in \alpha^2$ for the points $\mathbf{p}_0$ considered.

Consider the point $\mathbf{p}_0 = (0.3, 9.1)$ at which the matrix $\mathbf{A}_0$ has eigenvalues $\lambda_1 = -2.624$, $\lambda_2 = -1.472$, and $\lambda_3 = -7.096$. Calculations by formula (18) show that the pair λ_1, λ_2 gives the nearest value $\mathbf{p}'_{\min} = (-0.0008, 8.9990) \in \alpha^2$. The multiple eigenvalue λ' and the Jordan chain (the eigenvector $\mathbf{u}'_1$ and the associated vector $\mathbf{u}'_2$) evaluated at the point $\mathbf{p} = \mathbf{p}'_{\min}$ by approximate formulas (6) and (7) have the form

$$\lambda' = -2.00006,$$

$$\mathbf{U}' = [\mathbf{u}'_1, \mathbf{u}'_2] = \begin{pmatrix} 0.7044 & 0.1496 \\ -0.7044 & 0.0852 \\ 0.2346 & -0.1067 \end{pmatrix}. \tag{19}$$

For comparison, we give the exact values of $\mathbf{p}'_{\min}$, λ', and $\mathbf{U}'$:

$$\mathbf{p}'_{\min} = (0, 9), \quad \lambda' = -2,$$

$$[\mathbf{u}'_1, \mathbf{u}'_2] = \begin{pmatrix} 0.7044 & 0.1494 \\ -0.7044 & 0.0854 \\ 0.2348 & -0.1067 \end{pmatrix}. \tag{20}$$

Comparing (19) and (20), we see that the approximate values obtained with the use of Theorem 1 differ from the exact values only by the fourth digit.

5. Consider the stratum α^4 in the space of real matrices. Take a 10×10 Jordan matrix $\mathbf{A}_d \in \alpha^4$ with simple eigenvalues $\lambda = -4, -3, -2, -1, 0, 4$ and the Jordan 4×4 block with eigenvalue $\lambda = 2$. Consider the following perturbations of the matrix $\mathbf{A}_d$:

$$\mathbf{A}_0 = \mathbf{A}_d + \mathbf{D},$$

where $\mathbf{D}$ is a real 10×10 matrix whose elements are independent random variables with zero expectation and variance $\sigma^2 = 4 \times 10^{-4}$ ($\|\mathbf{D}\|_E \sim 0.2$). Taking as λ_1, λ_2, λ_3, λ_4 the simple eigenvalues of the matrix $\mathbf{A}_0$ nearest to $\lambda = 2$, we find the nearest real matrix $\mathbf{A}'_{\min} \in \alpha^4$ and the distance $\|\mathbf{A}'_{\min} - \mathbf{A}_0\|_E$ from $\mathbf{A}_0$ to the stratum α^4 by (10), (14). The mean value of the square of this distance obtained by numerical computations over 1000 random matrices $\mathbf{D}$ is 1.193×10^{-3}. Since the stratum α^4 has codimension 3, the expectation of $\|\mathbf{D}_n\|_E^2$, where $\mathbf{D}_n$ is the component of the perturbation matrix $\mathbf{D}$ normal to α^4, equals $3\sigma^2 = 1.2 \times 10^{-3}$; this agrees well with the numerical calculations.

Thus, the approximate formulas given by Theorems 1 and 2 make it possible to effectively analyze multiple eigenvalues corresponding to a single Jordan chain in the case of complex and real families of matrices. These formulas only involve the derivatives of the matrix $\mathbf{A}(\mathbf{p})$ with respect to the parameters at the point $\mathbf{p}_0$ and the simple eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{A}_0$. This information can easily be obtained with the use of standard programs for calculating eigenvalues of matrices; this makes the approach suggested constructive and applicable to numerical computations.

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